NONCOMMUTATIVE BANACH ALGEBRAS AND ALMOST PERIODIC FUNCTIONS[1]

BY
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1. Introduction

A structure theory is developed for a class of Banach algebras which we call inner product algebras (IP-algebras). We were led to these algebras by the algebra of almost periodic functions under convolution.

Let $A = \mathrm{AP}(G)$ be the set of all almost periodic functions on a topological group G considered as a Banach algebra under the norm $\|f\| = \sup |f(t)|$, pointwise addition, and convolution multiplication. This algebra is rich in structure. Not only is it a Banach algebra in the norm $\|f\|$, but also it is a pre-Hilbert space in the norm $|f| = (f, f)^{1/2}$, where the inner product is given by $(f, g) = M_t[f(t)\overline{g(t)}]$ (here M is the mean-value functional of von Neumann [8]). This pre-Hilbert space is, in general, not complete (even for G the real numbers). Denote the convolution of f and g by fg where $fg(s) = M_t[f(st^{-1})g(t)]$ [8, p. 456]. The two norms are connected [7], [8] by (1) $|f| \leqq \|f\|$ and (2) $\|fg\| \leqq |f|\,|g|$ for all $f, g \in A$. Also (3) $Af = 0$ implies $f = 0$. Moreover the natural involution $f \rightarrow f^*$ defined by $f^*(t) = \overline{f(t^{-1})}$ satisfies (4) $(fg, h) = (g, f^*h) = (f, hg^*)$ for all $f, g, h \in A$. Also (5) f lies in the closure of fA for each $f \in A$ [8, Theorem 17]. Our interest in $\mathrm{AP}(G)$ from the point of view of the general theory of Banach algebras began with the discovery that any Banach algebra with an involution which is a pre-Hilbert space satisfying conditions (1)–(5) (or even weaker conditions, see Theorem 4.9) is a semisimple dual Banach algebra.

A somewhat analogous situation was treated by Ambrose [1] who started with the L_2-algebra of a compact group as a model and abstracted to H^*-algebras. Likewise starting with $\mathrm{AP}(G)$ we abstract to what we call IP-algebras and right IP-algebras.[2] As in [1] our main goal is a structure theory for the algebras under consideration. We have, at the same time, been able to manage with requirements substantially weaker than those numbered above.

Let A be a Banach algebra which is also a pre-Hilbert space (A_h) in terms of the norm $|f|$. Suppose that, as in (1) and (3), convergence in the norm $\|f\|$ implies convergence in $|f|$ and $Af = 0$ implies $f = 0$. We call A a right IP-algebra if there exists a dense right ideal $\mathfrak{B}_r$ such that each $f \in \mathfrak{B}_r$ has a

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[2] Actually we consider an analogue of the right H^*-algebras of Smiley [13] as well.

right adjoint f^*, $(gf, h) = (g, hf^*)$ for all g, h, and right multiplication by f is a continuous mapping of A_h into A. By an IP-algebra we mean an algebra which is both a left and right IP-algebra. An advantage of requiring what is needed from (2) and (4) to hold only on a dense ideal rather than everywhere is that (unlike the H^*-algebra case) certain types of infinite direct sums of [right] IP-algebras are [right] IP-algebras. This admits a much larger variety of examples (see §2).

For structure theorems see Theorems 3.5, 4.3, 4.7, and 4.8. It is shown that any IP-algebra satisfying (5) is the direct topological sum of topologically simple IP-algebras each of which is continuously isomorphic to an algebra of completely continuous operators on a Hilbert space including all the finite-dimensional operators on that space.

2. Preliminaries and examples

Let A be an algebra over the complex field which is a Banach algebra under a norm $\| x \|$ and also a pre-Hilbert space in terms of a positive-definite inner product (x, y). Unless otherwise specified the topology on A is taken to be that provided by the norm $\| x \|$; we use A_h to designate A as a topological space under the norm $| x | = (x, x)^{1/2}$. Furthermore we let H denote the Hilbert space completion of A_h. It is not assumed that A_h is a normed algebra.

Let R_x $[L_x]$ denote the operation of right [left] multiplication by x, $R_x(y) = yx$. Let

$$\mathfrak{W}_r = \{y \in A \mid R_y \text{ is a continuous mapping of } A_h \text{ into } A\},$$

and define $\mathfrak{W}_l$ analogously. Consider $x \in \mathfrak{W}_r$, $z \in A$. There exists $a > 0$ such that $\| R_x(y) \| \leqq a | y |$, $y \in A$. Then $\| R_{xz}(y) \| \leqq (a \| z \|) | y |$, $y \in A$, so that $\mathfrak{W}_r$ is a right ideal of A.

We call an element x^* $[x']$ a *right* [*left*] *adjoint* of x if $(yx, z) = (y, zx^*)$ for all $y, z \in A$ $[(xy, z) = (y, x'z)$ for all $y, z \in A]$. In general no such elements need exist.

In these terms we formulate our basic definitions.

2.1. DEFINITIONS. We call A a *right IP-algebra* [*left IP-algebra*] if it satisfies the following conditions:

(a) For each $x \in A$, the functional $g_x(y) = (x, y)$ is continuous on A.

(b) $Ax = 0$ implies $x = 0$ $[xA = 0$ implies $x = 0]$.

(c) $\mathfrak{W}_r$ $[\mathfrak{W}_l]$ contains a dense right [left] ideal $\mathfrak{B}_r$ $[\mathfrak{B}_l]$ of A where each element of $\mathfrak{B}_r$ $[\mathfrak{B}_l]$ has a right [left] adjoint in A.

We call A an *IP-algebra* if it is both a right and a left IP-algebra (in terms of the same Banach algebra norm and inner product). Obviously every H^*-algebra is an IP-algebra.

We make some elementary observations on the definition of a right IP-algebra. It is trivial that the right adjoint of x is unique, if it exists. Suppose

x^* exists. Then $xx^* = 0$ implies $x = 0$. For if $xx^* = 0$, then $(yx, yx) = 0$ for all y, so that $Ax = 0$.

We consider next the significance of (a) from the point of view of linear space theory. Here (b) and (c) are irrelevant as are the ring properties of A, but the completeness of A in the norm $\| x \|$ is essential.

2.2. LEMMA. *Let A be a Banach space and pre-Hilbert space as above. Then* (a) *of Definition* 2.1 *holds if and only if there exists $M > 0$ such that $| x | \leqq M \| x \|$, for all $x \in A$.*

Proof. Suppose that $| x | \leqq M \| x \|$, $x \in A$. By the Cauchy-Schwarz inequality, $| (x, y) | \leqq M | x | \, \| y \|$ so that (a) holds. Suppose that (a) holds, and let H denote the completion of A in the norm $| f |$. Let $\| x_n - w \| \to 0$ in A and $| x_n - y | \to 0$ where $y \in H$. For any $v \in A$ we have, by (a), that $(v, w) = (v, y)$. Thus $y = w$. The closed graph theorem implies that, for some $M > 0$, $| x | \leqq M \| x \|$, $x \in A$.

