BROWNIAN MOTIONS ON A HALF LINE

Dedicated to W. Feller

BY

K. ITÔ AND H. P. MCKEAN, JR.[1]

Contents

Numbering. 1 means formula 1 of the present section; 2.1 means formula 1 of Section 2, etc.; the numbering of the diagrams is similar.

1. The classical Brownian motions

Consider the space of all (continuous) *sample paths* $w:[0, +\infty) \to R^1$

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with coordinates $\mathfrak{x}(t, w) = \mathfrak{x}(t)$ $(t \geqq 0)$, the field A of events

1. $$B = \overset{-1}{w}_{t_1 t_2 \cdots t_n}(A) = (w: (\mathfrak{x}(t_1), \mathfrak{x}(t_2), \cdots, \mathfrak{x}(t_n)) \in A)$$
$$0 < t_1 < t_2 < \cdots < t_n, \quad A \in \mathsf{B}(R^n), \quad n \geqq 1,$$[2]

and the Gauss kernel

2. $$g(t, a, b) = e^{-(b-a)^2/2t}/(2\pi t)^{1/2}, \quad (t, a, b) \in (0, +\infty) \times R^2.$$

Because of

3a. $$g(t, a, b) > 0,$$

3b. $$\int g(t, a, b)\, db = 1,$$

3c. $$g(t, a, b) = \int g(t - s, a, c) g(s, c, b)\, dc \qquad (t > s),$$

the function

4. $$P_a(B) = \int_A g(t_1, a, b_1) g(t_2 - t_1, b_1, b_2) \cdots g(t_n - t_{n-1}, b_{n-1}, b_n) \cdot db_1\, db_2 \cdots db_n$$

of $B = \overset{-1}{w}_{t_1 t_2 \cdots t_n}(A) \in \mathsf{A}$ is well-defined, nonnegative, additive, and of total mass $+1$ for each $a \in R^1$, and, as N. Wiener [1] discovered, the estimate

5. $$\int_{|a-b|>\varepsilon} g(t, a, b)\, db < \text{constant} \times \varepsilon^{-1} t^{1/2} e^{-\varepsilon^2/2t}, \qquad t \downarrow 0,$$

permits us to extend it to a nonnegative Borel measure $P_a(B)$ of total mass $+1$ on the Borel extension B of A (see P. Lévy [3] for an alternative proof).

Granting this, it is apparent that $P_a(\mathfrak{x}(0) \in db)$ is the unit mass at $b = a$. $P_a(B)$ is now interpreted as *the chance of the event B for paths starting at the point a* and the sample path $w: t \to \mathfrak{x}(t)$ with these probabilities imposed is called *standard Brownian motion starting at a.*

Given $t \geqq 0$, if $B \in \mathsf{B}$ and if w_t^+ denotes the shifted path $w_t^+: s \to \mathfrak{x}(t + s, w)$, then 4 implies

6. $$P_a(w_t^+ \in B \mid \mathfrak{x}(s): s \leqq t) = P_b(B), \qquad b = \mathfrak{x}(t),$$

i.e., the *law* of the *future $\mathfrak{x}(s): s > t$ conditional on the past $\mathfrak{x}(s): s \leqq t$* depends upon the *present* $b = \mathfrak{x}(s)$ alone (in short, *the Brownian traveller starts afresh at each constant time $t \geqq 0$*).

Because the Gauss kernel $g(t, a, b)$ is the fundamental solution of the heat flow problem

7. $$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial a^2}, \qquad (t, a) \in (0, +\infty) \times R^1,$$

[2] $\mathsf{B}(R^n)$ is the usual topological Borel field of the n-dimensional euclidean space R^n.

the operator $\mathfrak{G} = D^2/2$ acting on[3] $C^2(R^1)$ is said to *generate* the standard Brownian motion, and it is natural to seek other differential operators $\mathfrak{G}^\bullet$ giving rise via the fundamental solution of $\partial u/\partial t = \mathfrak{G}^\bullet u$ and the rule 4 to similar (stochastic) motions.

Consider, for example, the operator[4]

8. $$\mathfrak{G}^+ = \mathfrak{G} \mid C^2[0, +\infty) \cap (u{:}u^+(0) = 0):$$

the fundamental solution of $\partial u/\partial t = \mathfrak{G}^+ u$ is

9. $$g^+(t, a, b) = e^{-(b-a)^2/2t}/(2\pi t)^{1/2} + e^{-(b+a)^2/2t}/(2\pi t)^{1/2}, \qquad t > 0 \leqq a, b,$$

which satisfies 3a, 3b, and 3c, and the corresponding (*reflecting Brownian*) *motion* is identical in law to

10. $$\mathfrak{x}^+ = |\,\mathfrak{x}\,|,$$

where $\mathfrak{x}$ is a standard Brownian motion.

Consider next the operator

11. $$\mathfrak{G}^- = \mathfrak{G} \mid C^2[0, +\infty) \cap (u{:}u(0) = 0):$$

the fundamental solution of $\partial u/\partial t = \mathfrak{G}^- u$ is

12. $$g^-(t, a, b) = e^{-(b-a)^2/2t}/(2\pi t)^{1/2} - e^{-(b+a)^2/2t}/(2\pi t)^{1/2}, \qquad t > 0 \leqq a, b,$$

which satisfies 3 with

3b(bis). $$\int g^-(t, a, b)\, db < 1$$

in place of 3b, and the corresponding (*absorbing Brownian*) *motion* is identical in law to

13. $$\begin{aligned} \mathfrak{x}^-(t) &= \mathfrak{x}^+(t) \quad \text{if} \quad t < \mathfrak{m}_0, \\ &= \infty \qquad\ \ \text{if} \quad t \geqq \mathfrak{m}_0, \end{aligned}$$

where $\mathfrak{x}^+$ is the reflecting Brownian motion described above, $\mathfrak{m}_0$ is its passage time $\mathfrak{m}_0 = \min(t{:}\mathfrak{x}^+(t) = 0)$, and ∞ is an extra state adjoined to R^1.

Given $0 < \gamma < +\infty$, the operator

14. $$\mathfrak{G}^\gamma = \mathfrak{G} \mid C^2[0, +\infty) \cap (u{:}\gamma u(0) = u^+(0))$$

is also possible: the fundamental solution of $\partial u/\partial t = \mathfrak{G}^\gamma u$ is

15a. $$\begin{aligned} g^\gamma(t, a, b) &= g^\gamma(t, b, a) \\ &= g^-(t, a, b) + \int_0^t \frac{a}{(2\pi s^3)^{1/2}} e^{-a^2/2s}\, g^\gamma(t - s, 0, b)\, ds, \quad t > 0 \leqq a, b, \end{aligned}$$

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[3] $C^d(R^1)$ is the space of bounded continuous functions $f{:}R^1 \to R^1$ with d bounded continuous derivatives.

[4] $C^2[0, +\infty)$ is the space of functions $u \in C[0, +\infty)$ with $D^2u \in C(0, +\infty)$ and $(D^2u)(0) \equiv (D^2u)(0+)$ existing. $u^+(0) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1}[u(\varepsilon) - u(0)]$.

15b. $$g^{\gamma}(t, 0, 0) = 2 \int_0^{+\infty} e^{-\gamma c} \frac{c}{(2\pi t^3)^{1/2}} e^{-c^2/2t}\, dc, \qquad t > 0,$$

which satisfies 3 with 3b(bis) in place of 3b, and the corresponding (*elastic Brownian*) *motion* is identical in law to

16a. $$\mathfrak{x}^{\gamma}(t) = \mathfrak{x}^{+}(t) \quad \text{if} \quad t < \mathfrak{m}_\infty,$$
$$= \infty \quad \text{if} \quad t \geqq \mathfrak{m}_\infty,$$

16b. $$\mathfrak{m}_\infty = \mathfrak{t}^{-1}(\mathfrak{e}/\gamma),$$

where $\mathfrak{e}$ is an exponential holding time independent of the reflecting Brownian motion $\mathfrak{x}^+$ with law $P.(\mathfrak{e} > t) = e^{-t}$ and $\mathfrak{t}^{-1}$ is the inverse function of the *reflecting Brownian local time*:

17. $$\mathfrak{t}^{+}(t) = \lim_{\varepsilon \downarrow 0} (2\varepsilon)^{-1} \text{ measure } (s : \mathfrak{x}^+(s) < \varepsilon,\ s \leqq t)$$

(see Sections 3, 4, 14 for additional information about local times).

2. Feller's Brownian motions

W. Feller [1] discovered that the classical Brownian generators $\mathfrak{G}^{\pm}$ and $\mathfrak{G}^{\gamma}$ $(0 < \gamma < +\infty)$ of Section 1 are the simplest members of a wide class of restrictions $\mathfrak{G}^{\bullet}$ of $\mathfrak{G} \mid C^2[0, +\infty)$ which generate what could be called Brownian motions on $[0, +\infty)$. Feller found that the domain $D(\mathfrak{G}^{\bullet}) \subset C^2[0, +\infty)$ of such a generator could be described in terms of three nonnegative numbers p_1, p_2, p_3, and a nonnegative mass distribution $p_4(dl)$ $(l > 0)$ subject to[5]

1. $$p_1 + p_2 + p_3 + \int_{0+} (l \wedge 1) p_4(dl) = 1$$

as follows:

2. $$D(\mathfrak{G}^{\bullet}) = C^2[0, +\infty) \cap \Big(u : p_1 u(0) - p_2 u^+(0) + p_3(\mathfrak{G}u)(0) = \int_{0+} [u(l) - u(0)] p_4(dl)\Big).$$

M. Kac [1] cited the problem of describing the sample paths of the elastic Brownian motion $(p_3 = p_4 = 0 < p_1 p_2)$, and it was W. Feller's (private) suggestion that these should be the reflecting Brownian sample paths, killed at the instant some increasing function $\mathfrak{t}^+(\mathfrak{Z}^+ \cap [0, t])$ of the visiting set $\mathfrak{Z}^+ \equiv' (t : \mathfrak{x}^+(t) = 0)$ hits a certain level, that was the starting point of this paper.

P. Lévy's profound studies [3] had clarified the fine structure of the standard and reflecting Brownian motions and their local times, the papers of E. B. Dynkin [1] and G. Hunt [1] on Markov times provided an indispensable

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[5] $a \wedge b$ is the smaller of a and b. $\int_{0+}$ means $\int_{0<l<+\infty}$.

tool, H. Trotter [1] proved a deep result about local times, and W. Feller [2] had presented a (partial) description of the sample paths of the Brownian motion associated with $\mathfrak{G}^\bullet$ in the special case $p_4(0, +\infty) < +\infty$ (the case $p_4(0, +\infty) = +\infty$ was not discovered in Feller's original proof of 2, but this error was corrected by W. Feller [3] and A. D. Ventsell [1]).

It was left to use these ideas (and some new ones) to build up the sample paths of Feller's Brownian motions from the reflecting Brownian motion and its local time and (independent) exponential holding times and differential processes; that is the aim of the present paper.

3. Outline

Brownian motions on $[0, +\infty)$ are defined from a probabilistic point of view in Section 5, and a special case is disposed of in Section 6. Green operators

$$G^\bullet_\alpha: f \to E^\bullet_\cdot\left(\int_0 e^{-\alpha t} f(\mathfrak{x}^\bullet)\, dt\right)$$

and the generator $\mathfrak{G}^\bullet$ $(= \alpha - G^{\bullet\,-1}_\alpha)$ are introduced in Section 7 and computed in Section 8 using a method of E. B. Dynkin [1]. $\mathfrak{G}^\bullet$ turns out to be the restriction of $\mathfrak{G} \mid C^2[0, +\infty)$ to a domain $D(\mathfrak{G}^\bullet)$ as described in 2.2; it is the *simplest complete invariant* of the motion, i.e., the associated sample paths can be built up from

(a) *a reflecting Brownian motion* $\mathfrak{x}^+$,
(b) *a differential process* $\mathfrak{p}$ *with increasing sample paths based on* p_2 *and* p_4,
(c) *a stochastic clock* $\mathfrak{f}^{-1}$ *based on* $\mathfrak{x}^+$, $\mathfrak{p}$, *and* p_3,
(d) *a killing time based on* $\mathfrak{x}^+$, $\mathfrak{p}$, $\mathfrak{f}^{-1}$, *and* p_1

(see Sections 9–15)

Consider, for the sake of conversation, the case:

1. $$p_4(0, +\infty) = +\infty \quad \text{if} \quad p_2 = 0,$$

introduce the reflecting Brownian motion $\mathfrak{x}^+$ as described in Section 1 $(u^+(0) = 0)$, and let $\mathfrak{t}^+$ be P. Lévy's *mesure du voisinage* (*local time*)

2. $$\mathfrak{t}^+(t) = \lim_{\varepsilon \downarrow 0} (2\varepsilon)^{-1} \text{ measure } (s: \mathfrak{x}^+(s) < \varepsilon, s \leqq t)$$

as described in Section 4.

Given $p_1 = p_3 = 0$, if $\mathfrak{p}(dt \times dl)$ is a Poisson measure as described in Section 11 with mean $dt \times p_4\,(dl)$ indepedent of $\mathfrak{x}^+$, if $\mathfrak{p}$ is the (increasing) differential process

3. $$\mathfrak{p}(t) = p_2 t + \int_{0+} l\mathfrak{p}([0, t] \times dl), \qquad t \geqq 0,$$

and if $\mathfrak{p}^{-1}$ is its inverse function, then the desired motion is identical in law to[6]

4. $$\mathfrak{x}^{\bullet} = \mathfrak{p}\mathfrak{p}^{-1}\mathfrak{t}^{+} - \mathfrak{t}^{+} + \mathfrak{x}^{+},$$

which could be described as *a reflecting Brownian motion jumping out from $l = 0$ like the germ of the differential process $\mathfrak{p}$ run with the clock* $\mathfrak{p}^{-1}\mathfrak{t}^{+}$ (see Section 12 for pictures).

$\mathfrak{p}^{-1}\mathfrak{t}^{+}$ can be interpreted as a *local time* for the new sample path $\mathfrak{x}^{\bullet}$ (see Section 14), and, with its help, the description of the sample paths can be completed as follows: in case $p_1 = 0$, the desired motion is identical in law to

5a. $$\mathfrak{x}^{\bullet}(\mathfrak{f}^{-1}), \qquad \mathfrak{x}^{\bullet} = \mathfrak{p}\mathfrak{p}^{-1}\mathfrak{t}^{+} - \mathfrak{t}^{+} + \mathfrak{x}^{+},$$

where the stochastic clock $\mathfrak{f}^{-1}$ is the inverse function of

5b. $$\mathfrak{f} = t + p_3\,\mathfrak{p}^{-1}(\mathfrak{t}^{+}(t)),$$

while, in case $p_1 > 0$, it is identical in law to $\mathfrak{x}^{\bullet}(\mathfrak{f}^{-1})$ *killed* (i.e., sent off to an extra state ∞) at a time $\mathfrak{m}^{\bullet}_{\infty}$ ($< +\infty$) with conditional distribution

6. $$P.(\mathfrak{m}^{\bullet}_{\infty} > t \mid \mathfrak{x}^{\bullet}(\mathfrak{f}^{-1})) = e^{-p_1\mathfrak{p}^{-1}\mathfrak{t}^{+}\mathfrak{f}^{-1}}.$$

Here are two simple cases to be treated in Section 10.

Given $p_1 = p_4 = 0 < p_2\,p_3$ (i.e., $u^{+}(0) = (p_3/p_2)(\mathfrak{G}^{\bullet}u)(0)$), the desired motion is identical in law to

7a. $$\mathfrak{x}^{\bullet} = \mathfrak{x}^{+}(\mathfrak{f}^{-1}),$$

7b. $$\mathfrak{f} = t + (p_3/p_2)\mathfrak{t}^{+}.$$

$\mathfrak{f}^{-1}$ *counts standard time while $\mathfrak{x}^{\bullet}(t) > 0$ but runs slow on the barrier, and hence, compared to the reflecting Brownian motion, $\mathfrak{x}^{\bullet}$ lingers at $l = 0$ a little longer than it should*; as a matter of fact,

8. $$\text{measure } (s:\mathfrak{x}^{\bullet}(s) = 0,\ s \leqq t) = p_3\,\mathfrak{t}^{+}(\mathfrak{f}^{-1}(t)) > 0$$

if $t > \min\,(s:\mathfrak{x}^{\bullet}(s) = 0)$.

Given $p_3 = p_4 < p_1\,p_2$ (i.e., $(p_1/p_2)\,u(0) = u^{+}(0)$), the desired (elastic Brownian) motion is identical in law to a reflecting Brownian motion, killed at time $\mathfrak{m}^{\bullet}_{\infty}$ with conditional distribution

9. $$P.(\mathfrak{m}^{\bullet}_{\infty} > t \mid \mathfrak{x}^{+}) = e^{-(p_1/p_2)\,\mathfrak{t}^{+}(t)},$$

i.e., *killed on the barrier $l = 0$ at a rate $(p_1/p_2)\mathfrak{t}^{+}(dt){:}dt$ proportional to the local time.*

Brownian motions with similar barriers at both ends of $[-1, +1]$ or with a two-sided barrier on the line or the unit circle are studied in Sections 16 and 17, Section 18 treats a wider class of Brownian motions on $[0, +\infty)$, substantiating a conjecture of N. Ikeda, Section 19 describes the sample paths in case a diffusion operator $\mathfrak{G}u = u^{+}(dl)/e(dl)$ is used in place of

[6] $\mathfrak{p}\mathfrak{p}^{-1}\mathfrak{t}^{+}$ means $\mathfrak{p}(\mathfrak{p}^{-1}(\mathfrak{t}^{+}))$.

the reflecting Brownian generator $\mathfrak{G}^+$, and Section 20 indicates how to adapt the method to birth and death processes.

4. Standard Brownian motion: stopping times and local times

Before coming to Brownian motions on a half line, it is convenient to collect in one place some facts about the standard Brownian motion on the line (see K. Itô and H. P. McKean, Jr. [1] for the proofs and additional information).

Consider a standard Brownian motion with sample paths $w:t \to \mathfrak{x}(t)$, universal field $\mathbf{B}$, and probabilities $P_a(B)$ as described in Section 1, define[7] $\mathbf{B}_t = \mathbf{B}[\mathfrak{x}(s):s \leqq t]$, and, if $\mathfrak{m} = \mathfrak{m}(w)$ is a stopping time, i.e., if

1a. $$0 \leqq \mathfrak{m} \leqq +\infty,$$

1b. $$(\mathfrak{m} < t) \in \mathbf{B}_t, \qquad t \geqq 0,$$ [8]

then introduce the associated field

2. $$\mathbf{B}_{\mathfrak{m}+} = \mathbf{B} \cap (B:(\mathfrak{m} < t) \cap B \in \mathbf{B}_t, t \geqq 0).$$

$\mathbf{B}_{\mathfrak{m}+} = \bigcap_{s>t} \mathbf{B}_s$ in case $\mathfrak{m} \equiv t$; in general, $(\mathfrak{m} < t) \in \mathbf{B}_{\mathfrak{m}+}$ $(t \geqq 0)$, and, with the aid of

3a. $$\mathbf{B}_{\mathfrak{a}+} \subset \mathbf{B}_{\mathfrak{b}+}, \qquad \mathfrak{a} \leqq \mathfrak{b},$$

3b. $$\mathbf{B}_{\mathfrak{a}+} = \bigcap_{\varepsilon>0} \mathbf{B}_{\mathfrak{b}+}, \qquad \mathfrak{b} = \mathfrak{a} + \varepsilon,$$

it is not hard to see that $\mathbf{B}_{\mathfrak{m}+}$ measures the *past* $x(t):t \leqq \mathfrak{m}+$, i.e.,

4. $$\mathbf{B}_{\mathfrak{m}+} \supset \bigcap_{\varepsilon>0} \mathbf{B}[\mathfrak{x}(t \wedge (\mathfrak{m} + \varepsilon)):t \geqq 0].$$

E. B. Dynkin [1] and G. Hunt [1] discovered that *the Brownian traveller starts afresh at a stopping time*; this means that for each stopping time $\mathfrak{m}$, each $a \in R^1$, and each $B \in \mathbf{B}$,

5. $$P_a(w_{\mathfrak{m}}^+ \in B \mid \mathbf{B}_{\mathfrak{m}+}) = P_b(B), \qquad b = \mathfrak{x}(\mathfrak{m})$$

where $w_{\mathfrak{m}}^+$ denotes the shifted path[9] $w_{\mathfrak{m}}^+:t \to \mathfrak{x}(t + \mathfrak{m})$, $\mathfrak{x}(+\infty) \equiv \infty$, and $P_\infty(\mathfrak{x}(t) \equiv \infty, t \geqq 0) = 1$. Because $\mathfrak{m} \equiv t$ is a stopping time, 5 includes the *simple Markovian evolution* noted in 1.6; an alternative statement is that *conditional on* $\mathfrak{m} < +\infty$ *and on the present state* $b = \mathfrak{x}(\mathfrak{m})$, *the future* $\mathfrak{x}(t + \mathfrak{m}):t \geqq 0$ *is a standard Brownian motion, independent of* $\mathfrak{m}$ *and of the past* $\mathfrak{x}(t):t \leqq \mathfrak{m}+$.

Given $l > 0$, the *passage time* $\mathfrak{m}_l \doteq \min(t:\mathfrak{x}(t) = l)$ is a stopping time, and the motion $[\mathfrak{m}_l:l \geqq 0, P_0]$ is a differential process, homogeneous in the parameter l; it is, in fact, *the one-sided stable process* with exponent $\frac{1}{2}$, rate

[7] $\mathbf{B}[\mathfrak{q}(t):a \leqq t < b]$ means the smallest Borel subfield of $\mathbf{B}$ measuring the motion indicated inside the brackets.

[8] $(\mathfrak{m} < t)$ is short for $(w:\mathfrak{m} < t)$.

[9] ∞ is an extra state $\notin R^1$.

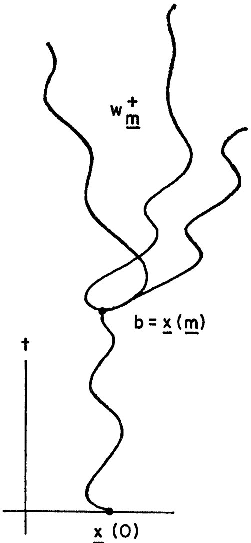


DIAGRAM 1

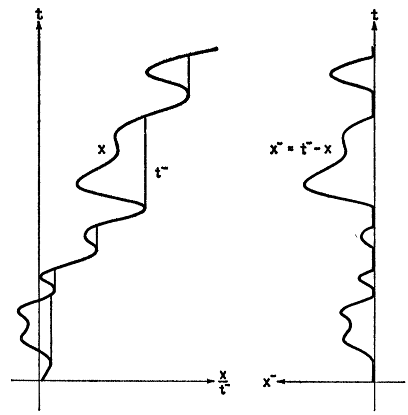


DIAGRAM 2

$\sqrt{2}$, and law

$$6. \qquad P_0(\mathfrak{m}_l \in dt) = \frac{l}{(2\pi t^3)^{1/2}} e^{-l^2/2t}\, dt$$

as P. Lévy [2] discovered.

$\mathfrak{m}.$, itself, is a sum of positive jumps (see Section 11 for information on this point), and its inverse function $\mathfrak{t}^-(t) = \max_{s \leq t} \mathfrak{x}(s)$ is continuous and flat outside a (Cantor-like) set of times of Hausdorff-Besicovitch dimension number $\frac{1}{2}$; the joint law

$$7. \qquad P_0[\mathfrak{x}(t) \in da, \mathfrak{t}^-(t) \in db] = 2\frac{2b-a}{(2\pi t^3)^{1/2}} e^{-(2b-a)^2/2t}\, da\, db, \qquad b \geqq 0,\ a \leqq b$$

is cited for future use.

Consider, next, the reflecting Brownian motion $\mathfrak{x}^+ = |\mathfrak{x}|$.

Given a reflecting Brownian stopping time $\mathfrak{m}$, i.e., a time $0 \leqq \mathfrak{m} \leqq +\infty$ with $(\mathfrak{m} < t) \in \mathsf{B}[\mathfrak{x}^+(s): s \leqq t]$ $(t \geqq 0)$, $\mathfrak{m}$ is likewise a standard Brownian stopping time, and it follows that, conditional on $\mathfrak{m} < +\infty$ and $b = \mathfrak{x}^+(\mathfrak{m})$, the shifted path $\mathfrak{x}^+(t + \mathfrak{m}): t \geqq 0$ is a reflecting Brownian motion, independent of $\mathfrak{m}$ and of the past $\mathfrak{x}^+(t): t \leqq \mathfrak{m}$; *in brief, the reflecting Brownian motion starts afresh at its stopping times.*

P. Lévy [3] observed that if $\mathfrak{x}$ is a standard Brownian motion starting at 0, then $\mathfrak{x}^- = \mathfrak{t}^- - \mathfrak{x}$ $(\mathfrak{t}^- = \max_{s \leqq t} \mathfrak{x}(s))$ is identical in law to the reflecting Brownian motion $\mathfrak{x}^+$ starting at 0. Diagram 2 is a mere caricature of the path, the actual visiting set $(t:\mathfrak{x} = 0)$ being a closed Cantor-like set of Lebesgue measure 0.

P. Lévy also indicated a proof of

8. $$P_0[\lim_{\varepsilon\downarrow 0}(2\varepsilon)^{-1}\text{ measure }(s:\mathfrak{x}^-(s) < \varepsilon, s \leqq t) = \mathfrak{t}^-(t), t \geqq 0] = 1,$$

which implies that $\mathfrak{t}^-$ is a function of $\mathfrak{x}^-$ alone, and deduced the existence of *the reflecting Brownian local time* (*mesure du voisinage*):

9. $$\mathfrak{t}^+(t) = \lim_{\varepsilon\downarrow 0}(2\varepsilon)^{-1}\text{ measure }(s:\mathfrak{x}^+(s) < \varepsilon, s \leqq t)$$

(see H. Trotter [1] for a complete proof). $\mathfrak{t}^+$ grows on the visiting set $\mathfrak{Z}^+ = (t:\mathfrak{x}^+(t) = 0)$; it is identical in law to $\mathfrak{t}^-$, and its inverse function $\mathfrak{t}^{-1}$ is identical in law to the standard Brownian passage times; especially, the joint law

10. $$P_0[\mathfrak{x}^+(t) \in da, \mathfrak{t}^+(t) \in db] = 2\,\frac{b+a}{(2\pi t^3)^{1/2}}\,e^{-(b+a)^2/2t}\,da\,db, \qquad a,b \geqq 0,$$

is deduced from the joint law of $\mathfrak{x}$ and $\mathfrak{t}^-$ above.

Skorokhod [1] has made the point that if $\mathfrak{x}$ is a standard Brownian motion, if $0 \leqq \mathfrak{x}^\bullet$ is continuous, if $0 \leqq \mathfrak{t}^\bullet$ is continuous, increasing, and flat outside $\mathfrak{Z}^\bullet = (t:\mathfrak{x}^\bullet = 0)$, and if $\mathfrak{x}^\bullet = \mathfrak{t}^\bullet - \mathfrak{x}$, then $\mathfrak{x}^\bullet = \mathfrak{x}^-$ and $\mathfrak{t}^\bullet = \mathfrak{t}^-$.

5. Brownian motions on $[0, +\infty)$

Given *probabilities* $P_a^\bullet(B)$ $(a \in [0, +\infty) \cup \infty)$ defined on the natural *universal field* $\mathbf{B}^\bullet$ of the *path space* comprising all *sample paths*

1a. $$w^\bullet : t \to \mathfrak{x}^\bullet(t) \equiv \mathfrak{x}^\bullet(t+) \in [0, +\infty) \cup \infty,$$

1b. $$\mathfrak{x}^\bullet(t) \equiv \infty, \qquad t \geqq \mathfrak{m}_\infty^\bullet \equiv \inf(t:\mathfrak{x}^\bullet = \infty)$$

and subject to

2a. *$P_a^\bullet(B)$ is a Borel function of a,*

2b. *$P_a^\bullet[\mathfrak{x}^\bullet(0) \in db]$ is the unit mass at $b = a$* $\quad (a \neq 0)$,

let us speak of the associated motion as

(a) *simple Markov* if it starts afresh at constant times:

3a. $$P_\cdot^\bullet(w_s^{\bullet+} \in B \mid \mathbf{B}_s^\bullet) = P_a^\bullet(B), \qquad s \geqq 0, B \in \mathbf{B}^\bullet, a = \mathfrak{x}^\bullet(s),$$

where $w_s^{\bullet+}$ is the shifted path $t \to \mathfrak{x}^\bullet(t + s)$ and $\mathbf{B}_s^\bullet$ is the field of $\mathfrak{x}^\bullet(t):t \leqq s$,

(b) *strict Markov* if it starts afresh at its stopping times:

3b. $$P_\cdot^\bullet(w_{\mathfrak{m}^\bullet}^{\bullet+} \in B \mid \mathbf{B}_{\mathfrak{m}^\bullet+}^\bullet) = P_a^\bullet(B), \qquad B \in \mathbf{B}^\bullet, a = \mathfrak{x}^\bullet(\mathfrak{m}^\bullet),$$

for each stopping time

4a. $$0 \leqq \mathfrak{m}^\bullet \leqq +\infty,$$

4b. $$(\mathfrak{m}^\bullet < t) \in \mathbf{B}_t^\bullet \quad (t \geqq 0),$$

where $\mathfrak{x}^\bullet(+\infty) \equiv \infty$ and $\mathbf{B}^\bullet_{\mathfrak{m}^\bullet+}$ is the field of events

5a. $$B \in \mathbf{B}^\bullet,$$

5b. $$B \cap (\mathfrak{m}^\bullet < t) \in \mathbf{B}^\bullet_t \quad (t \geqq 0),$$

(c) *a Brownian motion* if, in addition to (b), the stopped path

6a. $$\mathfrak{x}^\bullet(t): t < \mathfrak{m}_{0+} = \lim_{\varepsilon \downarrow 0} \inf (t: \mathfrak{x}^\bullet < \varepsilon), \qquad \mathfrak{x}^\bullet(0) = l > 0,$$

is identical in law to the stopped standard Brownian motion

6b $$\mathfrak{x}(t): t < \mathfrak{m}_0 = \min (t: \mathfrak{x} = 0), \qquad \mathfrak{x}(0) = l.$$

$E^\bullet_\cdot$ denotes the integral (expectation) based upon $P^\bullet_\cdot$, and $E^\bullet_\cdot(e, B) = E^\bullet_\cdot(B, e)$ denotes the integral of $e = e(w^\bullet)$ extended over B; the subscript . as in 3a and 3b stands for an unspecified point of $[0, +\infty) \cup \infty$ with the understanding that if several dots appear in a *single* formula, then it is the *same* point that is meant each time.

6. Special case: $p_+(0) < 1$

Given a Brownian motion as described above and a sample path $\mathfrak{x}^\bullet$ starting at $\mathfrak{x}^\bullet(0) = l > 0$, the *crossing time*

1. $$\mathfrak{m}^\bullet = \mathfrak{m}^\bullet_\varepsilon = \inf (t: \mathfrak{x}^\bullet(t) < \varepsilon), \qquad 0 < \varepsilon < l,$$

is a stopping time, $P^\bullet_l[\mathfrak{x}^\bullet(\mathfrak{m}^\bullet_\varepsilon) = \varepsilon] = 1$, $\mathfrak{m}^\bullet_{0+} = \lim_{\delta \downarrow 0} \mathfrak{m}^\bullet_\delta = \mathfrak{m}^\bullet_\varepsilon + \mathfrak{m}^\bullet_{0+}(w^{\bullet+}_{\mathfrak{m}^\bullet})$, and, since the stopped path $\mathfrak{x}^\bullet(t): t < \mathfrak{m}^\bullet_{0+}$ is standard Brownian,

2. $$\begin{aligned} &E^\bullet_l[e^{-\alpha \mathfrak{m}^\bullet_{0+}}, \mathfrak{x}^\bullet(\mathfrak{m}^\bullet_{0+}) \in B] \\ &= E^\bullet_l(e^{-\alpha \mathfrak{m}^\bullet_\varepsilon} E^\bullet_l[\exp(-\alpha \mathfrak{m}^\bullet_{0+}(w^{\bullet+}_{\mathfrak{m}^\bullet})), \mathfrak{x}^\bullet(\mathfrak{m}^\bullet_{0+}(w^{\bullet+}_{\mathfrak{m}^\bullet}), w^{\bullet+}_{\mathfrak{m}^\bullet}) \in B \mid \mathbf{B}^\bullet_{\mathfrak{m}^\bullet+}]) \\ &= E^\bullet_l(e^{-\alpha \mathfrak{m}^\bullet_\varepsilon}) E^\bullet_\varepsilon[e^{-\alpha \mathfrak{m}^\bullet_{0+}}, \mathfrak{x}^\bullet(\mathfrak{m}^\bullet_{0+}) \in B] \\ &\to E^\bullet_l(e^{-\alpha \mathfrak{m}^\bullet_{0+}}) P^\bullet_\varepsilon[\mathfrak{x}^\bullet(\mathfrak{m}^\bullet_{0+}) \in B] \qquad (\varepsilon \downarrow 0) \\ &= e^{-(2\alpha)^{1/2} l} P^\bullet_l[\mathfrak{x}^\bullet(\mathfrak{m}^\bullet_{0+}) \in B], \end{aligned}$$ [10]

i.e., $\mathfrak{x}^\bullet(\mathfrak{m}^\bullet_{0+})$ *is independent of* $\mathfrak{m}^\bullet_{0+}$, *and its law* $p_+(B) = P^\bullet_l[\mathfrak{x}^\bullet(\mathfrak{m}^\bullet_{0+}) \in B]$ *does not depend on* $l > 0$.

Consider the law $p(dl) \equiv P^\bullet_0[\mathfrak{x}^\bullet(0) \in dl]$, and, in case $p(0) = 1$, let $\mathfrak{e}$ be the *exit time* $\inf(t: \mathfrak{x}^\bullet(t) \neq 0)$.

Because

3a. $$p_+(0) = P^\bullet_l[\mathfrak{x}^\bullet(\mathfrak{m}^\bullet_{0+}) = 0, \mathfrak{x}^\bullet(0, w^{\bullet+}_{\mathfrak{m}^\bullet_{0+}}) = 0] = p_+(0)p(0), \qquad l > 0,$$

and

3b. $$p(0) = P^\bullet_0[\mathfrak{x}^\bullet(0) = 0, \mathfrak{x}^\bullet(0, w^{\bullet+}_0) = 0] = p(0)^2,$$

[10] $\mathfrak{m}^\bullet_{0+}$ is identical in law to the standard Brownian passage time $\mathfrak{m}_0 = \min(t: \mathfrak{x}(t) = 0)$, and hence $E^\bullet_l(\exp(-\alpha \mathfrak{m}^\bullet_{0+})) = \exp(-(2\alpha)^{1/2} l)$ (see 4.6).

the possibilities are

4a. $$p(0) = p_+(0) = 0,$$

4b. $$p(0) = 1 > p_+(0),$$

4c. $$p(0) = p_+(0) = 1.$$

4a is the simplest case. Diagram 1 shows the motion $[\mathfrak{x}^\bullet, P_0^\bullet]$: the jumps l_1, l_2, etc. are independent with common law $p_+(dl)$, the initial position l_0 is independent of l_1, l_2, etc. with law $p(dl)$, and the excursions leading back to $l = 0+$ are standard Brownian.

4b is more interesting. $\mathfrak{e}$ *is an exponential holding time independent of* $\mathfrak{x}^\bullet(\mathfrak{e})$ *with law* e^{-t/p_3} $(0 \leqq p_3 \leqq +\infty)$; indeed, if $s \geqq 0$, then $(\mathfrak{e} > s) \in \mathbf{B}_{s+}^\bullet = \bigcap_{t>s} \mathbf{B}_t^\bullet$, whence

5. $$P_0^\bullet(\mathfrak{e} > t + s) = P_0^\bullet(\mathfrak{e} > s, \mathfrak{e}(w_s^{\bullet +}) > t) = P_0^\bullet(\mathfrak{e} > s) P_0^\bullet(\mathfrak{e} > t)$$

and

6. $$\begin{aligned} P_0^\bullet[\mathfrak{e} > s, \mathfrak{x}^\bullet(\mathfrak{e}) \in dl] &= P_0^\bullet[\mathfrak{e} > s, \mathfrak{x}^\bullet(\mathfrak{e}(w_s^{\bullet +}) + s) \in dl] \\ &= P_0^\bullet(\mathfrak{e} > s) P_0^\bullet[\mathfrak{x}^\bullet(\mathfrak{e}) \in dl], \end{aligned}$$

completing the proof.

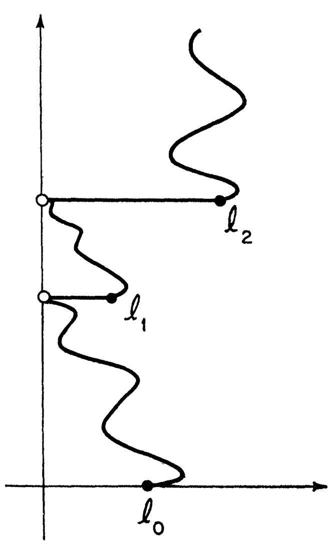


DIAGRAM 1

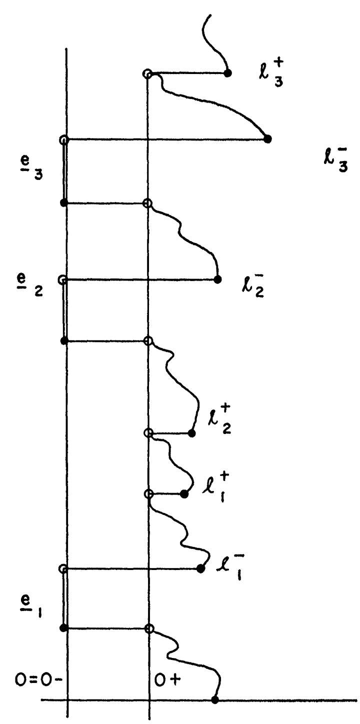


DIAGRAM 2

p_3 *has to be positive*; in the opposite case,

$$P_0^\bullet(\mathfrak{e} = 0) = p(0) = P_0^\bullet(\lim_{\varepsilon\downarrow 0} \mathfrak{m}_\varepsilon^\bullet = 0) = 1,$$

where now $\mathfrak{m}_\varepsilon^\bullet$ is the sum of the crossing time $\mathfrak{m}^\bullet = \inf(t: \mathfrak{x}^\bullet(t) > \varepsilon)$ and $\mathfrak{m}_{0+}^\bullet(w_{\mathfrak{m}^\bullet}^{\bullet +})$, and hence

7. $$1 = p(0) = P_0^\bullet(\lim_{\varepsilon\downarrow 0} \mathfrak{x}^\bullet(\mathfrak{m}_\varepsilon^\bullet) = 0) = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} P_0^\bullet(\mathfrak{x}^\bullet(\mathfrak{m}_\varepsilon^\bullet) < \delta) = \lim_{\delta\downarrow 0} p_+[0, \delta) = p_+(0),$$

contradicting $p_+(0) < 1$.

$p_-(dl) \equiv P_0^\bullet[\mathfrak{x}^\bullet(\mathfrak{e}) \in dl, \mathfrak{e} < +\infty]$ *attributes no mass to* $l = 0$ as is clear from

8a. $$P_0^\bullet(\mathfrak{e} > 0) = \lim_{t\downarrow 0} e^{-t/p_3} = 1$$

and

8b. $$p_-(0) = P_0^\bullet[\mathfrak{x}^\bullet(\mathfrak{e}) = 0, \mathfrak{e} < +\infty, \mathfrak{e}(w_{\mathfrak{e}}^{\bullet +}) = 0] \leqq P_0^\bullet(\mathfrak{e} = 0).$$

Diagram 2 is now evident; the jumps l_1^-, l_2^-, etc., l_1^+, l_2^+, etc., and the holding times $\mathfrak{e}_1$, $\mathfrak{e}_2$, etc. are independent with common laws $P(l_1^- \in dl) = p_-(dl)$, $P(l_1^+ \in dl) = p_+(dl)$, $P(\mathfrak{e}_1 > t) = e^{-t/p_3}$, and the excursions leading back to $l = 0+$ are standard Brownian.

4c occupies us in Sections 7–15; a further class of ramified simple Markov motions is studied in Section 18.

7. Green operators and generators: $p_+(0) = 1$

Consider the case $p_+(0) = 1$ (6.4c), and introduce the Green operators

1. $$G_\alpha^\bullet : f \in C[0, +\infty) \to E_\cdot^\bullet\left(\int_0^{\mathfrak{m}_\infty^\bullet} e^{-\alpha t} f(\mathfrak{x}^\bullet)\, dt\right), \qquad \alpha > 0.$$

Because $\mathfrak{m}^\bullet \equiv \mathfrak{m}_{0+}^\bullet = \lim_{\varepsilon\downarrow 0} \inf(t: \mathfrak{x}^\bullet(t) < \varepsilon)$ is a stopping time and $P_\cdot^\bullet(\mathfrak{x}^\bullet(\mathfrak{m}^\bullet) = 0) \equiv 1$,

2. $$\begin{aligned}(G_\alpha^\bullet f)(l) &= E_l^\bullet\left(\int_0^{\mathfrak{m}_{0+}^\bullet} e^{-\alpha t} f(\mathfrak{x}^\bullet)\, dt\right) \\ &\quad + E_l^\bullet\left(e^{-\alpha \mathfrak{m}_{0+}^\bullet} E_l^\bullet\left(\int_0^{\mathfrak{m}_\infty^\bullet(w_{\mathfrak{m}^\bullet}^{\bullet +})} e^{-\alpha t} f[\mathfrak{x}^\bullet(t + \mathfrak{m}^\bullet)]\, dt \mid \mathbf{B}_{\mathfrak{m}^\bullet +}^\bullet\right)\right) \\ &= (G_\alpha^- f)(l) + E_l^\bullet(e^{-\alpha \mathfrak{m}_{0+}^\bullet}) E_0\left(\int_0^{\mathfrak{m}_\infty^\bullet} e^{-\alpha t} f(\mathfrak{x}^\bullet)\, dt\right) \\ &= (G_\alpha^- f)(l) + e^{-(2\alpha)^{1/2} l} (G_\alpha^\bullet f)(0),\end{aligned}$$

where G_α^- is the Green operator for the (absorbing) Brownian motion with instant killing at $l = 0$:

3. $$(G_\alpha^- f)(a) = E_a\left(\int_0^{\mathfrak{m}_0} e^{-\alpha t} f(\mathfrak{x})\, dt\right)$$
$$= \int_0^{+\infty} \frac{e^{-(2\alpha)^{1/2}|b-a|} - e^{-(2\alpha)^{1/2}|b+a|}}{(2\alpha)^{1/2}} f\, db, \quad a \geqq 0;$$

especially, $G_\alpha^\bullet$ maps $C[0, +\infty)$ *into* $C^2[0, +\infty)$.