2.3. *Example.* Let G_0 be a compact topological group, and let $C(G_0)$ be the Banach space of all continuous complex-valued functions on G_0. Consider $C(G_0)$ as an algebra under convolution (with respect to Haar measure) where we set

$$(fg)(s) = \int f(st^{-1})g(t)\, dt, \qquad (f, g) = \int f(t)\overline{g(t)}\, dt,$$

and $f^*(t) = \overline{f(t^{-1})}$. Then $C(G_0)$ is a Banach algebra in the sup norm $\| f \|$ and a pre-Hilbert space in the norm $| f | = (f, f)^{1/2}$ satisfying the relations (1) through (5) of §1. In fact $C(G_0)$ is a dual algebra [5, p. 700] which is also an IP-algebra. From (1) and (2) we see that $| fg | \leqq \| fg \| \leqq | f | \, | g |$, so that $C(G_0)$ is a normed algebra in the norm $| f |$.

Now let G be any topological group, and consider $\mathrm{AP}(G)$ as described in §1. If G_0 is the Bohr compactification of G [10, p. 331], then $\mathrm{AP}(G)$ is isometrically isomorphic to $C(G_0)$ (with convolution multiplication) where the isomorphism preserves the inner product. Conversely, since all continuous functions on a compact group G_0 are almost periodic, $C(G_0)$ is the same as $\mathrm{AP}(G_0)$.

Let $A = C(G_0)$ or $\mathrm{AP}(G)$. It is readily seen that $\| f \| = \| f^* \|$ and $| f | = | f^* |$ for all $f \in A$. An important property of A is that the mappings L_f and R_f are completely continuous as transformations from either A or A_h into either A or A_h (see [5, §8] and [9]). In particular both A and A_h are CC algebras [5, p. 698]. The algebra A is a concrete model for the development of §5 below as well as for the notion of an IP-algebra. An interesting discussion of $\mathrm{AP}(G)$, for G abelian, which proceeds in a direction unrelated to the development here, was given by Helgason [3].

In general $\mathrm{AP}(G)$ as a pre-Hilbert space is not complete. Consider, for example, G the reals. If $\mathrm{AP}(G)$ were complete, the fact that $| f | \leqq \| f \|$ for all f would imply the existence of some $K > 0$ such that $\| f \| \leqq K | f |$

for all $f \in \mathrm{AP}(G)$. But consider the function

$$f_m(x) = e^{ix} + 2^{-1}e^{2ix} + \cdots + m^{-1}e^{mix}.$$

We have

$$|f_m|^2 = \sum_{n=1}^{m} n^{-2} \quad \text{and} \quad \| f_m \| = \sum_{n=1}^{m} n^{-1},$$

so that no such K can exist.

2.4. *Example.* Consider the Banach space l^1 of all sequences $a = \{a_n\}$ such that $\| a \| = \sum | a_n | < \infty$ made into a Banach algebra by defining, for $b = \{b_n\}$ the product by $ab = \{a_n b_n\}$. Let $\{\mu_n\}$ be any bounded sequence of positive numbers, $| \mu_n | \leqq K$ for all n. We obtain an IP-algebra if the inner product is taken as $(a, b) = \sum \mu_n a_n \bar{b}_n$. Clearly $| a |^2 \leqq K \| a \|^2$. The elements with only a finite number of nonzero coordinates form a dense set $\mathfrak{B}_r$ which, as can be seen by computation, lies in $\mathfrak{W}_r$. In general $\mathfrak{W}_r$ is not the entire algebra as easy examples show.

2.5. Definitions. Let $\{A_n\}$ be a sequence of Banach algebras where we denote the norm in A_n by $\| u \|_n$. Consider the collection A of all sequences $\alpha = \{\alpha_n\}$, $\alpha_n \in A_n$, such that $\| \alpha \| = \sum \| \alpha_n \|_n < \infty$. Define, for $\beta = \{\beta_n\}$ in A and a scalar μ, $\mu\alpha = \{\mu\alpha_n\}$, $\alpha + \beta = \{\alpha_n + \beta_n\}$, and $\alpha\beta = \{\alpha_n \beta_n\}$. Then A is a Banach algebra which we call the *l^1-sum of the Banach algebras A_n*.

Consider the collection A of all sequences $\{\alpha_n\}$, $\alpha_n \in A_n$, which "vanish at infinity", i.e., for each $\varepsilon > 0$ there exists N where $\| \alpha_n \|_n < \varepsilon$ for $n \geqq N$. Define, in A, the algebraic operations as above, and set $\| \alpha \| = \sup \| \alpha_n \|_n$. Then A is a Banach algebra which we call the $B(\infty)$ *sum of the Banach algebras A_n* (see [6, p. 411] and [10, p. 106]).

2.6. Lemma. *Let $\{A_n\}$ be a sequence of right IP-algebras. Then, with appropriate choices of inner products, their $B(\infty)$ sum and l^1-sum are right IP-algebras.*

Proof. Let $\| u \|_n$ denote the given Banach-algebra norm in A_n, $(u, v)_n$ the inner product there, and let $| u |_n = (u, u)_n^{1/2}$. For each n there is, by Lemma 2.2, a number $M_n > 0$ such that $| u |_n \leqq M_n \| u \|_n$, $u \in A_n$. Let $\mathfrak{B}_r^{(n)}$ be the right ideal demanded of A_n in (c) of Definition 2.1.

Consider first A, the $B(\infty)$ sum of the algebras A_n. Let $x = \{x_n\}$, $y = \{y_n\}$ be two elements of A where $x_n \in A_n$, $y_n \in A_n$, $n = 1, 2, \cdots$. We define an inner product in A by the rule

$$(x, y) = \sum_{n=1}^{\infty} (x_n, y_n)_n/(nM_n)^2.$$

Note that $|(x, y)| \leqq \pi^2 \| x \| \| y \|/6$ so that (a) of Definition 2.1 is fulfilled (see Lemma 2.2).

Trivially $Ax = 0$ implies $x = 0$. Define $\mathfrak{B}_r$ to be the collection of all $\{x_n\}$ where each $x_n \in \mathfrak{B}_r^{(n)}$ and only a finite number of the x_n are nonzero. Clearly $\mathfrak{B}_r$ is a dense right ideal of A. Let $x = \{x_n\}$ be an element of $\mathfrak{B}_r$

where $x_n = 0, n > N$. If we set $x^* = \{x_n^*\}$, we can readily obtain $(yx, z) = (y, zx^*)$ for all $y, z \in A$. We must show then existence of a constant $K > 0$ such that $\| yx \| \leqq K | y |$, for all $y \in A$. For each n there exists a number $t(n) > 0$ such that $\| zx_n \|_n \leqq t(n) | z |_n$, $z \in A_n$. Let $y = \{y_n\} \in A$. We have the following inequalities, where in each case Max is to be taken over the set $1, 2, \cdots, N$.

$$\| yx \| = \operatorname{Max} \| y_n x_n \|_n \leqq \operatorname{Max} t(n) | y_n |_n$$
$$\leqq [\operatorname{Max}(nt(n)M_n)] | y |.$$

Consider next A, the l^1-sum of the algebras A_n. Here we define

$$(x, y) = \textstyle\sum_{n=1}^{\infty} (x_n, y_n)_n / M_n^2 .$$

Then $|(x, y)| \leqq \| x \| \, \| y \|$ and $| x | \leqq \| x \|$. We proceed as in the $B(\infty)$ case and define $\mathfrak{B}_r$ in the same way. Using the same notation, for $x = \{x_n\} \in \mathfrak{B}_r$, $x_n = 0$, $n > N$, we have

$$\| yx \| = \textstyle\sum_{n=1}^{N} \| y_n x_n \|_n \leqq \sum_{n=1}^{N} t(n) | y_n |_n$$
$$\leqq \left(\textstyle\sum_{n=1}^{N} t(n) M_n\right) | y |.$$

2.7. *Example.* We give an example of an IP-algebra A where $x \to x^*$ is everywhere defined but $x \to x'$ is not defined on all of A. The algebra A will be the $B(\infty)$ sum of algebras A_n which we now describe.