Given $\alpha, \beta > 0$ and $f \in C[0, +\infty)$,

4. $(\alpha - \beta)\, G_\alpha^\bullet G_\beta^\bullet f$
$$= (\alpha - \beta)\, E_\cdot^\bullet\left(\int_0^{\mathfrak{m}_\infty^\bullet} e^{-\alpha t}\, (G_\beta^\bullet f)(\mathfrak{x}^\bullet)\, dt\right)$$
$$= (\alpha - \beta)\, E_\cdot^\bullet\left(\int_0^{\mathfrak{m}_\infty^\bullet} e^{-\alpha t}\, dt\, E_{\mathfrak{x}^\bullet(t)}^\bullet\left(\int_0^{\mathfrak{m}_\infty^\bullet} e^{-\beta s} f(\mathfrak{x}^\bullet)\, ds\right)\right)$$
$$= (\alpha - \beta)\, E_\cdot^\bullet\left(\int_0^{\mathfrak{m}_\infty^\bullet} e^{-(\alpha-\beta)t}\, dt \int_t^{\mathfrak{m}_\infty^\bullet} e^{-\beta s} f(\mathfrak{x}^\bullet)\, ds\right)$$
$$= E_\cdot^\bullet\left(\int_0^{\mathfrak{m}_\infty^\bullet} e^{-\beta s} f(\mathfrak{x}^\bullet)\, ds\, (\alpha - \beta) \int_0^s e^{-(\alpha-\beta)t}\, dt\right)$$
$$= G_\beta^\bullet f - G_\alpha^\bullet f,$$

i.e.,

5. $$G_\alpha^\bullet - G_\beta^\bullet + (\alpha - \beta) G_\alpha^\bullet G_\beta^\bullet = 0, \qquad \alpha, \beta > 0,$$

proving that the *range* $G_\alpha^\bullet\, C[0, +\infty) \equiv D(\mathfrak{G}^\bullet)$ and the *null-space* $G_\alpha^{\bullet\,-1}(0)$ are both independent of $\alpha > 0$; in fact, $G_\beta^{-1}(0) = \bigcap_{\alpha>0} G_\alpha^{\bullet-1}(0) = 0$ because if f belongs to it, then

6. $$0 = \lim_{\alpha\uparrow+\infty} \alpha(G_\alpha^\bullet f)(l) = \lim_{\alpha\uparrow+\infty} E_l^\bullet\left(\alpha\int_0^{\mathfrak{m}_\infty^\bullet} e^{-\alpha t} f(\mathfrak{x}^\bullet)\, dt\right) = f(l), \quad l \geqq 0,$$

thanks to $P_l^\bullet(\mathfrak{x}^\bullet(0+) = l) \equiv 1 \quad (l \geqq 0)$.

$G_\alpha^\bullet$ is now seen to be invertible, and another application of 5 implies that

7. $$\mathfrak{G}^\bullet \equiv \alpha - G_\alpha^{\bullet-1}: D(\mathfrak{G}^\bullet) \to C[0, +\infty)$$

is likewise independent of $\alpha > 0$.

$\mathfrak{G}^\bullet$ *is the generator cited in the section title; it is a contraction of* $\mathfrak{G} = D^2/2$ *acting on* $C^2[0, +\infty)$ because

8a. $$D(\mathfrak{G}^\bullet) = G_1^\bullet\, C[0, +\infty) \subset C^2[0, +\infty)$$

and

8b. $$(\alpha - \mathfrak{G}) G_\alpha^- = 1, \qquad \alpha > 0.$$

Given two Brownian motions with the *same generator*, their Green operators and hence their transition probabilities and laws in function space are the same, i.e., $\mathfrak{G}^\bullet$ *is a complete invariant of the Brownian motion.*

8. Generator and Green operators computed: $p_+(0) = 1$

$D(\mathfrak{G}^\bullet)$ can be described in terms of three nonnegative numbers p_1, p_2, p_3 and a nonnegative mass distribution $p_4(dl)$ $(l > 0)$ subject to

1a. $$p_1 + p_2 + p_3 + \int_{0+} (l \wedge 1)\, p_4(dl) = 1$$

and

1b. $$p_4(0, +\infty) = +\infty \quad \text{in case} \quad p_2 = p_3 = 0,$$

namely, $D(\mathfrak{G}^\bullet)$ is *the class of functions* $u \in C^2[0, +\infty)$ *subject to*[11]

2a. $$p_1\, u(0) + p_3(\mathfrak{G}u)(0) = p_2\, u^+(0) + \int_{0+} [u(l) - u(0)]\, p_4(dl),$$

as will now be proved.

1b is automatic from the rest because if $p_2 = p_3 = 0$ and $p_4(0, +\infty) < +\infty$, then an application of 2a to $u = \alpha G_\alpha^\bullet f \in D(\mathfrak{G}^\bullet)$ implies, on letting $\alpha \uparrow +\infty$, that

$$[p_1 + p_4(0, +\infty)]\, f(0) = \int_{0+} f p_4(dl) \qquad \text{for each } f \in C[0, +\infty),$$

which is absurd in view of 1a. Besides, it is enough to prove that

2b. $$D(\mathfrak{G}^\bullet) \subset C^2[0, +\infty) \cap \Big(u : p_1 u(0) + p_3(\mathfrak{G}u)(0) = p_2 u^+(0) + \int_{0+} [u(l) - u(0)]\, p_4(dl)\Big)$$

or some choice of p_1, p_2, p_3, p_4 subject to 1a, because, if u is a member of the second line, then so is the bounded solution $u^\bullet = G_1^\bullet(1 - \mathfrak{G})u - u$ of $\mathfrak{G}u^\bullet = u^\bullet$, and, expressing $u^\bullet$ as $c_1 e^{2^{1/2} l} + c_2 e^{-2^{1/2} l}$, it is found that $c_1 = c_2 = u^\bullet \equiv 0$, i.e., $u = G_1^\bullet(1 - \mathfrak{G})u \in D(\mathfrak{G}^\bullet)$.

Consider, for the proof of 2b, the *exit time*

3. $$\mathfrak{e} = \inf(t : \mathfrak{x}^\bullet(t) \neq 0)$$

and its law

4. $$P_0^\bullet(\mathfrak{e} > t) = e^{-t/k} \qquad (0 \leqq k \leqq +\infty),$$

and bear in mind that $\mathfrak{x}^\bullet(\mathfrak{e})$ is independent of $\mathfrak{e}$:

5. $$P_0^\bullet[\mathfrak{e} > t,\ \mathfrak{x}^\bullet(\mathfrak{e}) \in dl] = e^{-t/k}\, p(dl).$$

If $k = +\infty$ $(\mathfrak{e} \equiv +\infty)$, then $(\mathfrak{G}^\bullet u)(0) = 0$ for each $u \in D(\mathfrak{G}^\bullet)$, and 2b holds with $p_1 = p_2 = p_4 = 0$ and $p_3 = 1$.

---

[11] $\mathfrak{G} = D^2/2$.

If $0 < k < +\infty$, then

6. $$p(0) = P_0^\bullet[\mathfrak{x}^\bullet(\mathfrak{e}) = 0, \mathfrak{e}(w_\mathfrak{e}^{\bullet +}) = 0] \leqq P_0^\bullet(\mathfrak{e} = 0) = 0,$$

and choosing $u = G_\alpha^\bullet f \in D(\mathfrak{G}^\bullet)$, it appears that

7a. $$\begin{aligned} u(0) &= f(0)E_0^\bullet\left(\int_0^\mathfrak{e} e^{-\alpha t}\,dt\right) + E_0^\bullet[e^{-\alpha\mathfrak{e}}u(\mathfrak{x}^\bullet(\mathfrak{e})), \mathfrak{e} < \mathfrak{m}_\infty^\bullet] \\ &= \frac{\alpha u(0) - (\mathfrak{G}^\bullet u)(0)}{\alpha + k} + \frac{k}{\alpha + k}\int_{0+} up(dl), \end{aligned}$$ [12]

or, what is the same,

7b. $$u(0) + k^{-1}(\mathfrak{G}^\bullet u)(0) = \int_{0+} up(dl),$$

i.e., 2b holds with $p_1:p_2:p_3:p_4 = 1:0:k^{-1}:p$.

But, if $k = 0$ ($\mathfrak{e} \equiv 0$), the proof is less simple; the method used below is due to E. B. Dynkin [1].

$(\mathfrak{G}^\bullet u)(0) < -1$ for some $u \in D(\mathfrak{G}^\bullet)$ (if not, then $(\mathfrak{G}^\bullet u)(0) \equiv 0$, $f(0) = (1 - \mathfrak{G}^\bullet)G_1^\bullet f(0) = (G_1^\bullet f)(0)$ for each $f \in C[0, +\infty)$, and $P_0^\bullet(\mathfrak{e} = +\infty) = 1$), so, choosing $\varepsilon > 0$ so small that $(\mathfrak{G}^\bullet u)(l) < -1$ ($l \leqq \varepsilon$) and introducing the *crossing time* $\mathfrak{m}_\varepsilon^\bullet = \inf(t: \mathfrak{x}^\bullet(t) > \varepsilon)$, it is clear from

8. $$\begin{aligned} u(0) &= E_0^\bullet\left(\int_0^{\mathfrak{m}_\infty^\bullet} e^{-\alpha t}f(\mathfrak{x}^\bullet)\,dt\right), \qquad f = (\alpha - \mathfrak{G}^\bullet)u, \\ &= E_0^\bullet\left(\int_0^{\mathfrak{m}_\varepsilon^\bullet \wedge \mathfrak{m}_\infty^\bullet} e^{-\alpha t}(\alpha - \mathfrak{G}^\bullet)u(\mathfrak{x}^\bullet)\,dt\right) \\ &\qquad + E_0^\bullet[e^{-\alpha\mathfrak{m}_\varepsilon^\bullet}u(\mathfrak{x}^\bullet(\mathfrak{m}_\varepsilon^\bullet)), \mathfrak{m}_\varepsilon^\bullet < \mathfrak{m}_\infty^\bullet] \end{aligned}$$

that

9. $$E_0^\bullet(\mathfrak{m}_\varepsilon^\bullet \wedge \mathfrak{m}_\infty^\bullet) \leqq \lim_{\alpha \downarrow 0} E_0^\bullet\left(\int_0^{\mathfrak{m}_\varepsilon^\bullet \wedge \mathfrak{m}_\infty^\bullet} e^{-\alpha t}(\alpha - \mathfrak{G}^\bullet)u(\mathfrak{x}^\bullet)\,dt\right) < +\infty.$$

$(\mathfrak{G}^\bullet u)(0) < -1$ has no special advantage for the derivation of 8 which holds for each $u \in D(\mathfrak{G}^\bullet)$ and $\varepsilon > 0$; thus, keeping $\varepsilon > 0$ so small that $E_0^\bullet(\mathfrak{m}_\varepsilon^\bullet \wedge \mathfrak{m}_\infty^\bullet) < +\infty$ and letting $\alpha \downarrow 0$ in 8 implies

10. $$u(0) = -E_0^\bullet\left(\int_0^{\mathfrak{m}_\varepsilon^\bullet \wedge \mathfrak{m}_\infty^\bullet} (\mathfrak{G}^\bullet u)(\mathfrak{x}^\bullet)\,dt\right) + E_0^\bullet[u(\mathfrak{x}^\bullet(\mathfrak{m}_\varepsilon^\bullet)), \mathfrak{m}_\varepsilon^\bullet < +\infty], \qquad u \in D(\mathfrak{G}^\bullet),$$

and letting $\varepsilon \downarrow 0$ in 10 establishes E. B. Dynkin's *formula for the generator*:

11a. $$(\mathfrak{G}^\bullet u)(0) = \lim_{\varepsilon \downarrow 0} \int_{[\varepsilon,+\infty)\cup\infty} [u(l) - u(0)]p_\varepsilon(dl), \quad u \in D(\mathfrak{G}^\bullet), u(\infty) \equiv 0,$$

[12] $(\alpha - \mathfrak{G}^\bullet)G_\alpha^\bullet = 1$.

11b. $$p_\varepsilon(dl) = E_0^\bullet(\mathfrak{m}_\varepsilon^\bullet \wedge \mathfrak{m}_\infty^\bullet)^{-1} P_0^\bullet[\mathfrak{x}^\bullet(\mathfrak{m}_\varepsilon^\bullet \wedge \mathfrak{m}_\infty^\bullet) \in dl],$$

or, what is better for the present purpose,

12a. $$\lim_{\varepsilon \downarrow 0} \left[\frac{p_\varepsilon(\infty)}{D} u(0) + \frac{(\mathfrak{G}^\bullet u)(0)}{D} - \int_{[\varepsilon,+\infty)} u^\bullet(l)(l \wedge 1) \frac{p_\varepsilon(dl)}{D}\right] = 0,$$

12b. $$D = p_\varepsilon(\infty) + 1 + \int_{0+} (l \wedge 1) p_\varepsilon(dl),$$

12c. $$\begin{aligned} u^\bullet(l) &= \frac{u(l) - u(0)}{l \wedge 1} && \text{if } l > 0, \\ &= u^+(0) && \text{if } l = 0. \end{aligned}$$

Because $D(\mathfrak{G}^\bullet) \subset C^2[0, +\infty)$, $u^\bullet \in C[0, +\infty)$ and selecting $\varepsilon = \varepsilon_1 > \varepsilon_2 >$ etc. $\downarrow 0$ so as to have

13a. $$\lim_{\varepsilon \downarrow 0} p_\varepsilon(\infty)/D = p_1,$$

13b. $$\lim_{\varepsilon \downarrow 0} 1/D = p_3,$$

13c. $$\lim_{\varepsilon \downarrow 0} (l \wedge 1) p_3(dl)/D = p_*(dl)$$ [13]

existing, it is clear from 12 that

14a. $$p_1 u(0) + p_3(\mathfrak{G}^\bullet u)(0) = p_2 u^+(0) + \int_{(0,+\infty]} [u(l) - u(0)] p_4(dl),$$

14b. $$p_2 = p_*(0), \qquad p_4(dl) = p_*(dl)/(l \wedge 1) \quad (l > 0),$$

14c. $$p_1 + p_2 + p_3 + \int_{(0,+\infty]} (l \wedge 1) p_4(dl) = 1$$

for each $u \in D(\mathfrak{G}^\bullet)$ having a limit $u(+\infty)$ at $l = +\infty$.

But $p_4(+\infty) = 0$ because, if $f = e^{-n/l}$, then $u = G_1^\bullet f \in D(\mathfrak{G}^\bullet)$, $u(+\infty) = 1$, and at the same time $u(0)$, $u^+(0)$, $\mathfrak{G}^\bullet u(0)$, and $\int_{l<+\infty} [u(l) - u(0)] p_4(dl)$ are all small for large n, and this permits us to derive 14a anew for *each* $u \in D(\mathfrak{G}^\bullet)$, completing the proof of 2b.

Given $u \in D(\mathfrak{G}^\bullet)$ and inserting 7.2 into 2a, a little algebra justifies

15. $$(G_\alpha^\bullet f)(0) = \frac{p_2 \, 2 \int_{0+} e^{-(2\alpha)^{1/2} l} f(l)\, dl + p_3 f(0) + \int_{0+} (G_\alpha^- f)(l) p_4(dl)}{p_1 + (2\alpha)^{1/2} p_2 + \alpha p_3 + \int_{0+} [1 - e^{-(2\alpha)^{1/2} l}] p_4(dl)},$$

which finishes the computation of the Green operators.

9. Special case: $p_2 = 0 < p_3$ and $p_4 < +\infty$

Consider the special case

1a. $$p_2 = 0 < p_3,$$

[13] $\int_0 f(l \wedge 1) D^{-1} p_\varepsilon(dl)$ converges as $\varepsilon \downarrow 0$ to $\int f p_*(dl)$ extended over $[0, +\infty]$ for each $f \in C[0, +\infty]$.

1b. $$p_4 = p_4(0, +\infty) < +\infty,$$

and introduce a motion $\mathfrak{x}^\bullet$ based on a reflecting Brownian motion with sample paths $t \to \mathfrak{x}^+(t)$ and probabilities $P_a(B)$ $(a \geqq 0)$ as follows.

Given a sample path $\mathfrak{x}^+$ starting at a point of $[0, +\infty)$, let $\mathfrak{x}^\bullet = \mathfrak{x}^+$ up to the passage time $\mathfrak{m}_0 = \min(t{:}\,\mathfrak{x}^+(t) = 0)$; then make $\mathfrak{x}^\bullet$ wait at 0 for an exponential holding time $\mathfrak{e}_1$ with conditional law

1. $$P_\cdot(\mathfrak{e}_1 > t \mid \mathfrak{x}^+) = e^{-((p_1+p_4)/p_3)t};$$

at the end of that time let it jump to a point $l_1 \in (0, +\infty) \cup \infty$ with conditional law

2. $$\begin{aligned} P_\cdot(l_1 \in dl \mid \mathfrak{e}_1, \mathfrak{x}^+) &= p_4(dl)/(p_1 + p_4) \quad \text{if} \quad l > 0, \\ &= p_1/(p_1 + p_4) \qquad\quad \text{if} \quad l = 0, \end{aligned}$$

and, if $+\infty > l_1 > 0$, let it start afresh, while, if $l_1 = \infty$, let $\mathfrak{x}^\bullet = \infty$ at all later times (see Diagram 1).

Because $\mathfrak{x}^\bullet$ starts afresh at the passage time $\mathfrak{m}_0$,

3. $$\begin{aligned} (G_\alpha^\bullet f)(l) &= E_l\left(\int_0^{\mathfrak{m}_\infty^\bullet} e^{-\alpha t} f(\mathfrak{x}^\bullet)\,dt\right) \qquad (\mathfrak{m}_\infty^\bullet = \min\,(t{:}\,\mathfrak{x}^\bullet(t) = \infty)) \\ &= E_l\left(\int_0^{\mathfrak{m}_0} e^{-\alpha t} f(\mathfrak{x}^+)\,dt\right) + E_l(e^{-\alpha\mathfrak{m}_0})E_0\left(\int_0^{\mathfrak{m}_\infty^\bullet} e^{-\alpha t} f(\mathfrak{x}^\bullet)\,dt\right) \\ &= (G_\alpha^- f)(l) + e^{-(2\alpha)^{1/2} l}(G_\alpha^\bullet f)(0) \end{aligned}$$

as in 7.2, whence

4. $$\begin{aligned} (G_\alpha^\bullet f)(0) &= f(0)E_0\left(\int_0^{\mathfrak{e}_1} e^{-\alpha t}\,dt\right) + E_0(e^{-\alpha\mathfrak{e}_1})E_0[(G_\alpha^\bullet f)(l_1), \mathfrak{e}_1 < \mathfrak{m}_\infty^\bullet] \\ &= \frac{p_3 f(0)}{p_1 + \alpha p_3 + p_4} \\ &\quad + \frac{1}{p_1 + \alpha p_3 + p_4}\left[\int_{0+} (G_\alpha^- f)(l)p_4(dl) + \int_{0+} e^{-(2\alpha)^{1/2} l} p_4(dl)(G_\alpha^\bullet f)(0)\right], \end{aligned}$$

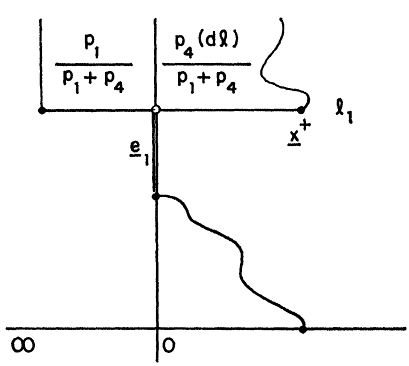


DIAGRAM 1

and, solving for $(G^\bullet_\alpha f)(0)$, one finds

5. $$(G^\bullet_\alpha f)(0) = \frac{p_3 f(0) + \int_{0+} (G^-_\alpha f)(l) p_4(dl)}{p_1 + \alpha p_3 + \int_{0+} [1 - e^{-(2\alpha)^{1/2} l}] p_4(dl)}.$$

Granting that the dot motion starts afresh at constant times (the reader will fill this gap), a comparison of 5 and 8.15 permits its identification as the Brownian motion associated with the operator $\mathfrak{G}^\bullet$ with domain

6. $$D(\mathfrak{G}^\bullet) = C^2[0, +\infty) \cap \Big(u: p_1 u(0) + p_3(\mathfrak{G}u)(0) = \int_{0+} [u(l) - u(0)] p_4(dl)\Big);$$

the proof that $\mathfrak{x}^\bullet$ is a Brownian motion can be based on the fact, used several times below, that if a motion is *simple* Markov and if its Green operators map $C[0, +\infty)$ into itself, then it is also *strict* Markov (see, for example K. Itô and H. P. McKean, Jr. [1]).

10. Special case: $p_2 > 0 = p_4$

Given a reflecting Brownian motion with sample paths $t \to \mathfrak{x}^+(t)$, probabilities $P_a(B)$, and local time

1. $$\mathfrak{t}^+(t) = \lim_{\varepsilon \downarrow 0} (2\varepsilon)^{-1} \text{measure}(s: \mathfrak{x}^+(s) < \varepsilon, s \leqq t),$$

it is possible to build up all the Brownian motions attached to the generators

2. $$\mathfrak{G}^\bullet = \mathfrak{G} \mid C^2[0, +\infty) \cap (u: p_1 u(0) - p_2 u^+(0) + p_3(\mathfrak{G}u)(0) = 0), \quad p_2 > 0$$

with the aid of an extra exponential holding time $\mathfrak{e}$ with conditional law

3. $$P_\cdot(\mathfrak{e} > t \mid \mathfrak{x}^+) = e^{-t}.$$

Beginning with the elastic Brownian case ($p_1 > 0 = p_3$), the desired motion is

4a. $$\mathfrak{x}^\bullet(t) = \mathfrak{x}^+(t) \quad \text{if} \quad t < \mathfrak{m}^\bullet_\infty,$$
$$= \infty \quad \text{if} \quad t \geqq \mathfrak{m}^\bullet_\infty,$$

4b. $$\mathfrak{m}^\bullet_\infty = \mathfrak{t}^{-1}((p_2/p_1)\mathfrak{e}) = \min(t: \mathfrak{t}^+(t) = (p_2/p_1)\mathfrak{e})$$

as stated in Sections 1 and 3.