Let A_n be the set of all infinite complex matrices $a = a(i, j)$, $i, j = 1, 2, \cdots$, such that $\sum | a(i, j) |^2 < \infty$. We define the norm in A_n by

$$\| a \|_n = [\textstyle\sum | a(i, j) |^2]^{1/2}.$$

Under the usual rules for matrix addition and multiplication we obtain a Banach algebra [1, p. 367]. We define the inner product for A_n by the rule

$$(a, b)_n = \textstyle\sum_{i,j=1}^{\infty} a(i, j) \overline{b(i, j)} \, \phi_n(i),$$

where $\phi_n(k) = k$ for $k = 1, \cdots, n$ and $\phi_n(k) = 1$ for $k > n$. Set $| a |_n^2 = (a, a)_n$. Here $| a |_n^2 \leqq n \| a \|_n^2$, and $\| a \|_n \leqq | a |_n$, $a \in A_n$. Routine computations show that if one defines, for $a \in A_n$, the matrices a^* and a' by the rules

$$a^*(i, j) = \overline{a(j, i)}, \qquad a'(i, j) = \overline{a(j, i)} \phi_n(j) / \phi_n(i),$$

then $(ba, c)_n = (b, ca^*)_n$ and $(ab, c)_n = (b, a'c)_n$ for all $b, c \in A_n$.

Now let A be the $B(\infty)$ sum of the algebras A_n. This gives us an IP-algebra, by Lemma 2.6, under a suitable choice of the inner product. Since here we have, for $a = \{a_n\} \in A$, $| a_n |_n \leqq n^{1/2} \| a_n \|_n$, we may choose the inner product as

$$(a, b) = \textstyle\sum_{n=1}^{\infty} n^{-3} (a_n, b_n)_n .$$

For $a = \{a_n\}$ we set $a^* = \{a_n^*\}$ and note that $\{a_n^*\}$ lies in the $B(\infty)$ sum since $\| a_n^* \|_n = \| a_n \|_n$. It is easy to verify that a^* is the right adjoint in A of a.

It is readily seen that any $a = \{a_n\} \in A$ with only a finite number of non-zero components has a left adjoint $a' = \{a_n'\}$. Yet we show that not every $a \in A$ has a left adjoint. Suppose otherwise. It follows from Theorem 3.2 shown below that there exists a constant $K > 0$ such that $\| a' \| \leqq K \| a \|$ for all $a \in A$. Now, for each $m = 1, 2, \cdots$, we consider an element $f^{(m)} \in A$ all of whose components except the m^{th} are zero and whose m^{th} component is the matrix $a(i, j)$ where $a(m, 1) = 1$ and all other entries are zero. Note $\| f^{(m)} \| = 1$. Observe that $(f^{(m)})'$ has all its components except the m^{th} zero and that the m^{th} component is the matrix $b(i, j)$ where $b(1, m) = m$ and all other $b(i, j) = 0$; observe that $\| (f^{(m)})' \| = m$. Since m is an arbitrary integer, this is a contradiction.

The phenomenon that $x \to x^*$ is discontinuous on A_h may be observed (in spite of the fact that the mapping is continuous and defined everywhere on A). For we have, in the above notation, $| f^{(m)} | / | (f^{(m)})^* | = m$.

2.8. *Example.* We give an example of an IP-algebra where neither of $x \to x^*$ and $x \to x'$ is everywhere defined. Let A_1 be an IP-algebra, given by 2.7, where $x \to x^*$ is everywhere defined and $x \to x'$ is not. By interchanging left and right in the development of Example 2.7, we can obtain an IP-algebra A_2 in which $x \to x'$ is everywhere defined but $x \to x^*$ is not. As the desired example take the direct sum of A_1 and A_2.

We now list definitions for some items used in the analysis below. Let B be a topological algebra. For any subset S of B we denote the left [right] annihilator of S in B by $\mathfrak{L}(S)$ [$\mathfrak{R}(S)$]. As in [2] we call B an *annihilator algebra* if $\mathfrak{L}(B) = \mathfrak{R}(B) = (0)$ and if $\mathfrak{L}(I) \neq (0)$ [$\mathfrak{R}(I) \neq (0)$] for each proper closed right [left] ideal I of B. As in [5] we call B a *dual algebra* if $\mathfrak{R}\mathfrak{L}(I) = I$ for every closed right ideal and $\mathfrak{L}\mathfrak{R}(I) = I$ for every closed left ideal.

3. Right IP-algebras

We begin with some minor details useful for the ensuing proofs. Given a right IP-algebra A there exists, by Lemma 2.2, a constant $M > 0$ such that $| x | \leqq M \| x \|$, $x \in A$. Consider the operator R_z, $R_z(x) = xz$, for $z \in \mathfrak{W}_r$. There exists a constant $a > 0$ such that $\| R_z(x) \| \leqq a | x |$, $x \in A$. Let $a(z)$ denote the least such constant. Since $| R_z(x) | \leqq M a(z) | x |$, $x \in A$, the operator R_z is a bounded operator on A_h, and its norm $| R_z |$ as an operator on A_h satisfies the relation

$$| R_z | \leqq M a(z), \qquad z \in \mathfrak{W}_r. \tag{3.1}$$

Let H be the Hilbert space completion of A_h. Since A is complete, R_z can be extended, for $z \in \mathfrak{W}_r$, by continuity to a bounded operator S_z of H

into A where $\| S_z(u)\| \leqq a(z)| u |$, $u \in H$ (see [14, p. 99]). Since $| S_z(u)| \leqq Ma(z)| u |$, S_z also defines a bounded linear operator of H into A_h.

For a subset $S \subset A$ we let $S^\perp = \{x \in A \mid (x, S) = 0\}$. Let I be a right ideal of A. For any $x \in I$, $y \in I^\perp$, and $z \in \mathfrak{B}_r$, we have $(x, yz) = (xz^*, y) = 0$. Thus $I^\perp \mathfrak{B}_r \subset I^\perp$. Since $\mathfrak{B}_r$ is dense in A and $I^\perp$ is closed in A by (a) of Definition 2.1, we see that $I^\perp$ is a right ideal of A.

3.1. Lemma. *Let I be a right ideal of a right IP-algebra A. Let K be a closed right ideal of A, $K \subset I$, and let $K^P = I \cap K^\perp$. Then*

$$I\mathfrak{B}_r \subset K \oplus K^P.$$

Proof. Let $f \in I$ and $d = \inf |f - u|^2$ as u ranges over K. There exists a sequence $\{h_n\}$ in K such that $d_n \downarrow d$ where $d_n = |f - h_n|^2$. Reasoning as in [7, pp. 57–58] we see that

$$|(v, f - h_n)| \leqq (d_n - d)^{1/2} | v | \tag{3.2}$$

for all $v \in K$ and that $\{h_n\}$ is a Cauchy sequence in A_h. Let $g \in \mathfrak{B}_r$. Then there exists $h \in A$ such that $\| h - h_n g \| \to 0$. Clearly $h \in K$. We write $fg = h + (fg - h)$ and show that $fg - h \in K^P$. Let $u \in K$. By (a) and (c) of the definition of a right IP-algebra, we have

$$|(u, fg - h)| = \lim |(u, fg - h_n g)| = \lim |(ug^*, f - h_n)|.$$

But $ug^* \in K$, and therefore, by (3.2), this limit is zero.

As in [10, p. 70] we say that a Banach algebra B has a *unique norm topology* if any two Banach-algebra norms for B are equivalent.

3.2. Theorem. *Let A be a right IP-algebra. Then*

(a) *A is semisimple.*

(b) *$\mathfrak{L}(\mathfrak{M}) \neq (0)$ for each modular maximal right ideal of A.*

(c) *Each nonzero right [left] ideal of A contains a minimal right [left] ideal of A.*

(d) *A has a unique norm topology.*

Proof. Let $z \in \mathfrak{W}_r$. Since $\| xz^2 | \leqq \| xz \| \, \| z \|$ for all $x \in A$, we see that

$$a(z^2) \leqq a(z)\| z \|, \qquad z \in \mathfrak{W}_r. \tag{3.3}$$

This is the case $n = 0$ of the following relation which can be shown, by an easy induction using (3.3), to hold for all positive integers n.

$$a(z^{2^{n+1}}) \leqq \| z^{2^n} \| \, a(z) \, \| z \|^{(2^n-1)}, \qquad z \in \mathfrak{W}_r. \tag{3.4}$$

For convenience set $F(n) = | R_f |$ where $f = z^{2^n}$. In view of (3.1) we have

$$F(n) \leqq Ma(z^{2^n}). \tag{3.5}$$

Next suppose that $z \in \mathfrak{W}_r$ satisfies the relation $z = z^*$. Then right multiplication by powers of z are bounded self-adjoint operators on A_h (or on the Hilbert space H). Therefore, for any such z, we obtain $F(n+1) = [F(n)]^2$. Moreover $F(n) = |(R_z)^{2^n}|$. From (3.4) and (3.5) we then obtain

$$(3.6) \qquad |(R_z)^{2^n}|^{2^{1-n}} = [F(n+1)]^{2^{-n}} \leqq \| z^{2^n} \|^{2^{-n}} [Ma(z)]^{2^{-n}} \| z \|^{(1-2^{-n})}.$$

Suppose that, in addition $z \in \mathrm{Rad}(A)$, the radical of A. Since A is a Banach algebra, $\| z^{2^n} \|^{2^{-n}} \to 0$, so that from (3.6) we see $|(R_z)^{2^n}|^{2^{-n}} \to 0$. By the theory of self-adjoint operators on a Hilbert space, $R_z = 0$. But then $z = 0$. In summary, if $z = z^*$ and $z \in \mathfrak{W}_r \cap \mathrm{Rad}(A)$, then $z = 0$.

Now consider any element $u \in \mathrm{Rad}(A)$ and any $v \in \mathfrak{B}_r$. The preceding guarantees that $(vu)(vu)^* = 0$. But then $vu = 0$ or $\mathfrak{B}_r u = 0$. Since $\mathfrak{B}_r$ is dense, we have $Au = 0$ or $u = 0$. Therefore A is semisimple.

Let $\mathfrak{M}$ be a modular maximal right ideal of A. We show that $\mathfrak{M}^\perp \neq (0)$. For suppose otherwise. Then an application of Lemma 3.1 to the case $I = A$ and $K = \mathfrak{M}$ gives $A\mathfrak{B}_r \subset \mathfrak{M}$. This implies that $\mathfrak{B}_r$ is contained in the primitive ideal $(\mathfrak{M}:A)$. Since $\mathfrak{B}_r$ is dense and since primitive ideals of A are closed, this is impossible. Whereas $\mathfrak{M}$ is maximal and $\mathfrak{M}^\perp$ is a right ideal, we can now state

$$(3.7) \qquad A = \mathfrak{M} \oplus \mathfrak{M}^\perp.$$

Let j be a left identity for A modulo $\mathfrak{M}$ where we write $j = u + v$ in the decomposition of (3.7). From $(1-j)A \subset \mathfrak{M}$ we obtain $(1-v)A \subset \mathfrak{M}$. For $x \in \mathfrak{M}^\perp$, $(1-v)x \in \mathfrak{M} \cap \mathfrak{M}^\perp = (0)$. Therefore $vx = x$ for all $x \in \mathfrak{M}^\perp$. Consequently $\mathfrak{M}^\perp = vA$ where $v^2 = v$. We can rewrite (3.7) as $A = \mathfrak{M} \oplus vA$. By the Peirce decomposition, $A = (1-v)A \oplus vA$. Recall that $(1-v)A \subset \mathfrak{M}$. It follows that $(1-v)A = \mathfrak{M}$ from which we deduce that $\mathfrak{L}(\mathfrak{M}) = Av \neq (0)$.

It follows from (3.7) that $\mathfrak{M}^\perp = vA$ is a minimal right ideal of A. If we start with a minimal right ideal eA of A, $e^2 = e$, then from the Peirce decomposition $A = (1-e)A \oplus eA$ we see that $(1-e)A$ is a modular maximal right ideal. Thus the modular maximal right ideals are precisely the ideals of the form $(1-e)A$ where $e^2 = e$ and eA is minimal. Let S be the socle [4, p. 64] of A. This two-sided ideal is the algebraic sum of the minimal right [left] ideals of A. As A is semisimple, $\mathfrak{L}(S) = \mathfrak{R}(S)$ ([2, p. 159] or [15, p. 354]). Suppose $y \in \mathfrak{R}(S)$. Then for each minimal left ideal Ae, $e^2 = e$, we have $y \in (1-e)A$. From this and (a) we see that $y = 0$. That (c) holds follows from [15, Lemma 4.1]. That (d) holds follows from a result of Rickart [10, Theorem 2.5.7].