With the aid of the conditional law

5. $$P_\cdot(\mathfrak{m}^\bullet_\infty > t \mid \mathfrak{x}^+) = P_\cdot(\mathfrak{e} > (p_1/p_2)\mathfrak{t}^+(t) \mid \mathfrak{x}^+) = e^{-(p_1/p_2)\mathfrak{t}^+(t)}$$

and the *addition rule*

6. $$\mathfrak{t}^+(t_2) = \mathfrak{t}^+(t_1) + \mathfrak{t}^+(t_2 - t_1, w^+_{t_1}), \qquad t_2 \geqq t_1,$$

it is clear that, if $db \subset [0, +\infty)$ and if $\mathfrak{m}_\infty^\bullet > t_1 \leqq t_2$, then

7a. $$P_\bullet[\mathfrak{x}^\bullet(t_2) \in db \mid \mathfrak{x}^+(s): s \leqq t_1,\ \mathfrak{m}_\infty^\bullet \wedge t_1,\ \mathfrak{m}_\infty^\bullet > t_1]$$
$$= \frac{P_\bullet[\mathfrak{x}^+(t_2) \in db,\ \mathfrak{m}_\infty^\bullet > t_2 \mid \mathfrak{x}^+(s): s \leqq t_1]}{P_\bullet(\mathfrak{m}_\infty^\bullet > t_1)}$$
$$= E_\bullet[\mathfrak{x}^+(t_2) \in db,\ e^{-(p_1/p_2)\mathfrak{t}^+(t_2)} \mid \mathfrak{x}^+(s): s \leqq t_1]e^{+(p_1/p_2)\mathfrak{t}^+(t_1)}$$
$$= E_\bullet[\mathfrak{x}^+(t_2) \in db,\ e^{-(p_1/p_2)\mathfrak{t}^+(t_2-t_1, w_{t_1}^+)} \mid \mathfrak{x}^+(s): s \leqq t_1]$$
$$= E_a[\mathfrak{x}^+(t_2 - t_1) \in db,\ e^{-(p_1/p_2)\mathfrak{t}^+(t_2-t_1)}], \qquad a = \mathfrak{x}^+(t_1),$$
$$= P_a[\mathfrak{x}^\bullet(t_2 - t_1) \in db], \qquad a = \mathfrak{x}^\bullet(t_1),$$

while, if $\mathfrak{m}_\infty^\bullet \leqq t_1$, then $\mathfrak{x}^\bullet(t_1) = \infty$, and

7b. $$P_\bullet[\mathfrak{x}^\bullet(t_2) \in db \mid \mathfrak{x}^+(s): s \leqq t_1,\ \mathfrak{m}_\infty^\bullet \wedge t_1,\ \mathfrak{m}_\infty^\bullet \leqq t_1]$$
$$= 0 = P_\infty[\mathfrak{x}^\bullet(t_2 - t_1) \in db].$$ [14]

Since $\mathfrak{x}^\bullet(s): s \leqq t_1$ is a Borel function of $\mathfrak{x}^+(s): s \leqq t_1$, $\mathfrak{m}_\infty^\bullet \wedge t_1$, and of the indicator of $(\mathfrak{m}_\infty^\bullet < t_1)$, it follows that

8. $$P_\bullet[\mathfrak{x}^\bullet(t_2) \in db \mid \mathfrak{x}^\bullet(s): s \leqq t_1] = P_a[\mathfrak{x}^\bullet(t_2 - t_1) \in db], \qquad a = \mathfrak{x}^\bullet(t_1),$$

establishing the simple Markovian nature of the dot motion.

Consider for the next step, its Green operators

$$G_\alpha^\bullet f = E_\bullet\left(\int_0^{\mathfrak{m}_\infty^\bullet} e^{-\alpha t} f(\mathfrak{x}^\bullet)\, dt\right),$$

and use the conditional law of $\mathfrak{m}_\infty^\bullet$ to check

9. $$G_\alpha^\bullet f = E_\bullet\left(\int_0^{+\infty} e^{-(p_1/p_2)\mathfrak{t}^+(s)} \frac{p_1}{p_2} \mathfrak{t}^+(ds) \int_0^s e^{-\alpha t} f(\mathfrak{x}^+)\, dt\right)$$
$$= E_\bullet\left(\int_0^{+\infty} e^{-\alpha t} f(\mathfrak{x}^+)\, dt \int_t^{+\infty} e^{-(p_1/p_2)\mathfrak{t}^+} \mathfrak{t}^+(ds)\right)$$
$$= E_\bullet\left(\int_0^{+\infty} e^{-\alpha t} e^{-(p_1/p_2)\mathfrak{t}^+} f(\mathfrak{x}^+)\, dt\right).$$

Because $\mathfrak{m}_0 = \min(t: \mathfrak{x}^+(t) = 0)$ is a stopping time and $\mathfrak{t}^+(t) = 0$ $(t \leqq \mathfrak{m}_0)$,

10. $$(G_\alpha^\bullet f)(l) = E_l\left(\int_0^{\mathfrak{m}_0} e^{-\alpha t} f(\mathfrak{x}^+)\, dt\right)$$
$$+ E_l\left(e^{-\alpha \mathfrak{m}_0} \int_0^{+\infty} e^{-\alpha t} \exp\{-(p_1/p_2)\mathfrak{t}^+(t, w_{\mathfrak{m}_0}^+)\} f[\mathfrak{x}^+(t + \mathfrak{m}_0)]\, dt\right)$$
$$= (G_\alpha^- f)(l) + E_l(e^{-\alpha \mathfrak{m}_0}) E_0\left(\int_0^{+\infty} e^{-\alpha t} e^{-(p_1/p_2)\mathfrak{t}^+} f(\mathfrak{x}^+)\, dt\right)$$
$$= (G_\alpha^- f)(l) + e^{-(2\alpha)^{1/2} l}(G_\alpha^\bullet f)(0), \qquad l \geqq 0,$$

[14] $P_\infty[\mathfrak{x}^\bullet \equiv \infty] = 1$ as usual.

and now the identification of the dot motion as the elastic Brownian motion will be complete as soon as it is verified that

11. $$(G^\bullet_\alpha f)(0) = \frac{p_2\, 2 \int_0^{+\infty} e^{-(2\alpha)^{1/2} l} f(l)\, dl}{p_1 + (2\alpha)^{1/2} p_2};$$

in fact, this will prove that the dot motion is simple Markov with the correct (elastic Brownian) Green operators, and the proof can be completed as at the end of Section 9.

But 11 is trivial; in fact, using the joint law 4.10,

12. $$\begin{aligned}(G^\bullet_\alpha f)(0) &= E_0\left(\int_0^{+\infty} e^{-\alpha t} e^{-(p_1/p_2)t^+} f(\mathfrak{x}^+)\, dt\right) \\ &= \int_0^{+\infty} e^{-\alpha t}\, dt \int_0^{+\infty} db \int_0^{+\infty} da\, 2 \frac{b+a}{(2\pi t^3)^{1/2}} e^{-(b+a)^2/2t} e^{-(p_1/p_2)b} f(a) \\ &= 2 \int_0^{+\infty} db \int_0^{+\infty} da\, e^{-(2\alpha)^{1/2}(b+a)} e^{-(p_1/p_2)b} f(a) \\ &= \frac{p_2\, 2 \int_{0+} e^{-(2\alpha)^{1/2} l} f(l)\, dl}{p_1 + (2\alpha)^{1/2} p_2}\end{aligned}$$

as stated.

Consider next, the case $p_3 > 0 = p_1$, and let us prove the desired motion to be[15]

13. $$\mathfrak{x}^\bullet = \mathfrak{x}^+(\mathfrak{f}^{-1}), \qquad \mathfrak{f} = t + (p_3/p_2)\mathfrak{t}^+.$$

Beginning, as before, with the proof that the dot motion is *simple Markov*, if $t_2 \geqq t_1$ and if $\mathfrak{m} = \mathfrak{f}^{-1}(t_1)$, then

(a) $(\mathfrak{m} < t) = (t_1 < \mathfrak{f}(t)) \in \mathsf{B}[\mathfrak{x}^+(s) : s \leqq t]$, i.e., $\mathfrak{m}$ *is a stopping time*;

(b) $\mathfrak{f}(\mathfrak{m} + s)) = \mathfrak{f}(\mathfrak{m}) + \mathfrak{f}(s, w_\mathfrak{m}^+) = t_1 + (t_2 - t_1)$ *if* $s = \mathfrak{f}^{-1}(t_2 - t_1, w_\mathfrak{m}^+)$ *and so* $\mathfrak{f}^{-1}(t_2) = \mathfrak{m} + s = \mathfrak{m} + \mathfrak{f}^{-1}(t_2 - t_1, w_\mathfrak{m}^+)$;

(c) $\mathfrak{x}^\bullet(t_2) = \mathfrak{x}^+[\mathfrak{f}^{-1}(t_2 - t_1, w_\mathfrak{m}^+) + \mathfrak{m}]$;

(d) $\mathfrak{x}^\bullet(s) : s \leqq t_1$ *is a Borel function of the stopped path* $t \to \mathfrak{x}^+(t \wedge \mathfrak{m})$ *and of* $\mathfrak{f}^{-1}(s) : s \leqq t_1$;

(e) $\mathfrak{f}^{-1}(s)$ *is the solution* r *of* $f(r) = s$ $(\leqq t_1 = f(\mathfrak{m}))$ *and, as such, it is likewise a Borel function of the stopped path*;

and now, using the strict Markovian nature of $\mathfrak{x}^+$, the law of $\mathfrak{x}^\bullet(t_2)$ conditional on $\mathsf{B}_{\mathfrak{m}+} \supset \mathsf{B}[\mathfrak{x}^\bullet(s) : s \leqq t_1]$ is found to be

14a. $$\begin{aligned}P_\cdot(\mathfrak{x}^+[\mathfrak{f}^{-1}(t_2 - t_1, w_\mathfrak{m}^+) + \mathfrak{m}] \in db \mid \mathsf{B}_{\mathfrak{m}+}) \\ = P_a(\mathfrak{x}^+[\mathfrak{f}^{-1}(t_2 - t_1)] \in db), &\qquad a = \mathfrak{x}^+(\mathfrak{m}), \\ = P_a(\mathfrak{x}^\bullet(t_2 - t_1) \in db), &\qquad a = \mathfrak{x}^\bullet(t_1),\end{aligned}$$

[15] $\mathfrak{f}^{-1}$ is the inverse function of $\mathfrak{f}$.

whence, taking the expectation of both sides conditional on $\mathsf{B}[\mathfrak{x}^\bullet(s):s \leqq t_1]$,

14b. $$P_\cdot(\mathfrak{x}^\bullet(t_2) \in db \mid \mathfrak{x}^\bullet(s):s \leqq t_1) = P_a(\mathfrak{x}^\bullet(t_2 - t_1) \in db), \qquad a = \mathfrak{x}^\bullet(t_1),$$

i.e., $\mathfrak{x}^\bullet = \mathfrak{x}^+(\mathfrak{f}^{-1})$ starts afresh at time t_1, as was to be proved.

Coming to the Green operators

$$G_\alpha^\bullet f = E_\cdot\left(\int_0^{+\infty} e^{-\alpha t} f(\mathfrak{x}^\bullet)\, dt\right),$$

since $\mathfrak{m}_0 = \min(t:\mathfrak{x}^+(t) = 0)$ is a stopping time and $\mathfrak{f}^{-1} \equiv t\ (t \leqq \mathfrak{m}_0)$,

15. $$\begin{aligned}(G_\alpha^\bullet f)(l) &= E_l\left(\int_0^{\mathfrak{m}_0} e^{-\alpha t} f(\mathfrak{x}^+)\, dt\right) \\ &\qquad + E_l\left(e^{-\alpha \mathfrak{m}_0} \int_0^{+\infty} e^{-\alpha t} f[\mathfrak{x}^+(\mathfrak{f}^{-1}(t, w_{\mathfrak{m}_0}^+) + \mathfrak{m}_0)]\, dt\right) \\ &= (G_\alpha^- f)(l) + e^{-(2\alpha)^{1/2} l}(G_\alpha^\bullet f)(0)\end{aligned}$$

as in the elastic Brownian case, and to complete the identification of $\mathfrak{x}^\bullet$ it is sufficient to check that

16. $$\begin{aligned}(G_\alpha^\bullet f)(0) &= E_0\left(\int_0^{+\infty} e^{-\alpha t} f[\mathfrak{x}^+(\mathfrak{f}^{-1})]\, dt\right) \\ &= E_0\left(\int_0^{+\infty} e^{-\alpha \mathfrak{f}} f(\mathfrak{x}^+)\mathfrak{f}(dt)\right) \\ &= E_0\left(\int_0^{+\infty} e^{-\alpha[t+(p_3/p_2)\mathfrak{t}^+]} f(\mathfrak{x}^+)\, dt\right) \\ &\qquad + f(0)E_0\left(\int_0^{+\infty} e^{-\alpha[t+(p_3/p_2)\mathfrak{t}^+]} \frac{p_3}{p_2}\, \mathfrak{t}^+(dt)\right) \quad {}^{16} \\ &= \frac{p_2\, 2\int_{0+} e^{-(2\alpha)^{1/2} l} f(l)\, dl}{(2\alpha)^{1/2} p_2 + \alpha p_3} \\ &\qquad + \frac{f(0)}{\alpha}\left[1 - E_0\left(\int_0^{+\infty} e^{-\alpha[t+(p_3/p_2)\mathfrak{t}^+]}\, dt\right)\right] \quad {}^{17,18} \\ &= \frac{p_2\, 2\int_{0+} e^{-(2\alpha)^{1/2} l} f(l)\, dl + p_3 f(0)}{(2\alpha)^{1/2} p_2 + \alpha p_3}\end{aligned}$$

as it should be.

Consider now the case $0 < p_1\, p_2\, p_3$; this time the motion is

---

[16] $\mathfrak{t}^+(dt) = 0$ off $\mathfrak{Z}^+ = (t:\mathfrak{x}^+(t) = 0)$.

[17] Use 12 with αp_3 in place of p_1.

[18] Do a partial integration under the expectation sign.

17a. $$\mathfrak{x}^\bullet(t) = \mathfrak{x}^+(\mathfrak{f}^{-1}) \quad \text{if} \quad t < \mathfrak{m}_\infty^\bullet, \qquad = \infty \quad \text{if} \quad t \geqq \mathfrak{m}_\infty^\bullet,$$

17b. $$\mathfrak{m}_\infty^\bullet = \mathfrak{f}[\mathfrak{t}^{-1}((p_2/p_1)\mathfrak{e})] = [\mathfrak{t}^+(\mathfrak{f}^{-1})]^{-1}((p_2/p_1)\mathfrak{e}),$$

as will still be proved.

$\mathfrak{x}^+(\mathfrak{f}^{-1})$ is a Brownian motion, its *local time*

18. $$\begin{aligned}\mathfrak{t}^\bullet(t) &= \text{measure } (s: \mathfrak{x}^+(\mathfrak{f}^{-1}) = 0, s \leqq t) \\ &= \text{measure } (s: \mathfrak{f}^{-1}(s) \in \mathfrak{Z}^+, s \leqq t) \\ &= \text{measure } \mathfrak{f}(\mathfrak{Z}^+) \cap [0, t] \\ &= \int_{\mathfrak{Z}^+ \cap [0, \mathfrak{f}^{-1}(t)]} \mathfrak{f}(ds) \\ &= (p_3/p_2)\mathfrak{t}^+[\mathfrak{f}^{-1}(t)] \quad ^{19}\end{aligned}$$

satisfies the addition rule 6, and, substituting them in place of $\mathfrak{x}^+$ and $\mathfrak{t}^+$ in the derivation of the simple Markovian nature of the elastic Brownian motion, it is found that the present motion is likewise simple Markov.

$G_\alpha^\bullet f = G_\alpha^- f + e^{-(2\alpha)^{1/2} l}(G_\alpha^\bullet f)(0)$ is derived as before, that the dot motion is Brownian follows, and now, using the evaluation 12 with $p_1 + \alpha\, p_3$ in place of p_1 in conjunction with the conditional law

19. $$\begin{aligned}P_\cdot(\mathfrak{m}_\infty^\bullet > t \mid \mathfrak{x}^+(\mathfrak{f}^{-1})) &= P_\cdot(\mathfrak{e} > (p_1/p_2)\mathfrak{t}^+(\mathfrak{f}^{-1}) \mid \mathfrak{x}^+(\mathfrak{f}^{-1})) \\ &= e^{-(p_1/p_2)\,\mathfrak{t}^+(\mathfrak{f}^{-1})} = e^{-(p_1/p_3)\,\mathfrak{t}^\bullet(t)},\end{aligned}$$

it develops that

20. $$\begin{aligned}(G_\alpha^\bullet f)(0) &= E_0\left(\int_0^{\mathfrak{m}_\infty^\bullet} e^{-\alpha t} f[\mathfrak{x}^+(\mathfrak{f}^{-1})]\, dt\right) \\ &= E_0\left(\int_0^{+\infty} e^{-\alpha t} e^{-(p_1/p_2)\,\mathfrak{t}^+(\mathfrak{f}^{-1})} f[\mathfrak{x}^+(\mathfrak{f}^{-1})]\, dt\right) \\ &= E_0\left(\int_0^{+\infty} e^{-\alpha t} e^{-((p_1+\alpha p_3)/p_2)\,\mathfrak{t}^+} f(\mathfrak{x}^+)\, dt\right) \\ &\qquad + f(0) E_0\left(\int_0^{+\infty} e^{-\alpha t} e^{-((p_1+\alpha p_3)/p_2)\,\mathfrak{t}^+} \frac{p_3}{p_2}\, \mathfrak{t}^+(dt)\right) \\ &= \frac{p_2\, 2 \int_{0+} e^{-(2\alpha)^{1/2} l} f(l)\, dl + p_3 f(0)}{p_1 + (2\alpha)^{1/2} p_2 + \alpha p_3},\end{aligned}$$

completing the proof.

A second description of the present motion is available: *it is the elastic*

[19] measure $(\mathfrak{Z}^+) = 0$. $\mathfrak{t}^+(dt) = 0$ outside $\mathfrak{Z}^+$.

Brownian motion $\mathfrak{x}^\bullet$ *described in* 4 *run with the new stochastic clock* $\mathfrak{f}^{-1}$ *which is the inverse function of*

21a. $$\mathfrak{f} = t + (p_3/p_2) \times \text{the elastic Brownian local time } \mathfrak{t}^\bullet,$$

21b. $$\begin{aligned} \mathfrak{t}^\bullet(t) &= \lim_{\varepsilon \downarrow 0} (2\varepsilon)^{-1} \text{ measure } (s: \mathfrak{x}^\bullet(s) < \varepsilon, s \leqq t) \\ &= \mathfrak{t}^+(t \wedge \mathfrak{m}_\infty^\bullet), \qquad \mathfrak{m}_\infty^\bullet = \min(t: \mathfrak{x}^\bullet = \infty). \end{aligned}$$

11. Increasing differential processes

Before describing the sample paths in the case $p_4 = p_4(0, +\infty) = +\infty$, it will be helpful to list some properties of differential processes with increasing sample paths.

Given a stochastic process with universal field B, probabilities P, and sample paths $t \to \mathfrak{p}(t)$:

1a. $$\mathfrak{p}(0) = 0,$$

1b. $$\mathfrak{p}(s) \leqq \mathfrak{p}(t), \qquad s \leqq t,$$

1c. $$\mathfrak{p}(t+) = \mathfrak{p}(t) < +\infty, \qquad t \geqq 0,$$

which is *differential* in the sense that *the shifted path* $\mathfrak{p}_+(t) \equiv \mathfrak{p}(t+s) - \mathfrak{p}(s)$ *is independent of its past* $\mathfrak{p}(t): t \leqq s$ *and identical in law to* $\mathfrak{p}$, P. Lévy [1][20] proved that

2a. $$E(e^{-\alpha \mathfrak{p}(t)}) = \exp\left\{-t\left[p_2 \alpha + \int_{0+} (1 - e^{-\alpha l}) p(dl)\right]\right\}, \qquad \alpha > 0,$$

2b. $$p_2 \geqq 0, \qquad p(dl) \geqq 0, \qquad \int_{0+} (l \wedge 1) p(dl) < +\infty$$

and expressed $\mathfrak{p}$ as

3. $$\mathfrak{p}(t) = p_2 t + \int_{0+} l \mathfrak{p}([0, t] \times dl), \qquad t \geqq 0,$$

in which $\mathfrak{p}(dt \times dl) =$ *the number of jumps of* $\mathfrak{p}$ *of magnitude* $\epsilon\, dl$ *occurring in time* dt is differential in the pair $(t, l) \in [0, +\infty) \times (0, +\infty)$ and Poisson distributed with mean $dt\, p(dl)$, i.e., if Q_1, Q_2, etc. are disjoint figures of $[0, +\infty) \times (0, +\infty)$, then $\mathfrak{p}(Q_1)$, $\mathfrak{p}(Q_2)$, etc., are independent, and

4. $$P(\mathfrak{p}(Q) = n) = (|Q|^n/n!)\, e^{-|Q|}, \qquad n \geqq 0,\ |Q| = \int_Q dt\, p(dl);$$

in short, $\mathfrak{p}(t)$ is *the (direct) integral* $\int_{0+} l\mathfrak{p}([0, t] \times dl)$ *of the differential Poisson processes* $\mathfrak{p}([0, t] \times dl)$ *with rates* $p(dl)$ *plus a linear part* $p_2 t$.

Given nonnegative p_2 and $p(dl)$ with $\int_{0+} (l \wedge 1) p(dl) < +\infty$ as in 2b, it is possible to make a Poisson measure $\mathfrak{p}(dt \times dl)$ with mean $dt\, p(dl)$ as de-

[20] See also K. Itô [1].

scribed above; the associated $\mathfrak{p}(t) = p_2 t + \int_{0+} l\mathfrak{p}([0, t] \times dl)$ is a differential process having 2a as its Lévy formula.

G. Hunt [1] discovered that if $\mathfrak{m}$ is a *stopping time*, i.e., if

$$5. \qquad (\mathfrak{m} < t) \in \mathsf{B}[\mathfrak{p}(s): s \leqq t] \times \mathsf{B}^{\bullet}, \qquad t \geqq 0,$$

for some field $\mathsf{B}^{\bullet}$ independent of $\mathfrak{p}$, then $\mathfrak{p}$ *starts afresh at time* $t = \mathfrak{m}$, *i.e., the shifted path* $\mathfrak{p}_{+}(t) \equiv \mathfrak{p}(t + \mathfrak{m}) - \mathfrak{p}(\mathfrak{m})$ *is independent of the past* $\mathfrak{p}(t): t \leqq \mathfrak{m}$ *and identical in law to* $\mathfrak{p}$ *itself.*

Given $a \geqq 0$, if P_a is the law that P induces on the space of sample paths $\mathfrak{q} \equiv \mathfrak{p} + a$, then

$$6. \quad P.(\mathfrak{q}(t_2) \in db \mid \mathfrak{q}(s): s \leqq t_1) = P_a(\mathfrak{q}(t_2 - t_1) \in db), \quad t_2 \geqq t_1, a = \mathfrak{q}(t_1),$$

the associated Green operators $f \to E(\int_0^{+\infty} e^{-\alpha t} f(\mathfrak{q})dt)$ map $C[0, +\infty)$ into itself, and the associated generator $\mathfrak{Q}$ is

$$7. \quad (\mathfrak{Q}f)(a) = p_2 f^{+}(a) + \int_{0+} [f(b + a) - f(a)]p(db), \qquad f \in C^1[0, +\infty).$$

Given $t \geqq 0$, $\mathfrak{p}([0, t] \times [\varepsilon, +\infty))$ is Poisson distributed and differential in ε with mean $tp[\varepsilon, +\infty)$; as such, it is identical in law to a standard Poisson process $\mathfrak{q}$ with unit jumps and unit rate run with the clock $tp[\varepsilon, +\infty)$, and, using the strong law of large numbers, it follows that

$$8. \qquad \lim_{\varepsilon \downarrow 0} \frac{\mathfrak{p}([0, t] \times [\varepsilon, +\infty))}{p[\varepsilon, +\infty)} = \lim_{\varepsilon \downarrow 0} \frac{\mathfrak{q}(tp[\varepsilon, +\infty))}{p[\varepsilon, +\infty)} = t,$$

which will be helpful to us in Section 14.