3.3. Corollary. *Let A be a right IP-algebra where, for each $x \in A$, x lies in the closure of xA. Then any closed right ideal R of A is the closure of the algebraic sum K of the minimal right ideals of A contained in R.*

Proof. If $K^\perp \cap R \neq (0)$, it contains, by Theorem 3.2, a minimal right ideal of A which must then be also in K. This is impossible. Lemma 3.1 now asserts that $R\mathfrak{B}_r \subset \bar{K}$. The closure hypothesis then shows that $R = \bar{K}$.

We take a closer look at a minimal left ideal.

3.4. LEMMA. *Let I be a minimal left ideal in a right IP-algebra. The two norms $| x |$ and $\| x \|$ are equivalent on I, and I is a Hilbert space in the norm $| x |$.*

Proof. Let $I = Ae$, $e^2 = e$. By the Gelfand-Mazur theorem,

$$eAe = \{\mu e \mid \mu \text{ complex}\}.$$

Thus $e\mathfrak{W}_r e = eAe$, and there exists $w \in \mathfrak{W}_r$ such that $ewe = e$. Set $f = we$. Clearly $f^2 = f$ and $Ae = Af$ where $f \in \mathfrak{W}_r$ (a right ideal). By Lemma 2.2, there exists $M > 0$ such that $| x | \leqq M \| x \|$, $x \in A$. Let $y = yf \in I$. Then $| y | \leqq M \| y \| \leqq Ma(f) | y |$. Thus the two norms are equivalent on I. Now I is closed in the topology of the norm $\| x \|$ and is a Banach space in that topology. Therefore it is complete in the topology of A_h.

For the notions of direct sum and topological direct sum of ideals in a Banach algebra see [10, p. 46].

3.5. THEOREM. *Let A be a right IP-algebra where A^2 is dense in A. Then the socle of A is dense in A, and A is the direct topological sum of its minimal closed two-sided ideals.*

Proof. Let I denote the closure of the socle S of A. For a modular maximal right ideal $\mathfrak{M}$ we can, by the proof of Theorem 3.2, write $A = \mathfrak{M} \oplus vA$ where $v^2 = v$, $\mathfrak{M} = (1 - v)A$ and $vA = \mathfrak{M}^{\perp}$. Since $\mathfrak{M}$ is a maximal right ideal, $(vA)^{\perp} = \mathfrak{M}$. Therefore $\mathfrak{M} \supset I^{\perp}$, and, as A is semisimple, $I^{\perp} = (0)$. It follows from Lemma 3.1 that $A^2 \subset I$. Our hypothesis on A^2 makes S dense in A.

Let Q be the right ideal of A which is the algebraic sum of the ideals vA where v is any idempotent as described in the preceding paragraph. The argument using these shows that Q is dense in A. We shall show that each element of Q possesses a *left* adjoint. First we consider v. For any $x, y \in A$ we can write $x = x_1 + x_2$, $y = y_1 + y_2$ where $x_1, y_1 \in \mathfrak{M}^{\perp}$ and $x_2, y_2 \in \mathfrak{M}$. A computation[3] in [11, p. 50] gives $(vx, y) = (x_1, y_1) = (x, vy)$. Therefore v is its own left adjoint. Next let $a \in vA$. The argument here is a modification of that of Saworotnow in [12, Theorem 1]. Clearly $va = a$. To see that a' exists we may assume that $av \neq 0$, for otherwise we consider $b = a + v$ where $bv \neq 0$. Now since vA is minimal and A is semisimple, vAv is a division algebra. By the Gelfand-Mazur theorem, there is a scalar μ such that $av = vav = \mu v$. Note $\mu \neq 0$. But $a^2 = vava = \mu a$. Then $\mu^{-1}a = f$ is an idempotent. Since vA is minimal, $vA = fA$. The Peirce decomposition $A = (1 - f)A \oplus fA$ makes $\mathfrak{N} = (1 - f)A$ a modular maximal right ideal of A. As in the proof of Theorem 3.2, $A = \mathfrak{N} \oplus \mathfrak{N}^{\perp}$, and we can write $f = z + v_1$, $z \in \mathfrak{N}$, $v_1 \in \mathfrak{N}^{\perp}$ obtaining $v_1^2 = v_1$ with $\mathfrak{N}^{\perp} = v_1 A$. By the above,

[3] Since $vA = M^{\perp}$, $(1 - v)A = M$ and $vx_2 = 0$, we have $vx = vx_1 = x_1$ and $(vx, y) = (x_1, y) = (x_1, y_1) = (x, y_1) = (x, vy)$.

$v_1' = v_1$. We may argue[4] as in [12, p. 57] to see that f is a nonzero scalar multiple of vv_1. Therefore f, and consequently a, possesses a left adjoint.

We now show that $K^\perp$ is a left ideal for any left ideal K of A. For let $x \in K, y \in K^\perp$, and $z \in Q$. Then $0 = (z'x, y) = (x, zy)$. Therefore $QK^\perp \subset K^\perp$. Since Q is dense in A and $K^\perp$ is closed, we see that $K^\perp$ is a left ideal.

Now let A_0 be the topological sum of the minimal closed two-sided ideals of A. We now can assert that $A_0^\perp$ is a two-sided ideal of A and wish to show $A_0^\perp = (0)$. Suppose otherwise. Then by Theorem 3.2, $A_0^\perp$ contains a minimal right ideal I of A. The arguments of [2, Theorem 5] show that $A_0^\perp$ contains a minimal closed two-sided ideal of A, which is impossible as $A_0^\perp \cap A_0 = (0)$. From this, Lemma 3.1 yields $A^2 \subset A_0$, so that we have $A_0 = A$. From the semisimplicity of A and the fact that the two-sided ideals in question are minimal closed ideals it is readily shown that we have a direct toplogical sum [10, Theorem 2.8.15], [2, Theorem 6].

4. On IP-algebras

We relate here IP-algebras to the more familiar annihilator algebras and dual algebras. Our key hypothesis is (as in Theorem 3.5) that A^2 is dense in A. Any IP-algebra with this property is an annihilator algebra (Theorem 4.4).

4.1. THEOREM. *Let A be an IP-algebra where A^2 is dense in A. Then there exists a dense two-sided ideal I such that each $x \in I$ possesses both a left and right adjoint.*

Proof. In the course of the proof of Theorem 3.5, it was shown that there exists a dense right ideal Q such that each $x \in Q$ possesses a left adjoint. Consider the two-sided ideal $I_1 = \mathfrak{B}_l Q$. Clearly I_1 is dense in A, and each element of I_1 possesses a left adjoint. Likewise there exists a dense two-sided ideal I_2 such that each element of I_2 possesses a right adjoint. Set $I = I_1 I_2$ to obtain the desired ideal.

4.2. LEMMA. *Let A be a right IP-algebra where $A\mathfrak{B}_r^*$ is dense in A_h. Then* (1) *x lies in the closure of xA in A_h for each $x \in A$, and* (2) *the closure in A_h of any right ideal I is $I^{\perp\perp}$.*

Proof. For a given $x \in A$ let M be the closure of xA in the Hilbert space completion H of A_h, and let N be the orthogonal complement of M in H. We write $x = u + v$ where $u \in M$ and $v \in N$. To establish (1) we must show that $v = 0$.

Let $z \in \mathfrak{B}_r$. Now $R_z(xA) \subset xA$, and, as noted above, S_z is a continuous mapping of H into A_h. Therefore $S_z(M) \subset M$. Let $\{v_n\}$ be a sequence in

[4] Since $va = a$ then $vf = f$. Also $0 = (z, v_1 A) = (v_1 z, A)$, so $v_1 z = 0$ and $v_1 f = v_1$. Then $0 \neq v_1 = v_1 f = v_1 vf$, so that $v_1 v \neq 0$. Thus $vv_1 = (v_1 v)' \neq 0$. Also $vv_1 = vv_1 f = vv_1 vf = \lambda f$ where $\lambda \neq 0$.

A where $|v - v_n| \to 0$. For each $w \in A$,

$$(xw, S_z(v)) = \lim (xw, v_n z) = \lim (xwz^*, v_n) = (xwz^*, v) = 0.$$

By the continuity of the inner product in H we have $S_z(v) \in N$. But $S_z(v) = xz - S_z(u)$. Thus $S_z(v) \in M \cap N = (0)$. Consequently $\lim (v_n z, w) = 0$ for all $w \in A$ which shows that $(v, wz^*) = 0$. Therefore v is orthogonal to $A\mathfrak{B}_r^*$. By hypothesis the latter set is dense in H so that $v = 0$.

We now show (2). Let K denote the closure in A_h of the right ideal I. Clearly K is closed in A by Lemma 2.2, and $I^{\perp\perp} \supset K$. Since $K^\perp \cap I^{\perp\perp} = (0)$ we learn from Lemma 3.1 that $K \supset I^{\perp\perp}\mathfrak{B}_r$. For each $x \in I^{\perp\perp}$, $x\mathfrak{B}_r$ is dense in xA in the topology of A_h by Lemma 2.2. Therefore $K \supset I^{\perp\perp}$.

4.3. THEOREM. *Let A be an IP-algebra where A^2 is dense in A. Then the closure in A_h of any right or left ideal I is $I^{\perp\perp}$.*

Proof. In order to utilize Lemma 4.2 we examine $A\mathfrak{B}_r^*$. First we show that $(A\mathfrak{B}_r^*)^\perp = (0)$. For if $(z, xw) = 0$ for all $x \in A$, $w \in \mathfrak{B}_r^*$, then $(zv, A) = 0$ for all $v \in \mathfrak{B}_r$, so that $z\mathfrak{B}_r = (0)$, and therefore $z = 0$. Now $A\mathfrak{B}_r^*$ is a left ideal of A; let K denote its closure in A. By Lemma 3.1, $\mathfrak{B}_l A \subset K \oplus K^\perp$. Inasmuch as $K^\perp = (0)$ and A^2 is dense, we see that $K = A$. It follows from Lemma 2.2 that $A\mathfrak{B}_r^*$ is dense in A_h. Therefore, by Lemma 4.2, the closure in A_h of any right ideal I is $I^{\perp\perp}$. By the interchange of left and right, the conclusion is also true for left ideals.

4.4. THEOREM. *Let A be an IP-algebra. Then A is an annihilator algebra if and only if A^2 is dense in A.*

Proof. It is readily seen that the condition on A^2 is necessary for A to be an annihilator algebra. Assume A^2 dense.

We use the one-sided ideals $\mathfrak{B}_l$ and $\mathfrak{B}_r$ of Definition 2.1; each $x \in \mathfrak{B}_r$ $[\mathfrak{B}_l]$ has a right [left] adjoint x^* $[x']$. Let K be a closed right ideal, $K \neq A$. We observe that $K^\perp \neq (0)$; for otherwise $K \supset A\mathfrak{B}_r$ by Lemma 3.1 which would make $K = A$ by our density hypothesis. Next we show that $K^\perp \cap \mathfrak{B}_l \neq (0)$. For otherwise, as $K^\perp$ is a right ideal, $K^\perp\mathfrak{B}_l = (0)$ which, since $\mathfrak{B}_l$ is dense, implies that $K^\perp = (0)$. Let $x \neq 0$, $x \in \mathfrak{B}_l \cap K^\perp$. Consider an arbitrary $z \in K$ and any $y \in \mathfrak{B}_r$. Note that $(xy, z) = (x, zy^*) = 0$. Thus $0 = (y, x'z)$. Since $\mathfrak{B}_r$ is dense, we see that $x'K = (0)$. Then $\mathfrak{L}(K) \neq (0)$. Likewise $\mathfrak{R}(I) \neq (0)$ for a closed right ideal $I \neq A$. Inasmuch as A is semisimple (Theorem 3.2), $\mathfrak{L}(A) = \mathfrak{R}(A) = (0)$.

We know no example of an IP-algebra where A^2 is not dense in A and have been unable to show A^2 is dense.[5] In that direction we offer the following.

[5] It is readily shown that A^2 is dense if A_h is complete. For then A and A_h are equivalent topologically, and $(A^2, w) = 0$ implies that $(A, w\mathfrak{B}_r) = 0$ and $w = 0$.

4.5. LEMMA. *In any right IP-algebra, A^3 is dense in A^2.*

Proof. Let B_0 denote the closure of A^2 in the Hilbert space completion H of A_h. Let $B_0^{\perp}$ be the orthogonal complement of B_0 in H. Take any $z \in \mathfrak{B}_r$. We show first that $S_z(B_0^{\perp}) = (0)$. For let $v \in B_0^{\perp}$ where $| v - w_n | \to 0$ with each $w_n \in A$. For any $x \in A$ we have

$$(x, S_z(v)) = \lim (x, w_n z) = \lim (xz^*, w_n) = (xz^*, v) = 0$$

as $xz^* \in B_0$. By the continuity of the inner product in H, $S_z(v) = 0$.

For any $x \in A$ write $x = u + v$ where $u \in B_0$ and $v \in B_0^{\perp}$. By the preceding paragraph, $xz = S_z(u)$ for each $z \in \mathfrak{B}_r$. Let $\{u_n\}$ be a sequence in A^2 where $| u - u_n | \to 0$. As noted in §3, S_z is a continuous mapping of H into A. Therefore $\| xz - u_n z \| \to 0$. Since $\mathfrak{B}_r$ is dense and $u_n \in A^2$, any element of A^2 is the limit of elements in A^3.

4.6. THEOREM. *Let A be a right IP-algebra, and B the closure of A^2. Then B is a right IP-algebra, and B^2 is dense in B. If A is an IP-algebra, then B is an annihilator algebra.*

Proof. By Lemma 4.5, A^3 is dense in A^2 from which one can deduce that A^4 is dense in A^2. This implies that B^2 is dense in B. We wish to show that B is a right IP-algebra. Let $x \in B$. If $Bx = 0$, then $A^2x = 0$ and, consequently, $x = 0$. Clearly $\mathfrak{B}_r^2$ is a dense right ideal of B. Moreover, for each $y \in \mathfrak{B}_r^2$, R_y is a continuous mapping of B (in the norm $| x |$) into B (in the norm $\| x \|$). Furthermore each $y \in \mathfrak{B}_r^2$ has a right adjoint clearly in $A^2 \subset B$. The last sentence now follows from Theorem 4.4.