Consider the special case $p(0, +\infty) < +\infty$ pictured in Diagram 1: the exponential holding times $\mathfrak{e}_1$, $\mathfrak{e}_2$, etc. between jumps are independent with common law $P(\mathfrak{e}_1 > t) = e^{-p(0, +\infty)t}$, the jumps l_1, l_2, etc. are likewise independent with common law $P(l_1 \in dl) = p(0, +\infty)^{-1}p(dl)$, and the slope of the slanting lines is $1/p_2$.

Consider, as a second example, the standard Brownian passage times $\mathfrak{m}_a =$

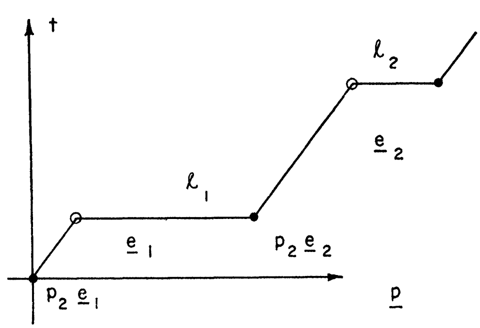


DIAGRAM 1

$\min(t\colon \mathfrak{x} = a)$ $(a \geqq 0)$ under the law $P = P_0$. Because the Brownian traveller starts afresh at its passage times, the shifted path $\mathfrak{m}_{b+a} - \mathfrak{m}_a = \mathfrak{m}_{b+a}(w^+_{\mathfrak{m}_a})$ is independent of $\mathfrak{m}_b : b \leqq a$ and identical in law to $\mathfrak{m}.$, i.e., $\mathfrak{m}.$ is differential (it is the *one-sided stable process with exponent* $\frac{1}{2}$ *and rate* $\sqrt{2}$ as noted in Section 4);

9a. $$p_2 = 0,$$

9b. $$p(dl) = dl/(2\pi l^3)^{1/2}$$

can be read off

10. $$E_0(e^{-\alpha\mathfrak{m}_a}) = e^{-(2\alpha)^{1/2}a} = \exp\left\{-a\int_{0+}(1 - e^{-\alpha l})\frac{dl}{(2\pi l^3)^{1/2}}\right\}.$$

$\mathfrak{m}_a$ is left-continuous, so in the direct integral $[0, a)$ must be used in place of $[0, a]$:

$$\mathfrak{m}_a = \int_{0+} l\mathfrak{p}([0, a) \times dl).$$

12. Sample paths: $p_1 = p_3 = 0 < p_4$ $(p_2 > 0/p_4 = +\infty)$

Given a reflecting Brownian motion with local time $\mathfrak{t}^+$, a nonnegative number p_2, and a nonnegative mass distribution $p_4(dl)$ $(l > 0)$ with $p_4 = p_4(0, +\infty) = +\infty$ in case $p_2 = 0$, introduce the Poisson measure $\mathfrak{p}(dt \times dl)$ with mean $dt\, p_4(dl)$, make up the associated differential process

1. $$\mathfrak{p}(t) = p_2 t + \int_{0+} l\mathfrak{p}([0, t] \times dl),$$

and consider the sample path[21]

2a. $$\mathfrak{x}^\bullet(t) = \mathfrak{p}\mathfrak{p}^{-1}\mathfrak{t}^+(t) - \mathfrak{t}^+(t) + \mathfrak{x}^+(t), \qquad t \geqq 0,$$

2b. $$\mathfrak{p}^{-1}(l) = \inf(t\colon \mathfrak{p}(t) > l)$$

and its alternative description

3. $$\mathfrak{x}^\bullet(t) = \mathfrak{p}\mathfrak{p}^{-1}\mathfrak{t}^-(t) + \mathfrak{x}^-(t), \qquad t \geqq 0,$$

in terms of the *standard Brownian motion* $\mathfrak{x}^- = -\mathfrak{t}^+ + \mathfrak{x}^+$ and its *minimum function* $\mathfrak{t}^-(t) = \mathfrak{t}^+(t) = -(\min_{s \leqq t} \mathfrak{x}^-(s) \wedge 0)$; it is to be proved that $\mathfrak{x}^\bullet$ *is the Brownian motion associated with*

4. $$p_2 u^+(0) + \int_{0+} [u(l) - u(0)]p_4(dl) = 0,$$

but before doing that let us look at some pictures of the sample path.

Consider the case $p_4 < +\infty$: the jumps l_1, l_2, etc. of $\mathfrak{p}$ are finite in number per unit time and can be labelled in their correct temporal order. $\mathfrak{p}$ and

[21] $\mathfrak{p}\mathfrak{p}^{-1}\mathfrak{t}^+(t)$ is short for $\mathfrak{p}(\mathfrak{p}^{-1}(\mathfrak{t}^+(t)))$.

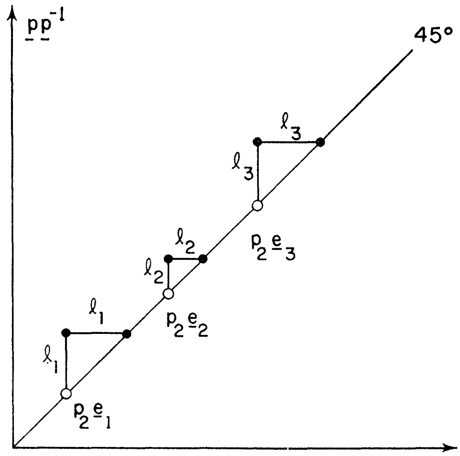
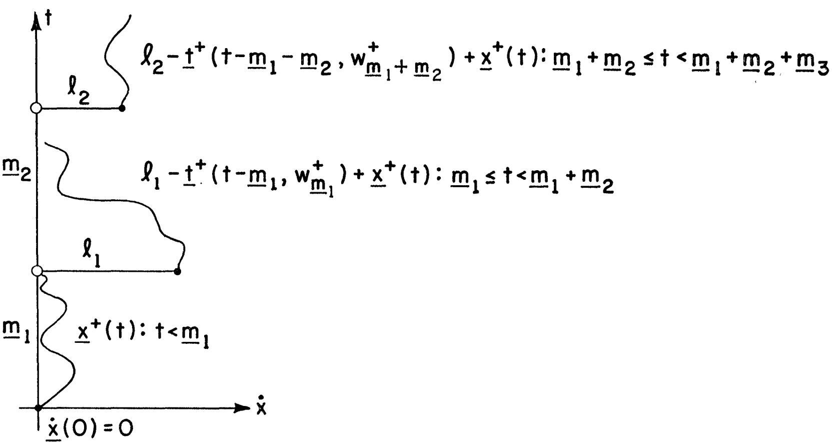


DIAGRAM 1



$$
\begin{aligned}
\mathfrak{m}_1 &= \mathfrak{t}^{-1}(p_2\,\mathfrak{e}_1) \\
\mathfrak{m}_2 &= \mathfrak{t}^{-1}(p_2\,\mathfrak{e}_2 + l_1\,,\, w^{+}_{\mathfrak{m}_1}) \\
\mathfrak{m}_3 &= \mathfrak{t}^{-1}(p_2\,\mathfrak{e}_3 + l_2\,,\, w^{+}_{\mathfrak{m}_1+\mathfrak{m}_2}) \\
&\text{etc.}
\end{aligned}
$$

DIAGRAM 2

$\mathfrak{p}^{-1}$ are seen in Diagram 11.1, $\mathfrak{p}\mathfrak{p}^{-1}$ in Diagram 1 of the present section, and the $\mathfrak{x}^{\bullet} = \mathfrak{p}\mathfrak{p}^{-1}\mathfrak{t}^{+} - \mathfrak{t}^{+} + \mathfrak{x}^{+}$ path in Diagram 2, in which $\mathfrak{t}^{-1}$ is left-continuous as usual and $\mathfrak{e}_1$, $\mathfrak{e}_2$, etc. are the exponential holding times between jumps of $\mathfrak{p}$.

Coming to the case $p_4 = +\infty$, $\mathfrak{p}(t)$ experiences an infinite number of jumps during each time interval $[t_1, t_2)$ $(t_1 < t_2)$, but

$$\mathfrak{p}([t_1, t_2) \times [\varepsilon, +\infty)) < +\infty \qquad (t_2 < +\infty, \varepsilon > 0),$$

and so it is legitimate to label the jumps as follows:

(a) arrange in separate rows the jumps occurring in (0, 1], (1, 2], etc.;

(b) in each row, arrange the jumps in order of magnitude beginning with the largest one;

(c) if several jumps of the same magnitude occur in a single row, arrange them in correct temporal order;

(d) number the rows as indicated below:

$$\begin{aligned} &l_1 \geqq l_3 \geqq l_6 \geqq l_{10}\,, \\ &l_2 \geqq l_5 \geqq l_9\,, \\ &l_4 \geqq l_8\,, \\ &l_7 \quad \text{etc.} \end{aligned}$$

Diagram 2 gives an approximate idea of the sample path in the case $p_2 = 0$. Diagram 3 ($p_2 = 0$, $\mathfrak{x}(0) = 0$) is based on the alternative description 3: the standard Brownian path $\mathfrak{x}^-$ has been slanted off to the left for the purposes of the picture, and the rule is to translate the excursions of $\mathfrak{x}^-$ between the endpoints of the flat stretches of $\mathfrak{p}^{-1}$ until the left legs of the hatched curvilinear triangles abut on the time axis and then to fill up the gaps with $\mathfrak{x}^\bullet = 0$. The picture is not so simple in case $p_2 > 0$: then $\mathfrak{Q} = (l : \mathfrak{p}\mathfrak{p}^{-1}(l) = l)$ has positive measure, and, on $\mathfrak{Q}^- = (t : \mathfrak{t}^-(t) \in \mathfrak{Q})$, $\mathfrak{x}^\bullet = \mathfrak{p}\mathfrak{p}^{-1}\mathfrak{t}^- - \mathfrak{x}^-$ reduces to the reflecting Brownian motion $\mathfrak{t}^- - \mathfrak{x}^- = \mathfrak{x}^+$.

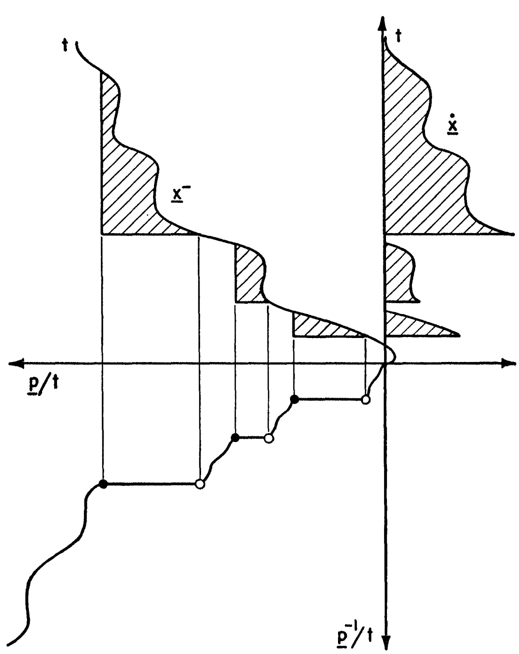


DIAGRAM 3

13. Simple Markovian character: $p_1 = p_3 = 0$ $(p_2 > 0/p_4 = +\infty)$

Consider the sample path

1. $$\mathfrak{x}^\bullet = \mathfrak{p}\mathfrak{p}^{-1}\mathfrak{t}^+ - \mathfrak{t}^+ + \mathfrak{x}^+ = \mathfrak{p}\mathfrak{p}^{-1}\mathfrak{t}^- + \mathfrak{x}^-$$

described in Section 12.

Given $t_2 \geqq t_1 \geqq 0$, if $\mathfrak{m} = \mathfrak{p}^{-1}\mathfrak{t}^-(t_1)$, if $\mathfrak{p}_+(t) = \mathfrak{p}(t + \mathfrak{m}) - \mathfrak{p}(\mathfrak{m})$, and if $\mathfrak{t}_+^-(t) = -\min_{s \leqq t}[\mathfrak{x}^-(s + t_1) - \mathfrak{x}^-(t_1)]$, then, as the reader will check,

2. $$\begin{aligned} \mathfrak{p}^{-1}\mathfrak{t}^-(t_2) - \mathfrak{p}^{-1}\mathfrak{t}^-(t_1) &= \inf(s:\mathfrak{p}(s) > \mathfrak{t}^-(t_2)) - \mathfrak{p}^{-1}\mathfrak{t}^-(t_1)) \\ &= \inf(s:\mathfrak{p}(s + \mathfrak{m}) > \mathfrak{t}^-(t_2)) \\ &= \inf(s:\mathfrak{p}_+(s) + \mathfrak{p}(\mathfrak{m}) > [\mathfrak{t}_+^-(t_2 - t_1) - \mathfrak{x}^-(t_1)] \vee \mathfrak{t}^-(t_1)) \\ &= \inf(s:\mathfrak{p}_+(s) > [\mathfrak{t}_+^-(t_2 - t_1) - \mathfrak{x}^\bullet(t_1)] \vee [\mathfrak{t}^-(t_1) - \mathfrak{p}(\mathfrak{m})]) \\ &= \inf(s:\mathfrak{p}_+(s) > [\mathfrak{t}_+^-(t_2 - t_1) - \mathfrak{x}^\bullet(t_1)] \vee 0), \end{aligned}$$ [22]

where the last step is justified as follows: $\mathfrak{a} = \mathfrak{t}^-(t_1) - \mathfrak{p}(\mathfrak{m}) \leqq 0$ since *either* $p_2 > 0$ *or* $p_4(0, +\infty) = +\infty$, $\mathfrak{p}^{-1}(0) = 0$, and it follows that *either* $\mathfrak{b} = \mathfrak{t}_+^-(t_2 - t_1) - \mathfrak{x}^\bullet(t_1) < 0$ and $\inf(s:\mathfrak{p}_+(s) > \mathfrak{a} \vee \mathfrak{b}) = \inf(s:\mathfrak{p}_+ > 0) = 0$ *or* $\mathfrak{b} \geqq 0$ and $\mathfrak{a} \vee \mathfrak{b} = \mathfrak{b}$.

Coming to the sample path, itself, an application of 2 implies

3. $$\begin{aligned} \mathfrak{x}^\bullet(t_2) &= \mathfrak{p}\mathfrak{p}^{-1}\mathfrak{t}^-(t_2) + \mathfrak{x}^-(t_2) \\ &= \mathfrak{p}(\mathfrak{p}_+^{-1}([\mathfrak{t}_+^-(t_2 - t_1) - \mathfrak{x}^\bullet(t_1)] \vee 0) + \mathfrak{m}) + \mathfrak{x}^-(t_2) \\ &= \mathfrak{p}_+\mathfrak{p}_+^{-1}([\mathfrak{t}_+^-(t_2 - t_1) - \mathfrak{x}^\bullet(t_1)] \vee 0) + \mathfrak{p}\mathfrak{p}^{-1}\mathfrak{t}^-(t_1) + \mathfrak{x}^-(t_2) \\ &= \mathfrak{p}_+\mathfrak{p}_+^{-1}([\mathfrak{t}_+^-(t_2 - t_1) - \mathfrak{x}^\bullet(t_1)] \vee 0) + [\mathfrak{x}^-(t_2) - \mathfrak{x}^-(t_1)] + \mathfrak{x}^\bullet(t_1) \\ &\equiv \mathfrak{p}_+\mathfrak{p}_+^{-1}\mathring{\mathfrak{t}}(t_2 - t_1) + \mathring{\mathfrak{x}}(t_2 - t_1). \end{aligned}$$

Consider this conditional on $\mathfrak{x}^\bullet(t_1) = a \geqq 0$.

Because of the differential character of the standard Brownian motion $\mathfrak{x}^-$,

4a. $$t \to \mathring{\mathfrak{x}}(t) = [\mathfrak{x}^-(t + t_1) - \mathfrak{x}^-(t_1)] + \mathfrak{x}^\bullet(t_1)$$

is likewise a standard Brownian motion starting at $\mathring{\mathfrak{x}}(0) = \mathfrak{x}^\bullet(t_1) = a$, independent of $\mathfrak{x}^-(s):s \leqq t_1$ and of $\mathfrak{p}$ (and hence independent of $\mathfrak{x}^\bullet(s):s \leqq t$, and of $\mathfrak{p}_+$ also) with minimum function

4b. $$\begin{aligned} -(\min_{s \leqq t} \mathring{\mathfrak{x}}(s) \wedge 0) &= -(\min_{s \leqq t} [\mathfrak{x}^-(s + t_1) - \mathfrak{x}^-(t_1)] + \mathfrak{x}^\bullet(t_1) \wedge 0) \\ &= [-\min_{s \leqq t} [\mathfrak{x}^-(s + t_1) - \mathfrak{x}^-(t_1)] - \mathfrak{x}^\bullet(t_1)] \vee 0 \\ &= [\mathfrak{t}_+^-(t) - \mathfrak{x}^\bullet(t_1)] \vee 0 \\ &= \mathring{\mathfrak{t}}(t). \end{aligned}$$

[22] $a \vee b$ is the larger of a and b.

Given $t \geqq 0$, the indicator of the event

5. $$(\mathfrak{m} > t) = (\mathfrak{p}^{-1}\mathfrak{t}^-(t_1) > t) = (\mathfrak{t}^-(t_1) > \mathfrak{p}(t))$$

is a Borel function of $\mathfrak{p}(s):s \leqq t$ and $\mathfrak{x}^-(s):s \leqq t_1$, and, since $\mathfrak{x}^-$ and $\mathfrak{p}$ are independent, $\mathfrak{m}$ is a *stopping time* for $\mathfrak{p}$, i.e., $\mathfrak{p}_+$ is identical in law to $\mathfrak{p}$ and independent of $\mathfrak{x}^-$ and of $\mathfrak{p}(s):s \leqq \mathfrak{m}$ and hence independent of $\mathfrak{x}^\bullet(s):s \leqq t_1$ and of $\mathring{\mathfrak{x}}$.

But now it is clear that, conditional on $\mathfrak{x}^\bullet(t_1) = a$, $\mathfrak{x}^\bullet(t_2)$ is independent of the past $\mathfrak{x}^\bullet(s):s \leqq t_1$ with law

6. $$P_a[\mathfrak{x}^\bullet(t) \in db], \qquad a = \mathfrak{x}^\bullet(t_1), \quad t = t_2 - t_1$$

as was to be proved.

14. Local times: $p_1 = p_3 = 0$ $(p_2 > 0/p_4 = +\infty)$

Because the reflecting Brownian local time $\mathfrak{t}^+$ was central to the construction of the Brownian motions in the case $p_4 = 0$ treated in Section 10, one expects that a similar local time $\mathfrak{t}^\bullet$ based upon the path $\mathfrak{x}^\bullet = \mathfrak{p}\mathfrak{p}^{-1}\mathfrak{t}^+ - \mathfrak{t}^+ + \mathfrak{x}^+$ should figure in the general case; the purpose of this section is to prove its existence.

Given $p_2 > 0$, the contention is that the *local time*

1a. $$\mathfrak{t}^\bullet(t) = \lim_{\varepsilon \downarrow 0}(2\varepsilon p_2)^{-1} \text{ measure } (s:\mathfrak{x}^\bullet(s) < \varepsilon, s \leqq t), \qquad t \geqq 0$$

exists and can be expressed as

1b. $$\begin{aligned} \mathfrak{t}^\bullet(t) &= p_2^{-1}\mathfrak{t}^+(\mathfrak{Q}^+ \cap [0, t]) \\ &= p_2^{-1} \mid \mathfrak{Q} \cap [0, \mathfrak{t}^+(t)]| \\ &= \mathfrak{p}^{-1}\mathfrak{t}^+(t), \end{aligned}$$

in which

2a. $$\mathfrak{Q} = (t:\mathfrak{p}\mathfrak{p}^{-1}(t) = t),$$

2b. $$\mathfrak{Q}^+ = (t:\mathfrak{t}^+(t) \in \mathfrak{Q}).$$

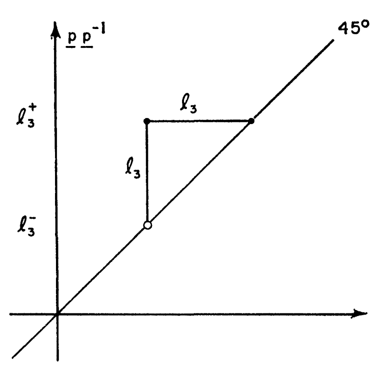


DIAGRAM 1

Consider, for the proof, the intervals $[l_1^-, l_1^+)$, $[l_2^-, l_2^+)$, etc. of the complement of $\mathfrak{Q}$, and note that the complement of $\mathfrak{Q}^+$ is the union of the intervals $[\mathfrak{t}^{-1}(l_1^-), \mathfrak{t}^{-1}(l_1^+))$, $[\mathfrak{t}^{-1}(l_2^-), \mathfrak{t}^{-1}(l_2^+))$, etc., whence[23] $\partial\mathfrak{Q}^+$ is *countable*.