As in [10, p. 101] we call A *topologically simple* if the only closed two-sided ideals of A are A and (0). The above shows that any topologically simple IP-algebra is an annihilator algebra.

4.7. THEOREM. *Let A be an IP-algebra where, for each $x \in A$, x lies in the closure of xA and in the closure of Ax. Then any closed two-sided ideal of A is an annihilator IP-algebra, and A is the topological direct sum of topologically simple annihilator IP-algebras.*

Proof. Let I be a closed right ideal of A. Suppose that $| x_n - x | \to 0$ where each $x_n \in I$. For each $y \in \mathfrak{B}_r$ we have $\| x_n y - xy \| \to 0$, so that $I \supset x\mathfrak{B}_r$. Our hypotheses show that $x \in I$ so that I is also closed in A_h. Therefore the closed left and right ideals in A are identical with those in A_h. In particular, by Theorem 4.3, $I = I^{\perp\perp}$ for any such ideal I.

Let K be a closed two-sided ideal of A. Note that $K^{\perp}$ is also a two-sided ideal of A. Let $x \in K$, and suppose that x possesses a right adjoint x^* in A. For each $y \in K^{\perp}$ we have $yx = 0$. Hence $0 = (yx, z) = (y, zx^*)$ for all $z \in A$. Therefore $Ax^* \subset K^{\perp\perp} = K$, and consequently $x^* \in K$ (this argument is taken from [1, Lemma 2.5]).

We verify that K is a right IP-algebra. Observe that $\mathfrak{B}_r K$ is a dense right ideal of K and can be used to satisfy (c) of Definition 2.1. That K is a semisimple annihilator algebra follows from Theorem 4.4 and [10, Theorem 2.8.12]. The final conclusion is a consequence of the structure theory of [2].

It is natural to consider the topologically simple case next. For this we adopt the following notation. Given a Hilbert space E let $\mathfrak{F}(E)$ $[\mathfrak{K}(E)]$ be the algebra of all finite-dimensional [completely continuous] bounded linear operations on E.

4.8. Theorem. *Let A be a topologically simple IP-algebra. Then there exist a Hilbert space E and a continuous isomorphism T of A onto a dense subset of $\mathfrak{K}(E)$ where $T(A) \supset \mathfrak{F}(E)$, and, whenever x' exists, $T(x')$ is the adjoint operator of $T(x)$.*

Proof. As already observed, A is an annihilator algebra. By Lemma 3.4 a minimal left ideal $E = Ae$, $e^2 = e$, of A is a Hilbert space in the norm $|x|$. A continuous isomorphism T of A onto a dense set of $\mathfrak{K}(E)$ containing $\mathfrak{F}(E)$ is set up, according to [2, Theorems 9 and 10], by defining $T(b)(xe) = bxe$. Suppose that b' exists. Then $(T(b)(xe), ye) = (xe, T(b')(ye))$ in terms of the inner product of E.

Every semisimple dual Banach algebra is an annihilator algebra [2]. So far as we know it is an open problem to decide whether or not the converse holds for semisimple Banach algebras. In order to obtain A as a dual algebra we have been compelled by our methods to assume that either the left or right adjoint exists for all elements of A. In all the work to this point the adjoint operations need only be defined for suitable dense sets. But all the hypotheses here are fulfilled by AP(G), for example.

4.9. Theorem. *Let A be a right IP-algebra where, for each $x \in A$, the closure of xA contains x and x has a left adjoint x'. Then A is a dual algebra.*

Proof. Consider a left ideal K. We have (see [5, p. 697]) that

$$Kx = 0 \quad \leftrightarrow \quad (A, Kx) = 0 \quad \leftrightarrow \quad (K'A, x) = 0,$$

while the last is equivalent to $(K', x) = 0$ since K' lies in the closure of $K'A$. Therefore $\mathfrak{R}(K) = (K')^{\perp}$. Now let I be a right ideal. Since I' is a left ideal, we have $\mathfrak{L}(I) = [\mathfrak{R}(I')]' = (I^{\perp})'$. But $\mathfrak{L}(I)$ is itself a left ideal so that $\mathfrak{R}\mathfrak{L}(I) = I^{\perp\perp}$. Suppose that I is closed. It follows readily from Lemma 3.1 that $I^{\perp\perp}\mathfrak{B}_r \subset I$. This implies here that $I^{\perp\perp} = I$, and so $I = \mathfrak{R}\mathfrak{L}(I)$.

It follows from Theorem 3.2 (d) that $x \to x'$ is bicontinuous on A. Let K be a closed left ideal. Then K' is a closed right ideal, so that $\mathfrak{R}\mathfrak{L}(K) = [\mathfrak{L}\mathfrak{R}(K')]' = K$, and A is a dual algebra.

5. On A_h, a normed algebra

We shall assume that A_h is a normed algebra and, under suitable conditions, compare the ideals in A_h with those of its completion H.

The specific assumptions on a right IP-algebra A which will be assumed in §5 (after the axiomatic investigation of Theorem 5.1) are

(1) A is a normed algebra in the norm $|x|$.
(2) Each element of A has a right adjoint.
(3) A^2 is dense in A_h.
(4) The mapping $x \to x^*$ is continuous on A_h.

5.1. THEOREM. *Let A be a right IP-algebra satisfying* (2) *and* (3). *Suppose that each $x \in A^2$ has a left adjoint and $x' = x^*$. Then* (4) *is valid, and $x' = x^*$ for all $x \in A$.*

Proof. Let $x, y, w, v \in A^2$. It is easy to see that $(xy, uv) = ((uv)^*, (xy)^*)$. By linearity we see that $(x, y) = (y^*, x^*)$ for all $x, y \in A^4$. Note that A^4 is dense in A^2 by Lemma 4.5, and therefore A^4 is dense in A_h. On A^4 we have, in particular, $|x| = |x^*|$. We shall show that $|x| = |x^*|$ for all x. If A_h were complete, this would be immediately clear; since it is not, in general, we must rely on a more complicated argument.

Let $x \in A$, and choose a sequence $\{x_n\} \in A^4$ with $|x - x_n| \to 0$. Then $\{x_n\}$ is a Cauchy sequence in A_h, and

$$|(z, x_n^*) - (z, x_m^*)| \leqq |z|\,|x_m - x_n| \to 0 \quad \text{as} \quad n, m \to \infty.$$

Therefore $f(z) = \lim\, (z, x_n^*)$ exists, and $|f(z)| \leqq |x|\,|z|$, so that $f(z)$ is continuous on A_h. Also note that $|f(z) - (z, x_m^*)| \leqq |z|\,|x - x_m|$.