Because $\mathfrak{t}^+$ is continuous and $\mathfrak{x}^\bullet = \mathfrak{x}^+$ on $\mathfrak{Q}^+$,

3. $$\begin{aligned}\lim_{\varepsilon\downarrow 0}(2\varepsilon)^{-1} \text{ measure } (s:\mathfrak{x}^\bullet(s) < \varepsilon,\ s \in \mathfrak{Q}^+ \cap [0, t]) &= \lim_{\varepsilon\downarrow 0}(2\varepsilon)^{-1} \text{ measure } (s:\mathfrak{x}^+(s) < \varepsilon,\ s \in \mathfrak{Q}^+ \cap [0, t]) \\ &= \mathfrak{t}^+(\mathfrak{Q}^+ \cap [0, t]).\end{aligned}$$

Consider, next,

4. $$\begin{aligned}\mathfrak{Q}_\varepsilon^- &= (t: t \notin \mathfrak{Q}^+,\ \mathfrak{x}^\bullet < \varepsilon) \\ &= \cup_{n\geqq 1} [\mathfrak{t}^{-1}(l_n^-), \mathfrak{t}^{-1}(l_n^+)) \cap (t: l_n^+ - \mathfrak{t}^+ < \varepsilon) \\ &= \cup_{n\geqq 1} [\mathfrak{t}^{-1}(l_n^-), \mathfrak{t}^{-1}(l_n^+)) \cap [\mathfrak{t}^{-1}(l_n^+ - \varepsilon), +\infty) \\ &= \cup_{n\geqq 1} [\mathfrak{t}^{-1}(l_n^- \vee (l_n^+ - \varepsilon)), \mathfrak{t}^{-1}(l_n^+)).\end{aligned}$$

Because $\mathfrak{t}^{-1}$ is left-continuous, $\cap_{\varepsilon>0} \mathfrak{Q}_\varepsilon^- = \emptyset$, and, seeing as $\partial\mathfrak{Q}_\varepsilon^-$ is countable and $\mathfrak{t}^+$ is continuous, it develops, much as in 3, that

5. $$\begin{aligned}\overline{\lim}_{\varepsilon\downarrow 0}(2\varepsilon)^{-1} \text{ measure } (s:\mathfrak{x}^\bullet(s) < \varepsilon,\ s \in [0, t] - \mathfrak{Q}^+) &\leqq \overline{\lim}_{\varepsilon\downarrow 0}(2\varepsilon)^{-1} \text{ measure } (s:\mathfrak{x}^+(s) < \varepsilon,\ s \in \mathfrak{Q}_\delta^- \cap [0, t]) \\ &= \mathfrak{t}^+(\mathfrak{Q}_\delta^- \cap [0, t]) \\ &\downarrow 0 \quad (\delta \downarrow 0),\end{aligned}$$

which justifies the definition 1a and the first line of 1b; the second line of 1b is immediate from the definition of $\mathfrak{Q}^+$, and, as to the third line,

6. $$\begin{aligned}\mathfrak{p}\mathfrak{p}^{-1} &= t, & t &\in \mathfrak{Q}, \\ &= l_n^+, & t &\in [l_n^-, l_n^+) \quad (n \geqq 1), \\ &= p_2\mathfrak{p}^{-1} + \int_{0+} l\mathfrak{p}([0, \mathfrak{p}^{-1}] \times dl), \end{aligned}$$

and, picking out the continuous part on both sides, it is clear that

7. $$\begin{aligned}\mathfrak{p}^{-1}(dt) &= p_2^{-1}\, dt \quad \text{on} \quad \mathfrak{Q}, \\ &= 0 \qquad\quad \text{off} \quad \mathfrak{Q},\end{aligned}$$

completing the proof.

$\mathfrak{p}^{-1}\mathfrak{t}^+$ can still be interpreted as a local time in case $p_2 = 0$ $(p_4 = +\infty)$:

8. $$\mathfrak{p}^{-1}\mathfrak{t}^+(t) = \lim_{\varepsilon\downarrow 0} \frac{\sum_{l_n>\varepsilon} \text{measure } (s:\mathfrak{x}^\bullet(s) < \varepsilon,\ s \in [\mathfrak{t}^{-1}(l_n^-), \mathfrak{t}^{-1}(l_n^+) \cap [0, t])}{\varepsilon^2 p_4[\varepsilon, +\infty)}.$$

[23] $\partial\mathfrak{Q}^+$ denotes the boundary of $\mathfrak{Q}^+$.

Consider, for the proof, the scaled *visiting times*:

9. $$\mathfrak{d}_n = \varepsilon^{-2} \text{ measure } (s : \mathfrak{x}^{\bullet}(s) < \varepsilon,\ s \in [\mathfrak{t}^{-1}(l_n^-), \mathfrak{t}^{-1}(l_n^+))).$$

Conditional on $\mathfrak{p}$ (i.e., conditional on $l_1^{\pm}$, $l_2^{\pm}$, etc.), the visiting times $\mathfrak{d}_n$ are independent because $\mathfrak{x}^+$ starts from scratch at the place $\mathfrak{x}^+(\mathfrak{m}) = 0$ at time $\mathfrak{m} = \mathfrak{t}^{-1}(l_n^-)$ $(n \geqq 1)$; in addition, if $l_n > \varepsilon$, then $\mathfrak{d}_n$ is identical in law to measure $(s : \mathfrak{x}(s) > 0,\ s < \mathfrak{m}_1)$, where $\mathfrak{x}$ is a standard Brownian motion starting at 0 and $\mathfrak{m}_1$ is its passage time to 1, as will now be verified.

Given $\sigma > 0$, the *scaling*

10. $$\mathfrak{x}(t) \to \sigma\mathfrak{x}(t/\sigma^2)$$

preserves the Wiener measure for standard Brownian paths starting at 0 and sends

11a. $$\mathfrak{x}^+(t) \to \sigma\mathfrak{x}^+(t/\sigma^2),$$

11b. $$\mathfrak{t}^+(t) \to \sigma\mathfrak{t}^+(t/\sigma^2),$$

11c. $$\mathfrak{t}^{-1}(t) \to \sigma^2\mathfrak{t}^{-1}(t/\sigma),$$

12a. $$\mathfrak{x}^-(t) \to \sigma\mathfrak{x}^-(t/\sigma^2),$$

12b. $$\mathfrak{t}^-(t) \to \sigma\mathfrak{t}^-(t/\sigma^2),$$

12c. $$\mathfrak{m}_l \to \sigma^2\mathfrak{m}_{l/\sigma},$$

where $\mathfrak{x}^-$ is the standard Brownian motion $\mathfrak{t}^+ - \mathfrak{x}^+$, $\mathfrak{t}^- = \mathfrak{t}^+ = \max_{s \leqq t} \mathfrak{x}^-(s)$, and $\mathfrak{m}_l = \min(t : \mathfrak{x}^- = l)$, and, using $\mathfrak{a} \equiv \mathfrak{b}$ to indicate that $\mathfrak{a}$ and $\mathfrak{b}$ are identical in law, it follows from the rules 11 and 12 that in case $l_n > \varepsilon$,

13. $$\begin{aligned}
\mathfrak{d}_n &\equiv \varepsilon^{-2} \text{ measure } (s : l_n - \mathfrak{t}^+(s) + \mathfrak{x}^+(s) < \varepsilon,\ s < \mathfrak{t}^{-1}(l_n)), \quad \mathfrak{x}^+(0) = 0, \\
&\equiv \varepsilon^{-2} \text{ measure } (s : 1 - \mathfrak{t}^+(s/\sigma^2) + \mathfrak{x}^+(s/\sigma^2) < \varepsilon/\sigma,\ s/\sigma^2 < \mathfrak{t}^{-1}(1)), \\
&\qquad\qquad \sigma = l_n, \\
&= \varepsilon^{-2}l_n^2 \text{ measure } (s : 1 - \mathfrak{t}^+(s) + \mathfrak{x}^+(s) < \varepsilon/l_n,\ s < \mathfrak{t}^{-1}(1)) \\
&= \varepsilon^{-2}l_n^2 \text{ measure } (s : \mathfrak{x}^-(s) > 1 - \varepsilon/l_n,\ s < \mathfrak{m}_1) \\
&= \varepsilon^{-2}l_n^2 \text{ measure } (s : \mathfrak{x}^-(s) > 1 - \varepsilon/l_n,\ \mathfrak{m}_{1-\varepsilon/l_n} \leqq s < \mathfrak{m}_1) \\
&\equiv \varepsilon^{-2}l_n^2 \text{ measure } (s : \mathfrak{x}(s) > 0,\ s < \mathfrak{m}_{\varepsilon/l_n}), \qquad \mathfrak{x}(0) = 0, \\
&\equiv \text{measure } (s : \mathfrak{x}(s) > 0,\ s < \mathfrak{m}_1),
\end{aligned}$$

where the scaling 10 was used in step 2 $(\sigma = l_n)$ and in step 7 $(\sigma = \varepsilon/l_n)$.

Coming back to 8, the strong law of large numbers combined with the rule

14. $$\lim_{\varepsilon \downarrow 0} p_4[\varepsilon, +\infty)^{-1}\mathfrak{p}([0, t] \times [\varepsilon, +\infty)) = t \qquad (t \geqq 0)$$

(see Section 11) and the simple evaluation

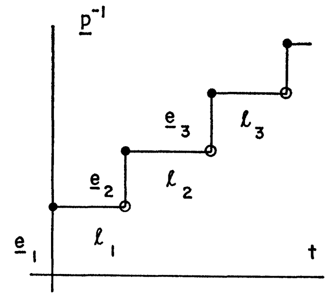


DIAGRAM 2

15.
$$\begin{aligned} E_0(\mathfrak{d}_1) &= \int_0^{+\infty} dt\, P_1[\mathfrak{x}(t) < 1,\ \mathfrak{m}_0 > t] \\ &= \int_0^{+\infty} dt \int_0^1 \frac{e^{-(a-1)^2/2t} - e^{-(a+1)^2/2t}}{(2\pi t)^{1/2}}\, da \\ &= \int_0^1 2a\, da = 1, \end{aligned}$$

justifies

16.
$$\begin{aligned} &\lim_{\varepsilon\downarrow 0} \frac{\sum_{l_n>\varepsilon} \text{measure } (s:\mathfrak{x}^\bullet(s) < \varepsilon,\ s \in [\mathfrak{t}^{-1}(l_n^-),\ \mathfrak{t}^{-1}(l_n^+)) \cap [0, t])}{\varepsilon^2 p_4[\varepsilon, +\infty)} \\ &\qquad = \lim_{\varepsilon\downarrow 0} \sum_{\substack{l_n>\varepsilon \\ \mathfrak{t}^{-1}(l_n^+)\leqq t}} \mathfrak{d}_n/p_4[\varepsilon, +\infty) \\ &\qquad = E_0(\mathfrak{d}_1) \lim_{\varepsilon\downarrow 0} \frac{\#(l_n : l_n > \varepsilon, \mathfrak{t}^{-1}(l_n^+) \leqq t)}{p_4[\varepsilon, +\infty)} \quad {}^{24} \\ &\qquad = \lim_{\varepsilon\downarrow 0} \frac{\#(l_n : l_n > \varepsilon, l_n^+ < \mathfrak{t}^+(t))}{p_4[\varepsilon, +\infty)} \\ &\qquad = \lim_{\varepsilon\downarrow 0} \frac{\mathfrak{p}([0, \mathfrak{p}^{-1}\mathfrak{t}^+(t)] \times [\varepsilon, +\infty))}{p_4[\varepsilon, +\infty)} \\ &\qquad = \mathfrak{p}^{-1}\mathfrak{t}^+(t), \end{aligned}$$

where the use of $l_n^+ < \mathfrak{t}^+(t)$ in place of $\mathfrak{t}^{-1}(l_n^+) \leqq t$ in step 3 is justified because both describe the same class of jumps plus or minus a single jump and $p_4[\varepsilon, +\infty) \uparrow +\infty$ as $\varepsilon \downarrow 0$; a picture helps to see that $l_n^+ < \mathfrak{t}^+$ and $\mathfrak{p}^{-1}(l_n^+) < \mathfrak{p}^{-1}(\mathfrak{t}^+)$ are identical as needed in step 4.

$\mathfrak{p}^{-1}\mathfrak{t}^+$ *cannot be computed from the sample path if* $p_2 = 0$ *and* $p_4 < +\infty$, as is clear from Diagram 2 in which

17. $\mathfrak{p}^{-1}\mathfrak{t}^+(t) = \mathfrak{e}_1 + \cdots + \mathfrak{e}_n, \quad \mathfrak{t}^{-1}(l_1 + \cdots + l_{n-1}) \leqq t < \mathfrak{t}^{-1}(l_1 + \cdots + l_n),$

and $\mathfrak{x}^\bullet$ is independent of the holding times $\mathfrak{e}_1$, $\mathfrak{e}_2$, etc. But it still has some features of a local time: it is the sum of n independent holding times $\mathfrak{e}$ with

[24] $\#(l_n$:etc.) denotes the number of jumps l_n with the properties described inside.

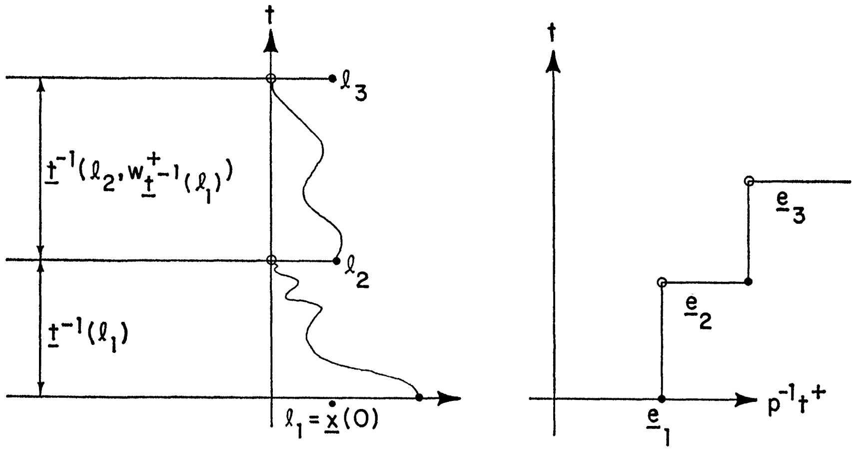


DIAGRAM 3

common conditional law $P_\cdot(\mathfrak{e}_1 > t \mid \mathfrak{x}^\bullet) = e^{-p_4 t}$, where n is the number of times that the sample path approaches 0 before time t (see Diagram 3).

15. Sample paths and Green operators: $p_1 u(0) + p_3(\mathfrak{G}u)(0) = p_2 u^+(0) + \int_{0+} [u(l) - u(0)]p_4(dl)$ $(p_2 > 0/p_4 = +\infty)$

Consider the motion $\mathfrak{x}^\bullet = \mathfrak{p}\mathfrak{p}^{-1}\mathfrak{t}^+ - \mathfrak{t}^+ + \mathfrak{x}^+$ and its local time $\mathfrak{t}^\bullet = \mathfrak{p}^{-1}\mathfrak{t}^+$, and let us use them to build up the sample paths in the general case $(p_2 > 0/p_4 = +\infty)$ imitating the prescription of Section 10:

1a. $$\mathfrak{y}^\bullet(t) = \mathfrak{x}^\bullet[\mathfrak{f}^{-1}(t)] \quad \text{if} \quad t < \mathfrak{m}_\infty^\bullet,$$
$$= \infty \quad \text{if} \quad t \geqq \mathfrak{m}_\infty^\bullet,$$

1b. $$\mathfrak{f}(t) = t + p_3\,\mathfrak{t}^\bullet(t),$$

1c. $$P_\cdot(\mathfrak{m}_\infty^\bullet > t \mid \mathfrak{x}^\bullet) = e^{-p_1 t^\bullet[\mathfrak{f}^{-1}(t)]}.$$

Given $l \geqq 0$,

2. $$(G_\alpha^\bullet f)(l) = E_l\left(\int_0^{\mathfrak{m}_\infty^\bullet} e^{-\alpha t} f(\mathfrak{y}^\bullet)\,dt\right)$$
$$= E_l\left(\int_0^{+\infty} e^{-\alpha t} e^{-p_1 \mathfrak{t}^\bullet(\mathfrak{f}^{-1})} f[\mathfrak{x}^\bullet(\mathfrak{f}^{-1})]\,dt\right)$$
$$= E_l\left[\int_0^{+\infty} e^{-\alpha \mathfrak{f}} e^{-p_1 \mathfrak{t}^\bullet} f(\mathfrak{x}^\bullet) \mathfrak{f}(dt)\right]$$
$$= E_l\left[\int_0^{\mathfrak{m}_0} e^{-\alpha t} f(\mathfrak{x}^+)\,dt\right]$$
$$+ E_l(e^{-\alpha \mathfrak{m}_0}) E_0\left[\int_0^{+\infty} e^{-\alpha t} e^{-p_1 \mathfrak{t}^\bullet} f(\mathfrak{x}^\bullet)\,\mathfrak{f}\,(dt)\right] \quad [25]$$
$$= (G_\alpha^- f)(l) + e^{-(2\alpha)^{1/2} l}(G_\alpha^\bullet f)(0),$$

[25] $\mathfrak{x}^\bullet = \mathfrak{x}^+$ and $\mathfrak{t}^\bullet = 0$ up to time $\mathfrak{m}_0 = \min(t : \mathfrak{x}^+ = 0)$, and $\mathfrak{x}^\bullet$ starts afresh at that moment.

especially, the Green operators map $C[0, +\infty)$ into itself in the special case $p_1 = p_3 = 0$ ($\mathfrak{y}^\bullet = \mathfrak{x}^\bullet$), and, since $\mathfrak{x}^\bullet$ starts afresh at constant times, it follows that it must be a Brownian motion. $\mathfrak{y}^\bullet$ is likewise a Brownian motion as is clear on arguing as in Section 10 with $\mathfrak{x}^\bullet$ and $\mathfrak{t}^\bullet$ in place of $\mathfrak{x}^+$ and $\mathfrak{t}^+$, and now, for the identification of its generator as the contraction of $\mathfrak{G} = D^2/2$ to

$$3. \quad D(\mathfrak{G}^\bullet) = C^2[0, +\infty) \cap \Big(u\colon p_1 u(0) + p_3(\mathfrak{G}u)(0) = p_2 u^+(0) + \int_{0+} [u(l) - u(0)]p_4(dl)\Big),$$

it suffices to make the evaluation

$$4. \quad e = (G_\alpha^\bullet f)(0) = E_0\left[\int_0^{+\infty} e^{-\alpha\mathfrak{f}} e^{-p_1 \mathfrak{t}^\bullet} f(\mathfrak{x}^\bullet)\mathfrak{f}\,(dt)\right] = \frac{p_2 2\int_{0+} e^{-(2\alpha)^{1/2}l} f(l)\,dl + p_3 f(0) + \int_{0+} (G_\alpha^- f)(l)p_4(dl)}{p_1 + (2\alpha)^{1/2}p_2 + \alpha p_3 + \int_{0+} [1 - e^{-(2\alpha)^{1/2}l}]p_4(dl)}.$$

e is decomposed into simpler integrals in several steps (see the explanation below):

$$\begin{aligned}
5. \quad e &= E_0\left[\int_0^{+\infty} e^{-\alpha t} e^{-(p_1+\alpha p_3)\mathfrak{t}^\bullet} f(\mathfrak{x}^\bullet)\,dt\right] \\
&\qquad + p_3 f(0) E_0\left[\int_0^{+\infty} e^{-\alpha t} e^{-(p_1+\alpha p_3)\mathfrak{t}^\bullet}\mathfrak{t}^\bullet(dt)\right] \\
&= \sum_{n\geq 1} E_0\left[\int_{[\mathfrak{t}^{-1}(l_n^-),\mathfrak{t}^{-1}(l_n^+))} e^{-\alpha t}\, e^{-(p_1+\alpha p_3)\mathfrak{p}^{-1}\mathfrak{t}^+} f(l_n^+ - \mathfrak{t}^+ + \mathfrak{x}^+)\,dt\right] \\
&\quad + E_0\left[\int_0^{+\infty} e^{-\alpha t} e^{-(p_1+\alpha p_3)\mathfrak{p}^{-1}\mathfrak{t}^+} f(\mathfrak{x}^+)\,dt\right] \\
&\quad - \sum_{n\geq 1} E_0\left[\int_{[\mathfrak{t}^{-1}(l_n^-),\mathfrak{t}^{-1}(l_n^+))} e^{-\alpha t} e^{-(p_1+\alpha p_3)\mathfrak{p}^{-1}\mathfrak{t}^+} f(\mathfrak{x}^+)\,dt\right] \\
&\quad + p_3 f(0) E_0\left[\int_0^{+\infty} e^{-\alpha t} e^{-(p_1+\alpha p_3)\mathfrak{p}^{-1}\mathfrak{t}^+}\mathfrak{p}^{-1}\mathfrak{t}^+(dt)\right] \\
&= \sum_{n\geq 1} E_0\Big[e^{-\alpha\mathfrak{t}^{-1}(l_n^-)}\, e^{-(p_1+\alpha\mathfrak{p}_3)p^{-1}(l_n^+)} \\
&\qquad\qquad \cdot E_0\Big(\int_0^{\mathfrak{t}^{-1}(l_n)} e^{-\alpha t} f\,[l_n - \mathfrak{t}^+ + \mathfrak{x}^+]\,dt \,\big|\, l_n\Big)\Big] \\
&\quad + E_0\left[\int_0^{+\infty} e^{-\alpha t} e^{-(p_1+\alpha p_3)\mathfrak{p}^{-1}\mathfrak{t}^+} f(\mathfrak{x}^+)\,dt\right] \\
&\quad - \sum_{n\geq 1} E_0\Big[e^{-\alpha\mathfrak{t}^{-1}(l_n^-)}\, e^{-(p_1+\alpha p_3)\mathfrak{p}^{-1}(l_n^+)} E_0\Big(\int_0^{\mathfrak{t}^{-1}(l_n)} e^{-\alpha t} f(\mathfrak{x}^+)\,dt \,\big|\, l_n\Big)\Big] \\
&\quad + \frac{p_3 f(0)}{p_1 + \alpha p_3}\left[1 - \alpha E_0\Big(\int_0^{+\infty} e^{-\alpha t} e^{-(p_1+\alpha p_3)\mathfrak{p}^{-1}\mathfrak{t}^+}\,dt\Big)\right] \text{[26]} \\
&= e_1 + e_2 - e_3 + e_4,
\end{aligned}$$

[26] $p_3/(p_1 + \alpha p_3) = 0$ if $p_3 = 0$.

where $\mathfrak{t}^\bullet(dt) = 0$ *outside* $\mathfrak{Z}^\bullet \equiv (t: \mathfrak{x}^\bullet = 0)$ was used in step 1, $[0, +\infty)$ was split into $\mathfrak{Q}^+ + \bigcup_{n \geq 1} [\mathfrak{t}^{-1}(l_n^-), \mathfrak{t}^{-1}(l_n^+))$ in step 2, and $\mathfrak{p}\mathfrak{p}^{-1}\mathfrak{t}^+$ was evaluated as $\mathfrak{t}^+$ or l_n^+ according as $t \in \mathfrak{Q}^+$ or $\mathfrak{t}^{-1}(l_n^-) \leqq t < \mathfrak{t}^{-1}(l_n^+)$, and, in step 3, it was noted that, conditional on $\mathfrak{p}$, the standard Brownian traveller starts afresh at time $\mathfrak{m} = \mathfrak{t}^{-1}(l_n^-)$ at the place $l = 0$; the addition rule

$$\mathfrak{t}^{-1}(l_n^+) = \mathfrak{t}^{-1}(l_n^-) + \mathfrak{t}^{-1}(l_n, w^+_{\mathfrak{t}^{-1}(l_n^-)})$$

was also used in step 3, and a partial (time) integration was performed under the expectation sign in e_4.