Since $|x - x_n| \to 0$, then $|xy - x_n y| \to 0$ for all $y \in \mathfrak{W}_r$. If we choose $y \in \mathfrak{W}_r^3$, then we also know that $|y^*x^* - y^*x_n^*| \to 0$ and that y' exists and is equal to y^*. Then, for such y and any $z \in A$,

$$(yz, x_n^*) = (z, y^*x_n^*) \to (z, y^*x^*) = (yz, x^*).$$

Therefore $f(w) = (w, x^*)$ for all $w \in \mathfrak{W}_r^3 A$. But $\mathfrak{W}_r^3 A$ is dense in A^4 and therefore in A_h. Since $f(w)$ and (w, x^*) are both continuous functionals on A_h, $f(w) = (w, x^*)$ for all w. Then

$$\begin{aligned} |(|x^*|^2 - |x_n^*|^2)| &\leqq |f(x^*) - (x^*, x_n^*)| + |f(x_n^*) - (x_n^*, x_n^*)| \\ &\leqq (|x^*| + |x_n|)(|x - x_n|) \to 0. \end{aligned}$$

Thus $|x^*| = \lim |x_n^*| = \lim |x_n| = |x|$.

Now that we know $(x, x) = (x^*, x^*)$ for all x, we see easily that also $(x, y) = (y^*, x^*)$ for all $x, y \in A$. Then, for any $x, y, z \in A$ we have $(xy, z) = (z^*, y^*x^*) = (z^*x, y^*) = (y, x^*z)$. This shows that x' exists for all x and is equal to x^*.

5.2. LEMMA. *H is a right H^*-algebra.*

Proof. For this notion see [13]. The given involution of assumption (2) may, by (4), be extended to be an involution (which we also denote by $x \to x^*$) on H. The only verification which is at all necessary is to show that, for $u \in H$, $Hu = 0$ implies $u = 0$. Suppose that $Hu = 0$, and let $\{u_n\}$ be a sequence in A, $|u - u_n| \to 0$. We have $|u^* - u_n^*| \to 0$. For all $g, h \in A$, $(u_n^* g, h) \to (u^*g, h) = 0$. But then $(u_n^*, hg^*) \to 0$ which makes u^* orthogonal to A^2. Therefore, by (3), $u = 0$. From this it follows, in particular, that H is semisimple.

5.3. THEOREM. *If A_h is topologically simple, then so is H.*

Proof. Let I be a closed two-sided ideal of H, $I \neq (0)$. If we show that $I \cap A \neq (0)$, then $I \cap A = A$ and $I = H$.

Suppose $I \cap A = (0)$. Let $x \in I$ and $y \in \mathfrak{W}_r$. There exists a sequence $\{x_n\}$ in A such that $|x - x_n| \to 0$. The sequence $x_n y$ converges in both norms, hence to an element of A. But $|xy - x_n y| \to 0$. Therefore $xy \in I \cap A = (0)$. This shows that $I\mathfrak{W}_r = (0)$. Inasmuch as $\mathfrak{W}_r$ is dense in A_h, we see that $IH = (0)$. Since H is a right H^*-algebra, this yields $I = (0)$, which is impossible.

Now the nature of topologically simple right H^*-algebras is described in [13]. Thus A can be realized as a suitable matrix algebra.

5.4. THEOREM. *Suppose that, for each $x \in A$, the operator R_x is a completely continuous operator on A_h. Then the minimal right, left, and two-sided ideals of A are the same as those of H.*

Proof. It is not difficult to show that, for each $x \in A$, the operator L_x is also completely continuous on A_h. For let T denote the involution $x \to x^*$; note that T is continuous on A_h and that $L_x = TR_{x^*} T$. For each $x \in A$, R_x can be extended by continuity from A_h to H. It is readily seen that, so extended, it is completely continuous. Next let $y \in H$. The operation R_y of right multiplication by y is completely continuous as an operator on H being the uniform limit of such operators.

Recall that A is semisimple (Theorem 3.2). It follows from the Riesz theory (see [5, p. 698]) that the minimal right and left ideals are finite-dimensional. Let eA, $e^2 = e$, be a minimal right ideal of A. Inasmuch as eA is finite-dimensional, $eA = eH$. Moreover H is semisimple by Theorem 3.2 or [13]. Thus the minimal one-sided ideals of A are minimal one-sided ideals of H. In this vein we mention that any right [left] ideal I of A which is finite-dimensional is automatically a right [left] ideal of H.

Consider now any minimal right ideal I of A. Let $[I]$ be the intersection of all two-sided ideals of A containing I. By the reasoning of the proof of [2, Theorem 5], $[I]$ is a minimal two-sided ideal of A. Moreover, by Theorem 3.2, every two-sided ideal contains a minimal right ideal, so that all minimal two-sided ideals of A are of this form. Given the minimal right ideal $I = eA$, $e^2 = e$, we note that AeA is [5, p. 698] a finite-dimensional two-sided ideal containing I. It follows that all minimal two-sided ideals of A are finite-dimensional and are minimal two-sided ideals of H.

Recall that H is semisimple. Then the reasoning which we have employed shows that the minimal right, left, and two-sided ideals of H are finite-dimensional. Our task is to show that these ideals are all already in A.

To this end we examine first the socle S of A (see the proof of Theorem 3.2). We show that $S^{\perp} = (0)$. Let $y \in S^{\perp}$, and let I be a minimal right ideal of A. Inasmuch as $xx^* = 0$ implies $x = 0$, a lemma of Rickart [10, Lemma 4.10.1] shows that we can write $I = eA$ where $e^2 = e = e^*$. Since $Ae \subset S$, we have $(x, ye) = (xe, y) = 0$ for all $x \in A$. Thus $yI = (0)$, so that $y \in \mathfrak{L}(S)$. But, as noted in the proof of Theorem 3.2, $\mathfrak{L}(S) = (0)$.

Here $\mathfrak{L}(S)$ is the left annihilator of S in A. We wish to consider also the left annihilator $\mathfrak{Q}(S)$ of S in H. We show that $\mathfrak{Q}(S) = (0)$. It follows from Theorem 3.2 (d) that $x \to x^*$ is bicontinuous on A. Therefore $\mathfrak{B}_r^*$ is dense in A, so that $A\mathfrak{B}_r^*$ is dense in A^2 in the topology of the norm $\| x \|$ and therefore *a fortiori* in A_h. But by hypothesis, A^2 is dense in A_h. Lemma 4.2 then applies to show that the closure of S in A_h is $S^{\perp\perp}$. Since $S^{\perp} = (0)$, S is dense in A_h and therefore in H. Let $w \in \mathfrak{Q}(S)$. The semisimplicity of H now gives $w = 0$.

Let M be a minimal two-sided ideal of H. We know that $M \cap A$ is a finite-dimensional ideal of A, thus an ideal of H. Therefore $M \cap A = M$ or $M \cap A = (0)$. We rule out the latter possibility. Suppose that $M \cap A = (0)$. Let Ae be a minimal left ideal of A, $e^2 = e$. Since Ae is a left ideal of H, $MAe = (0)$. Then $M \subset \mathfrak{Q}(S) = (0)$, which is impossible. Now we have $M \cap A = M$ or $A \supset M$.

Consider next a minimal right ideal I of H. We have shown that the intersection K of all the two-sided ideals of H containing I is a minimal two-sided ideal of H. As just established, $K \subset A$. Thus $I \subset A$.

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