To compute e_1, substitute the standard Brownian motion $\mathfrak{x}^- = \mathfrak{t}^+ - \mathfrak{x}^+$ and its passage times $\mathfrak{m}_l = \mathfrak{t}^{-1}(l)$ into the conditional expectation and integrate them out, next integrate out $\mathfrak{t}^{-1}(l_n^-)$ conditional on $\mathfrak{p}$, express the integral in terms of the Poisson measure $\mathfrak{p}(dt \times dl)$, and use the differential character of the latter to integrate it out also:

$$6.\quad e_1 = \sum_{n \geq 1} E_0\left[e^{-\alpha \mathfrak{t}^{-1}(l_n^-)} e^{-(p_1+\alpha p_3)\mathfrak{p}^{-1}(l_n^+)} E_0\left(\int_0^{\mathfrak{m}_{l_n}} e^{-\alpha t} f(l_n - \mathfrak{x}^-)\, dt \mid l_n\right)\right]$$

$$= \sum_{n \geq 1} E_0[e^{-\alpha \mathfrak{t}^{-1}(l_n^-)} e^{-(p_1+\alpha p_3)\mathfrak{p}^{-1}(l_n^+)} (G_\alpha^- f)(l_n)]$$

$$= \sum_{n \geq 1} E_0[e^{-(2\alpha)^{1/2} l_n^-} e^{-(p_1+\alpha p_3)\mathfrak{p}^{-1}(l_n^+)} (G_\alpha^- f)(l_n)]$$

$$= E_0\left[\int_{[0,+\infty)\times(0,+\infty)} \mathfrak{p}(dt \times dl) e^{-(2\alpha)^{1/2}\mathfrak{p}(t-)} e^{-(p_1+\alpha p_3)t} (G_\alpha^- f)(l)\right]$$

$$= \lim_{\varepsilon \downarrow 0} E_0\left[\int_{[\varepsilon,+\infty)\times(0,+\infty)} \mathfrak{p}(dt \times dl) e^{-(2\alpha)^{1/2}\mathfrak{p}(t-\varepsilon)} e^{-(p_1+\alpha p_3)t} (G_\alpha^- f)(l)\right]$$

$$= \int_{[0,+\infty)\times(0,+\infty)} dt\, p_4(dl) \exp\left\{-t\left[p_2(2\alpha)^{1/2} + \int_{0+}(1 - e^{-(2\alpha)^{1/2} l}) p_4(dl)\right]\right\} \cdot e^{-(p_1+\alpha p_3)t} (G_\alpha^- f)(l)$$

$$= \frac{\int_{0+} (G_\alpha^- f)(l) p_4(dl)}{p_1 + (2\alpha)^{1/2} p_2 + \alpha p_3 + \int_{0+} (1 - e^{-(2\alpha)^{1/2} l}) p_4(dl)}.$$

To compute e_2, use the joint law $\dfrac{2(b+a)}{(2\pi t^3)^{1/2}} e^{-(b+a)^2/2t}\, da\, db$ of $\mathfrak{x}^+$ and $\mathfrak{t}^+$:

$$7.\quad e_2 = E_0\left[\int_0^{+\infty} e^{-\alpha t}\, dt \int_0^{+\infty} db \int_0^{+\infty} da\; 2\frac{b+a}{(2\pi t^3)^{1/2}} e^{-(b+a)^2/2t} \cdot e^{-(p_1+\alpha p_3)\mathfrak{p}^{-1}(b)} f(a)\right]$$

$$= E_0\left[\int_0^{+\infty} e^{-(2\alpha)^{1/2}b} e^{-(p_1+\alpha p_3)\mathfrak{p}^{-1}(b)}\, db\right] 2\int_{0+} e^{-(2\alpha)^{1/2}a} f(a)\, da$$

$$= E_0\left[\int_0^{+\infty} e^{-(2\alpha)^{1/2}\mathfrak{p}} e^{-(p_1+\alpha p_3)t}\, \mathfrak{p}(dt)\right] 2\int_{0+} e^{-(2\alpha)^{1/2}l} f(l)\, dl$$

$$= \frac{p_1+\alpha p_3}{(2\alpha)^{1/2}} E_0\left[\int_0^{+\infty} e^{-(p_1+\alpha p_3)t}(1-e^{-(2\alpha)^{1/2}\mathfrak{p}})\, dt\right] 2\int_{0+} e^{-(2\alpha)^{1/2}l} f(l)\, dl$$

$$= \frac{(2\alpha)^{1/2}p_2 + \int_{0+}(1-e^{-(2\alpha)^{1/2}l})p_4(dl)}{p_1+(2\alpha)^{1/2}p_2+\alpha p_3+\int_{0+}(1-e^{-(2\alpha)^{1/2}l})p_4(dl)} \frac{2}{(2\alpha)^{1/2}}\int_{0+} e^{-(2\alpha)^{1/2}l} f(l)\, dl.$$

To compute e_3, use the same manipulations as for e_1 together with the lemma[27]

8. $$E_0\left(\int_0^{\mathfrak{t}^{-1}(l)} e^{-\alpha t} f(\mathfrak{x}^+)\, dt\right) = (G_\alpha^+ f)(0)[1-e^{-(2\alpha)^{1/2}l}],$$

obtaining

9. $$e_3 = \sum_{n\geq 1} E_0[e^{-(2\alpha)^{1/2}l_n^-} e^{-(p_1+\alpha p_3)\mathfrak{p}^{-1}(l_n^+)}(1-e^{-(2\alpha)^{1/2}l_n})](G_\alpha^+ f)(0)$$

$$= E_0\left[\int_{[0,+\infty)\times(0,+\infty)} \mathfrak{p}(dt\times dl) e^{-(2\alpha)^{1/2}\mathfrak{p}(t-)} e^{-(p_1+\alpha p_3)t}(1-e^{-(2\alpha)^{1/2}l})\right] \cdot (G_\alpha^+ f)(0)$$

$$= \frac{\int_{0+}(1-e^{-(2\alpha)^{1/2}l})p_4(dl)}{p_1+(2\alpha)^{1/2}p_2+\alpha p_3+\int_{0+}(1-e^{-(2\alpha)^{1/2}l})p_4(dl)} \frac{2}{(2\alpha)^{1/2}}\int_{0+} e^{-(2\alpha)^{1/2}l} f(l)\, dl.$$

To compute e_4, use 6 with $f = 1$:

10. $$e_4 = \frac{p_3 f(0)}{p_1+\alpha p_3}\left[1 - \frac{(2\alpha)^{1/2}p_2 + \int_{0+}(1-e^{-(2\alpha)^{1/2}l})p_4(dl)}{p_1+(2\alpha)^{1/2}p_2+\alpha p_3+\int_{0+}(1-e^{-(2\alpha)^{1/2}l})p_4(dl)}\right]$$

$$= \frac{p_3 f(0)}{p_1+(2\alpha)^{1/2}p_2+\alpha p_3+\int_{0+}(1-e^{-(2\alpha)^{1/2}l})p_4(dl)}.$$

Combining 5, 6, 7, 9, and 10 verifies 4, and that finishes the proof.

16. Bounded interval: $[-1, +1]$

A Brownian motion on $[-1, +1]$ is defined as in Section 5 except that

1. $$|\mathfrak{x}^\bullet| \leqq 1, \qquad t < \mathfrak{m}_\infty^\bullet,$$

[27] G_α^+ is the reflecting Brownian Green operator.

and the stopped path

2a. $$\mathfrak{x}^\bullet(t): t < \mathfrak{e}^\bullet = \lim_{\varepsilon \downarrow 0} \inf(t: |\mathfrak{x}^\bullet| > 1 - \varepsilon), \quad -1 < \mathfrak{x}^\bullet(0) = l < +1$$

is now identical in law to the stopped standard Brownian path

2b. $$\mathfrak{x}(t): t < \mathfrak{e} = \min(t: |\mathfrak{x}| = 1), \qquad \mathfrak{x}(0) = l.$$

Except in the case $P^\bullet_\cdot[|\mathfrak{x}^\bullet(\mathfrak{e}^\bullet)| = 1] < 1$ which can be treated as in Section 6, $C[-1, +1]$ is mapped into itself under the Green operators, $\mathfrak{G}^\bullet_\cdot$ can be defined as before, and $D(\mathfrak{G}^\bullet)$ can be described in terms of six nonnegative numbers $p_{\pm 1}$, $p_{\pm 2}$, $p_{\pm 3}$ and two nonnegative mass distributions $p_{\pm 4}(dl)$ subject to

3a. $$p_{-1} + p_{-2} + p_{-3} + \int_{-1}^{+1} (1 + l) p_{-4}(dl) = 1, \qquad p_{-4}(-1) = 0,$$

3b. $$p_{+1} + p_{+2} + p_{+3} + \int_{-1}^{+1} (1 - l) p_{+4}(dl) = 1, \qquad p_{+4}(+1) = 0,$$

4a. $$p_{-4}(-1, +1] = +\infty \quad \text{in case} \quad p_{-2} = p_{-3} = 0,$$

4b. $$p_{+4}[-1, +1) = +\infty \quad \text{in case} \quad p_{+2} = p_{+3} = 0$$

as follows. $D(\mathfrak{G}^\bullet)$ is the class of functions $u \in C^2[-1, +1]$ subject to

5a. $$p_{-1} u(-1) - p_{-2} u^+(-1) + p_{-3}(\mathfrak{G}u)(-1) = \int_{-1}^{+1} [u(l) - u(-1)]\, p_{-4}(dl),$$

5b. $$p_{+1} u(+1) + p_{+2} u^-(+1) + p_{+3}(\mathfrak{G}u)(+1) = \int_{-1}^{+1} [u(l) - u(+1)] p_{+4}(dl).$$ [28]

$\mathfrak{G}^\bullet$ is the contraction of $\mathfrak{G} = D^2/2$ to $D(\mathfrak{G}^\bullet)$,

6. $$(G^\bullet_\alpha f)(l) = (G^-_\alpha f)(l) + e_-(l)(G^\bullet_\alpha f)(-1) + e_+(l)(G^\bullet_\alpha f)(+1), \qquad |l| \leq 1,$$

in which

7a. $$(G^-_\alpha f)(a) = E_a\left(\int_0^{\mathfrak{e}} e^{-\alpha t} f(\mathfrak{x})\, dt\right) = 2 \int_{-1}^{+1} G(a, b) f(b)\, db,$$ [29]

7b. $$G(a, b) = G(b, a) = \frac{\sinh(2\alpha)^{1/2}(1 + a) \sinh (2\alpha)^{1/2}(1 - b)}{(2\alpha)^{1/2}}, \quad a \leq b,$$

[28] $u^-(+1) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1}[u(1) - u(1 - \varepsilon)]$.

[29] $P_\cdot$, $E_\cdot$, $\mathfrak{x}$, $\mathfrak{m}$ are the standard Brownian probabilities, expectations, sample paths, and passage times.

is the Green operator for the Brownian motion with instant killing at ± 1 and

8a. $$e_-(l) = \frac{\sinh (2\alpha)^{1/2}(1-l)}{\sinh 2(2\alpha)^{1/2}} = E_l\,(e^{-\alpha \mathfrak{m}_{-1}}, \mathfrak{m}_{-1} < \mathfrak{m}_{+1}),$$

8b. $$e_+(l) = \frac{\sinh (2\alpha)^{1/2}(1+l)}{\sinh 2(2\alpha)^{1/2}} = E_l(e^{-\alpha \mathfrak{m}_{+1}}, \mathfrak{m}_{+1} < \mathfrak{m}_{-1}),$$

and, substituting 6 into 5 and solving for $(G_\alpha^\bullet f)(\pm 1)$, one obtains

9. $$\begin{bmatrix} (G_\alpha^\bullet f)(-1) \\ (G_\alpha^\bullet f)(+1) \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}^{-1} \cdot \begin{bmatrix} p_{-2}(G_\alpha^- f)^+(-1) + p_{-3} f(-1) + \int_{-1}^{+1} (G_\alpha^- f)(l) p_{-4}(dl) \\ -p_{+2}(G_\alpha^- f)^-(+1) + p_{+3} f(+1) + \int_{-1}^{+1} (G_\alpha^- f)(l) p_{+4}(dl) \end{bmatrix},$$

where the exponent -1 indicates the inverse of $\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}$, and

10a. $$e_{11} = p_{-1} - e_-^+(-1) p_{-2} + \alpha p_{-3} + \int_{-1}^{+1} (1 - e_-) p_{-4}(dl),$$

10b. $$e_{12} = -p_{-2} e_+^+(-1) - \int_{-1}^{+1} e_+ \, p_{-4}(dl),$$

10c. $$e_{21} = p_{+2} e_-^-(+1) - \int_{-1}^{+1} e_- \, p_{+4}(dl),$$

10d. $$e_{22} = p_{+1} + e_+^-(+1) p_{+2} + \alpha p_{+3} + \int_{-1}^{+1} (1 - e_+) p_{+4}(dl),$$

all of which is due to W. Feller [1], [3]; the proofs can be carried out as in Section 8.

Coming to the sample paths, let us confine our attention to the case $p_{-4}(-1, +1] = p_{+4}[-1, +1) = +\infty$, leaving the opposite case to the reader.

Given a standard Brownian motion with sample paths $t \to \mathfrak{x}(t)$ and probabilities $P_a(B)$, if f is the map: $R^1 \to [-1, +1]$ defined by folding the line at ± 1, ± 3, ± 5, etc. as in Diagram 1, then $\mathfrak{x}^+ = f(\mathfrak{x})$ is the (reflecting) Brownian motion on $[-1, +1]$ associated with 5 in the special case $p_{\pm 1} = p_{\pm 3} = p_{\pm 4} = 0$ $(u^+(-1) = u^-(+1) = 0)$; the dot sample path will be made up using $\mathfrak{x}^+$ and its local times

11a. $\mathfrak{t}^-(t) = \lim_{\varepsilon \downarrow 0}(2\varepsilon)^{-1}$ measure $(s : \mathfrak{x}^+(s) < -1 + \varepsilon, \ s \leqq t)$,

11b. $\mathfrak{t}^+(t) = \lim_{\varepsilon \downarrow 0}(2\varepsilon)^{-1}$ measure $(s : \mathfrak{x}^+(s) > 1 - \varepsilon, \ s \leqq t)$,

a pair of independent Poisson measures $\mathfrak{p}_\pm(dt \times dl)$ with means $dt \; p_{\pm 4}(dl)$,

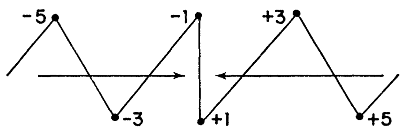


DIAGRAM 1

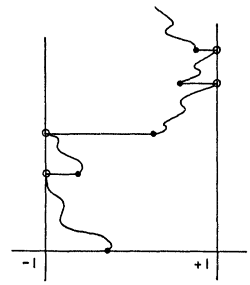


DIAGRAM 2

and the associated differential processes

12a. $$\mathfrak{p}_-(t) = p_{-2}t + \int_{l>-1} (1 + l)\mathfrak{p}_-([0, t] \times dl),$$

12b. $$\mathfrak{p}_+(t) = p_{+2}t + \int_{l<+1} (1 - l)\mathfrak{p}_+([0, t] \times dl).$$

Diagram 2 depicts the sample paths associated with 5 if $p_{\pm1} = p_{\pm3} = 0$: $\mathfrak{x}^\bullet$ and $\mathfrak{x}^+$ agree up to time $\mathfrak{m}_{\pm1} = \min(t:|\mathfrak{x}^+| = 1)$; if $\mathfrak{m}_{-1} < \mathfrak{m}_{+1}$, as in the picture, then $\mathfrak{x}^\bullet$ changes over into $\mathfrak{p}_-\mathfrak{p}_-^{-1}\mathfrak{t}^- - \mathfrak{t}^- + \mathfrak{x}^+$ until it hits $+1$, at which instant it changes over into $-\mathfrak{p}_+\mathfrak{p}_+^{-1}\mathfrak{t}^+ + \mathfrak{t}^+ + \mathfrak{x}^+$ until it hits -1 for the second time, etc.

If $p_{-1} = p_{+1} = 0$ and $p_{-3} + p_{+3} > 0$, then the desired motion is as in Diagram 2 but run with the new clock $\mathfrak{f}^{-1}$ which is the inverse function of

13a. $$\mathfrak{f} = t + p_{-3}\mathfrak{t}^{\bullet-} + p_{+3}\mathfrak{t}^{\bullet+},$$

13b. $$\mathfrak{t}^{\bullet\pm} = \mathfrak{p}_\pm^{-1}\mathfrak{t}^{\pm},$$

while, if $p_{-1} + p_{+1} > 0$, then one has just to kill the above motion $\mathfrak{x}^\bullet(\mathfrak{f}^{-1})$ at time $\mathfrak{m}_\infty^\bullet$ with conditional law

14. $$P_\cdot(\mathfrak{m}_\infty^\bullet > t \mid \mathfrak{x}^\bullet) = e^{-[p_{-1}\mathfrak{t}^{\bullet-}(\mathfrak{f}^{-1}) + p_{+1}\mathfrak{t}^{\bullet+}(\mathfrak{f}^{-1})]};$$

the proofs are left to the industrious reader.

17. Two-sided barriers

A Brownian motion on R^1 with a *two-sided barrier* at $l = 0$ is defined as in Section 5 except that

1. $$\mathfrak{x}^\bullet \in R^1, \qquad t < \mathfrak{m}_\infty^\bullet,$$

and the stopped path

2a. $\mathfrak{x}^\bullet(t):t < \mathfrak{e}^\bullet = \lim_{\varepsilon\downarrow0}\inf(t:|\mathfrak{x}^\bullet| < \varepsilon), \qquad \mathfrak{x}^\bullet(0) = l \in R^1 - 0$

is identical in law to the stopped standard Brownian motion

2b. $$\mathfrak{x}(t):t < \mathfrak{e} = \min(t:\mathfrak{x} = 0), \qquad \mathfrak{x}(0) = l.$$

Except in the case $P_{\cdot}^{\bullet}[\mathfrak{x}^{\bullet}(\mathfrak{e}^{\bullet}) = 0] < 1$, which is ignored as before, $C(R^1)$ is mapped into itself under the Green operators, $\mathfrak{G}^{\bullet}$ is the contraction of $\mathfrak{G} = D^2/2$ to [30]

$$3.\quad D(\mathfrak{G}^{\bullet}) = C^{\bullet 2}(R^1) \cap \Big(u: p_1 u(0) + p_{-2}u^-(0) - p_{+2}u^+(0) + p_3(\mathfrak{G}u)(0\pm) = \int_{|l|>0} [u(l) - u(0)]p_4(dl)\Big)$$

for some nonnegative numbers p_1, $p_{\pm 2}$, p_3 and some nonnegative mass distribution $p_4(dl)$ subject to

$$4a.\quad p_1 + p_{-2} + p_{+2} + p_3 + \int (|l| \wedge 1)p_4\,(dl) = 1, \qquad p_4(0) = 0,$$

$$4b.\quad p_4(R^1) = +\infty \quad \text{in case} \quad p_{\pm 2} = p_3 = 0,$$

and the Green operators are

$$5.\quad (G_\alpha^{\bullet} f)(l) = (G_\alpha^- f)(l) + e^{-(2\alpha)^{1/2}|l|}(G_\alpha^{\bullet} f)(0),$$

where

$$6.\quad (G_\alpha^- f)(a) = \int_{ab>0} \frac{e^{-(2\alpha)^{1/2}|b-a|} - e^{-(2\alpha)^{1/2}|b+a|}}{(2\alpha)^{1/2}} f(b)\,db$$

is the Green operator for the Brownian motion with instant killing at $l = 0$ and

$$7a.\quad (G_\alpha^{\bullet} f)(0) = \frac{-p_{-2}(G_\alpha^- f)^-(0) + p_{+2}(G_\alpha^- f)^+(0) + p_3 f(0) + \int_{|l|>0} (G_\alpha^- f)(l)p_4(dl)}{p_1 + (2\alpha)^{1/2}(p_{-2} + p_{+2}) + \alpha p_3 + \int_{|l|>0} (1 - e^{-(2\alpha)^{1/2}|l|})p_4(dl)},$$

$$7b.\quad \pm(G_\alpha^- f)^{\pm}(0) = 2\int_{\pm l>0} e^{-(2\alpha)^{1/2}|l|} f(l)\,dl.$$

Coming to the sample paths, P. Lévy [3] proved that if $t \to \mathfrak{x}(t)$ is a standard Brownian path starting at 0 and if $\mathfrak{Z}_1$, $\mathfrak{Z}_2$, etc. are the (open) intervals of the complement of $\mathfrak{Z} = (t: \mathfrak{x} = 0)$, then the *signs* e_1, e_2, etc. of the *excursions* $\mathfrak{x}(t): t \in \mathfrak{Z}_1$, etc., are independent Bernouilli trials with common law $P_0(e_1 = \pm 1) = \frac{1}{2}$ (standard coin-tossing game), independent of $\mathfrak{Z}$ and of the (unsigned) *scaled excursions*

$$8.\quad \mathfrak{x}_1(t) = |\mathfrak{Z}_1|^{-1/2}\,|\mathfrak{x}(t\,|\mathfrak{Z}_1| + \inf \mathfrak{Z}_1)|, \qquad 0 \leqq t \leqq 1,$$

etc.

which are independent, identical in law, and likewise independent of $\mathfrak{Z}$ (see Diagram 1).

Given $p_{-2} + p_{+2} > 0$, it is not difficult to see that if e_1, e_2, etc. is now a *skew coin-tossing game* independent of the scaled excursions and of $\mathfrak{Z}$ (i.e.,

[30] $C^{\bullet 2}(R^1) = C^2(-\infty, 0] \cap C^2[0, +\infty) \cap (u: u''(0-) = u''(0+))$.

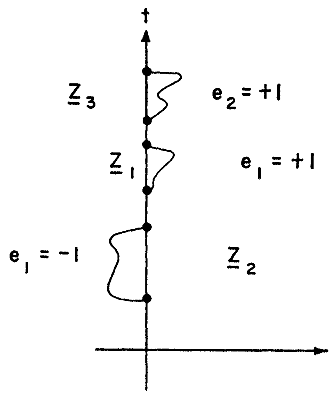


DIAGRAM 1

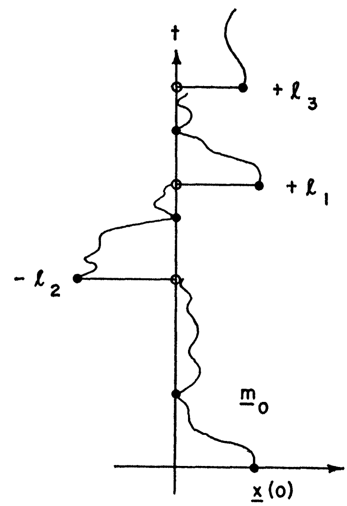


DIAGRAM 2

independent of $|\mathfrak{x}|$) with law

9. $$P_0(e_1 = -1):P_0(e_1 = +1) = p_{-2} : p_{+2}\,;$$

then the *skew Brownian motion*

10. $$\begin{aligned} \mathfrak{x}^\bullet(t) &= e_n\,|\mathfrak{x}(t)| && \text{if } t \in \mathfrak{Z}_n\,, \quad n \geqq 1, \\ &= 0 && \text{if } t \in \mathfrak{Z}, \end{aligned}$$

starts afresh at each *constant* time $t \geqq 0$; in addition, its Green operators decompose as in 5, and evaluating $(G^\bullet_\alpha f)(0)$ as [31]

11. $$\begin{aligned} (G^\bullet_\alpha f)(0) &= \sum_{n \geqq 1} E_0\left(\int_{\mathfrak{Z}_n} e^{-\alpha t} f(e_n\,|\mathfrak{x}|)\,dt\right) \\ &= \sum_{n \geqq 1}\left(\frac{p_{-2}}{p_{-2}+p_{+2}}\,E_0\left[\int_{\mathfrak{Z}_n} e^{-\alpha t} f(-|\mathfrak{x}|)\,dt\right]\right. \\ &\qquad\qquad \left. + \frac{p_{+2}}{p_{-2}+p_{+2}}\,E_0\left[\int_{\mathfrak{Z}_n} e^{-\alpha t} f(+|\mathfrak{x}|)\,dt\right]\right) \\ &= \frac{p_{-2}}{p_{-2}+p_{+2}}\,E_0\left[\int_0^{+\infty} e^{-\alpha t} f(-|\mathfrak{x}|)\,dt\right] \\ &\qquad\qquad + \frac{p_{+2}}{p_{-2}+p_{+2}}\,E_0\left[\int_0^{+\infty} e^{-\alpha t} f(+|\mathfrak{x}|)\,dt\right] \\ &= \frac{2p_{-2}\int^{0-} e^{-(2\alpha)^{1/2} l} f(l)\,dl + 2p_{+2}\int_{0+} e^{-(2\alpha)^{1/2} l} f(l)\,dl}{(2\alpha)^{1/2}(p_{-2}+p_{+2})} \\ &= \frac{-p_{-2}(G^-_\alpha f)^-(0) + p_{+2}(G^-_\alpha f)^+(0)}{(2\alpha)^{1/2}(p_{-2}+p_{+2})}, \end{aligned}$$

[31] $|\,\mathfrak{Z}\,| = 0$.

one identifies 10 as the Brownian motion associated with 3 in the special case $p_1 = p_3 = p_4 = 0$ $(p_{-2}\,u^-(0) = p_{+2}\,u^+(0))$.

Coming to the case $p_1 = p_{\pm 2} = p_3 = 0$ $(p_4(R^1 - 0) = +\infty)$, if $\mathfrak{p}(dt \times dl)$ is a Poisson measure with mean $dt\,p_4(dl)$ independent of the standard Brownian motion $\mathfrak{x}$, if $[l_1^-, l_1^+), [l_2^-, l_2^+)$, etc. are the flat stretches of the inverse function $\mathfrak{p}^{-1}$ of $\mathfrak{p}(t) = \int |l|\,\mathfrak{p}([0, t] \times dl)$, and if $\mathfrak{t}^+$ is the local time at 0 of the (independent) reflecting Brownian motion $\mathfrak{x}^+ = |\mathfrak{x}|$, then the desired motion is

$$
\begin{aligned}
12. \qquad \mathfrak{x}^\bullet(t) &= \mathfrak{x}(t) && \text{if} \quad t < \mathfrak{m}_0 = \min(t : \mathfrak{x} = 0), \\
&= \pm[\mathfrak{p}\mathfrak{p}^{-1}\mathfrak{t}^+ - \mathfrak{t}^+ + \mathfrak{x}^+] && \text{if} \quad t \in \mathfrak{Q}^+, \\
&= 0 && \text{if} \quad \mathfrak{m}_0 \leqq t \notin \mathfrak{Q}^+,
\end{aligned}
$$

where $\mathfrak{Q}^+ = \cup_{n \geq 1} [\mathfrak{t}^{-1}(l_n^-), \mathfrak{t}^{-1}(l_n^-))$, and the ambiguous sign in the second line is *positive* during the interval $[\mathfrak{t}^{-1}(l_n^-), \mathfrak{t}^{-1}(l_n^+))$ if $l_n = l_n^+ - l_n^-$ is a jump of $\mathfrak{p}(dt \times dl \cap (0, +\infty])$ and *negative* otherwise (see Diagram 2).

Granting that 12 is simple Markov (the proof is left to the reader), it is enough for its identification to evaluate[32]

$$
\begin{aligned}
13. \quad (G_\alpha^\bullet f)(0) &= \sum_{n \geqq 1} E_0\left(\int_{\mathfrak{t}^{-1}(l_n^-)}^{\mathfrak{t}^{-1}(l_n^+)} e^{-\alpha t} f[\pm(l_n^+ - \mathfrak{t}^+ + \mathfrak{x}^+)]\,dt\right) \\
&= \sum_{n \geqq 1} E_0[e^{-(2\alpha)^{1/2} l_n^-}(G_\alpha^- f)(\pm l_n)] \\
&= E_0\left[\int_0^{+\infty} \int_{R^1 - 0} \mathfrak{p}(dt \times dl) e^{-(2\alpha)^{1/2}\mathfrak{p}(t-)}(G_\alpha^- f)(l)\right] \\
&= \int_{|l|>0} (G_\alpha^- f)(l) p_4(dl) \Big/ \int_{|l|>0} (1 - e^{-(2\alpha)^{1/2} l}) p_4(dl)
\end{aligned}
$$

with the aid of the tricks developed in Section 15.

Coming to the case $p_1 = p_3 = 0$, it suffices to combine the special cases $p_1 = p_3 = p_4 = 0$ and $p_1 = p_{\pm 2} = p_3 = 0$ as follows.

Given $\mathfrak{p}(dt \times dl)$, $\mathfrak{x}$, and $\mathfrak{t}^+$ as above, if $\mathfrak{x}_2^\bullet$ is the skew Brownian motion based upon $p_{\pm 2}$ and $\mathfrak{x}$, if $\mathfrak{x}_4^\bullet$ is the motion of 12 based upon $\mathfrak{p}^\bullet(t) = \int |l|\,\mathfrak{p}([0, t] \times dl)$ and $\mathfrak{x}$, if $[l_1^-, l_1^+), [l_2^-, l_2^+)$, etc. are the flat stretches of the inverse function of $\mathfrak{p} = p_2 t + \mathfrak{p}^\bullet$ $(p_2 = p_{-2} + p_{+2})$, and if $\mathfrak{Q}^+ = \cup_{n \geq 1} [\mathfrak{t}^{-1}(l_n^-), t^{-1}(l_n^+))$, then the desired motion is

$$
\begin{aligned}
14. \qquad \mathfrak{x}^\bullet(t) &= \mathfrak{x}(t) && \text{if} \quad t < \mathfrak{m}_0, \\
&= \mathfrak{x}_4^\bullet(t^\bullet) && \text{if} \quad t \in \mathfrak{Q}^+, t^\bullet = |\mathfrak{Q}^+ \cap [0, t)|, \\
&= \mathfrak{x}_2^\bullet(t) && \text{if} \quad t \in [\mathfrak{m}_0, +\infty) - \mathfrak{Q}^+;
\end{aligned}
$$

the reader will check that this sample path starts afresh at each *constant*

---

[32] $|\,[0, +\infty) - \mathfrak{Q}^+\,| = 0$ because $\mathfrak{p}(t)$ has no linear part $(p_2 t)$.

time $t \geqq 0$ and will complete its identification with the aid of

$$15.\quad (G_\alpha^\bullet f)(0) = E_0\left[\int_0^{+\infty} e^{-\alpha t} f(\mathfrak{x}^\bullet)\,dt\right]$$

$$= \sum_{n\geqq 1} E_0\left[\int_{\mathfrak{t}^{-1}(l_n^-)}^{\mathfrak{t}^{-1}(l_n^+)} e^{-\alpha t} f[\mathfrak{x}_4^\bullet(t)]\,dt\right]$$

$$+ E_0\left[\int_0^{+\infty} e^{-\alpha t} f(\mathfrak{x}_2^\bullet)\,dt\right] - \sum_{n\geqq 1} E_0\left[\int_{\mathfrak{t}^{-1}(l_n^-)}^{\mathfrak{t}^{-1}(l_n^+)} e^{-\alpha t} f(\mathfrak{x}_2^\bullet)\,dt\right]$$

$$= \sum_{n\geqq 1} E_0[e^{-\alpha \mathfrak{t}^{-1}(l_n^-)}(G_\alpha^- f)(\pm l_n)]$$

$$+ E_0\left[\int_0^{+\infty} e^{-\alpha t} f(\mathfrak{x}_2^\bullet)\,dt\right]\left(1 - \sum_{n\geqq 1} E_0[e^{-\alpha \mathfrak{t}^{-1}(l_n^-)} - e^{-\alpha \mathfrak{t}^{-1}(l_n^+)}]\right)$$

$$= E_0\left[\int_0^{+\infty}\int_{R^1-0} \mathfrak{p}(dt\times dl)e^{-(2\alpha)^{1/2}\mathfrak{p}(t-)}(G_\alpha^- f)(l)\right]$$

$$+ E_0\left[\int_0^{+\infty} e^{-\alpha t} f(\mathfrak{x}_2^\bullet)\,dt\right]$$

$$\times\left(1 - E_0\left[\int_0^{+\infty}\int_{R^1-0} \mathfrak{p}(dt\times dl)e^{-(2\alpha)^{1/2}\mathfrak{p}(t-)}e^{-(2\alpha)^{1/2}|l|}\right]\right)$$

$$= \frac{-p_{-2}(G_\alpha^- f)^-(0) + p_{+2}(G_\alpha^- f)^+(0) + \int (G_\alpha^- f)(l)p_4(dl)}{(2\alpha)^{1/2}p_2 + \int (1 - e^{-(2\alpha)^{1/2}|l|})p_4(dl)}.$$

If $p_3 > 0 = p_1$, it is clear that the desired motion is the sample path $\mathfrak{x}^\bullet$ of 14 run with the stochastic clock $\mathfrak{f}^{-1}$ inverse to $\mathfrak{f} = t + p_3\,\mathfrak{p}^{-1}\mathfrak{t}^+$ (see Section 14 for the interpretation of $\mathfrak{p}^{-1}\mathfrak{t}^+$ as a local time), while, if $p_1 > 0$ also, the motion $\mathfrak{x}^\bullet(\mathfrak{f}^{-1})$ has to be annihilated at time $\mathfrak{m}_\infty^\bullet$ with conditional law

$$16.\qquad P_\cdot(\mathfrak{m}_\infty^\bullet > t \mid \mathfrak{x}^\bullet(\mathfrak{f}^{-1})) = e^{-p_1\mathfrak{p}^{-1}\mathfrak{t}^+\mathfrak{f}^{-1}(t)}.$$

The reader is invited to furnish the proofs.

Brownian motions with the same kind of two-sided barrier can be defined on the unit circle $S^1 = [0, 1)$ as W. Feller [1], [3] pointed out.

Given a standard Brownian motion on R^1, its projection onto[33] $S^1 = R^1/Z^1$ is the so-called *standard circular Brownian motion*; its generator is the contraction of $\mathfrak{G} = D^2/2$ to $C^2(S^1)$.

Consider now the general circular Brownian motion with a two-sided barrier at $l = 0$ (i.e., the obvious circular analogue of a Brownian motion with two-sided barrier on R^1), and, as before, single out the case

$$17.\qquad P_\cdot^\bullet[\mathfrak{x}^\bullet(\mathfrak{e}^\bullet) = 0] = 1, \qquad \mathfrak{e}^\bullet = \lim_{\varepsilon\downarrow 0}\inf(t: |\mathfrak{x}^\bullet| < \varepsilon).$$

[33] Z^1 is the integers.

$\mathfrak{G}^\bullet$ is the contraction of $\mathfrak{G} = D_2/2$ to[34]

$$18.\quad D(\mathfrak{G}^\bullet) = C^{\bullet 2}(S^1) \cap \Big(u: p_1 u(0) + p_{-2}u^-(0) - p_{+2}u^+(0) + p_3(\mathfrak{G}u)(0\pm) = \int [u(l) - u(0)]p_4(dl)\Big)$$

for some nonnegative numbers p_1, $p_{\pm 2}$, p_3 and some nonnegative mass distribution $p_4(dl)$ subject to

$$19a.\quad p_1 + p_{-2} + p_{+2} + p_3 + \int_0^1 l(1-l)p_4(dl) = 1, \quad p_4(0) = p_4(1) = 0,$$

$$19b.\quad p_4(S^1) = +\infty \quad \text{in case} \quad p_{\pm 2} = p_3 = 0,$$

and an application of 18 to

$$20a.\quad (G_\alpha^\bullet f)(l) = (G_\alpha^- f)(l) + \frac{\sinh (2\alpha)^{1/2} l + \sinh (2\alpha)^{1/2}(1-l)}{\sinh (2\alpha)^{1/2}} (G_\alpha^\bullet f)(0), \qquad 0 \leqq l < 1,$$

$$20b.\quad (G_\alpha^- f)(a) = 2\int_0^1 G(a,b)f(b)\,db, \qquad 0 \leqq a < 1,$$

$$20c.\quad G(a,b) = G(b,a) = \frac{\sinh (2\alpha)^{1/2} a \sinh (2\alpha)^{1/2}(1-b)}{(2\alpha)^{1/2}\sinh (2\alpha)^{1/2}}, \qquad 0 \leqq a \leqq b < 1,$$

establishes the formula

$$21.\quad (G_\alpha^\bullet f)(0) = \left[2p_{-2}\int_0^1 \frac{\sinh (2\alpha)^{1/2}(1-l)}{\sinh (2\alpha)^{1/2}} f(l)\,dl + 2p_{+2}\int_0^1 \frac{\sinh (2\alpha)^{1/2} l}{\sinh (2\alpha)^{1/2}} f(l)\,dl + p_3 f(0) + \int_0^1 (G_\alpha^- f)(l)p_4(dl)\right] \Big/ \left[p_1 + (2\alpha)^{1/2}\frac{\cosh (2\alpha)^{1/2} - 1}{\sinh (2\alpha)^{1/2}}(p_{-2} + p_{+2}) + \alpha p_3 + \int_0^1 \left(1 - \frac{\sinh (2\alpha)^{1/2} l + \sinh (2\alpha)^{1/2}(1-l)}{\sinh (2\alpha)^{1/2}}\right) p_4(dl)\right].$$

Given a standard circular Brownian motion $\mathfrak{x}$ with local time

$$22.\quad \mathfrak{t}(t) = \lim_{\varepsilon \downarrow 0}(2\varepsilon)^{-1} \text{measure}\,(s: |\mathfrak{x}(s)| < \varepsilon,\ s \leqq t)$$

and a (circular) differential process $\mathfrak{p}$ based on $p_{\pm 2}$ and p_4, it is possible to build up the circular Brownian sample paths as in the linear case, but a second method suggests itself: *the method of images.*

Consider for this purpose a Brownian motion on R^1 with two-sided barriers at the integers having as its generator the contraction of $\mathfrak{G} = D^2/2$ to the class of functions $u \in C(R^1) \cap C^2(R^1 - Z^1)$ such that

[34] $C^{\bullet 2}(S^1) = C(S^1) \cap C^2(S^1 - 0) \cap (u: u''(0-) = u''(0+))$.

23a. $$(\mathfrak{G}u)(n-) = (\mathfrak{G}u)(n+),$$

23b. $$p_1 u(n) + p_{-2} u^-(n) - p_{+2} u^+(n) + p_3(\mathfrak{G}u)(n\pm) = \int_0^1 [u(l+n) - u(n)] p_4(dl)$$

at each integer $n = 0, \pm 1, \pm 2$, etc. (the reader is invited to build up the sample paths for himself). Because the barriers are periodic, the projection of this motion onto $S^1 = R^1/Z^1$ is (simple) Markov, and its identification as the desired circular Brownian motion is immediate.

18. Simple Brownian motions

Given a *simple Brownian motion* on $[0, +\infty)$, described as in Section 5 except that *it need not start afresh at nonconstant stopping times*,

1. $$(G_\alpha^\bullet f)(l) = (G_\alpha^- f)(l) + e^{-(2\alpha)^{1/2} l}(G_\alpha^\bullet f)(0+), \qquad l > 0,$$

as will now be proved with a view to the classification of all such Brownian motions.

Given $\alpha > 0$, a nonnegative Borel function f, and $t_2 \geqq t_1 \geqq 0$,

2. $$\begin{aligned} E_\cdot^\bullet[e^{-\alpha t_2}(G_\alpha^\bullet f)(\mathfrak{x}^\bullet(t_2)) \mid \mathsf{B}_{t_1}^\bullet] &= e^{-\alpha t_2} E_l^\bullet[(G_\alpha^\bullet f)(\mathfrak{x}^\bullet(t))], \qquad l = \mathfrak{x}^\bullet(t_1), t = t_2 - t_1, \\ &= e^{-\alpha t_2} \int_0^{+\infty} e^{-\alpha s}\, ds\, E_l^\bullet(E_{\mathfrak{x}^\bullet(t)}^\bullet[f(\mathfrak{x}^\bullet(s))]) \\ &= e^{-\alpha t_2} \int_0^{+\infty} e^{-\alpha s}\, ds\, E_l^\bullet[f(\mathfrak{x}^\bullet(t+s))] \\ &= e^{-\alpha t_1} \int_t^{+\infty} e^{-\alpha s}\, ds\, E_l^\bullet[f(\mathfrak{x}^\bullet(s))] \\ &\leqq e^{-\alpha t_1}(G_\alpha^\bullet f)(\mathfrak{x}^\bullet(t_1)), \end{aligned}$$

i.e., $e^{-\alpha t}(G_\alpha^\bullet f)(\mathfrak{x}^\bullet)$ is a (nonnegative) *supermartingale*; as such, it possesses one-sided limits as[35] $t = k2^{-n} \downarrow s$ $(s \geqq 0)$, and it follows that if $l > \varepsilon > 0$ and if $\mathfrak{m}^\bullet$ is the crossing time $\inf(t: \mathfrak{x}^\bullet < \varepsilon)$, then

3. $$\begin{aligned} (G_\alpha^\bullet f)(l) &= E_l^\bullet\left[\int_0^{\mathfrak{m}^\bullet} e^{-\alpha t} f(\mathfrak{x}^\bullet)\, dt\right] \\ &\quad + \lim_{n\uparrow+\infty} \textstyle\sum_{k\geqq 0} E_l^\bullet\Big[(k-1)2^{-n} \leqq \mathfrak{m}^\bullet < k2^{-n}, \\ &\qquad e^{-\alpha k 2^{-n}} \displaystyle\int_0^{\mathfrak{m}_\infty^\bullet(w_{k2}^+{}^{-n})} e^{-\alpha t} f(\mathfrak{x}^\bullet(t + k2^{-n}))\, dt\Big] \\ &= E_l^\bullet\left[\int_0^{\mathfrak{m}^\bullet} e^{-\alpha t} f(\mathfrak{x}^\bullet)\, dt\right] \\ &\quad + \lim_{n\uparrow+\infty} \textstyle\sum_{k\geqq 0} E_l^\bullet[(k-1)2^{-n} \leqq \mathfrak{m}^\bullet < k2^{-n}, \\ &\qquad e^{-\alpha k 2^{-n}}(G_\alpha^\bullet f)(\mathfrak{x}^\bullet(k2^{-n}))] \end{aligned}$$

[35] See J. L. Doob [1].

$$= E_l\left[\int_0^{\mathfrak{m}} e^{-\alpha t} f(\mathfrak{x})\,dt\right] + E_l[e^{-\alpha\mathfrak{m}} \lim_{k2^{-n}\downarrow\mathfrak{m}} (G_\alpha^\bullet f)(\mathfrak{x}(k2^{-n}))],$$

where $\mathfrak{x}$ is a standard Brownian motion, $E.$ its expectation, and $\mathfrak{m}$ its passage time $\min(t:\mathfrak{x} = \varepsilon)$.

But, in the standard Brownian case, $\lim_{k2^{-n}\downarrow\mathfrak{m}}(G_\alpha^\bullet f)(\mathfrak{x}(k2^{-n}))$ is measurable over $\mathsf{B}_{\mathfrak{m}+}$ and also independent of $\mathsf{B}_{\mathfrak{m}+}$ (i.e., it is measurable over $\mathsf{B}[\mathfrak{x}(t + \mathfrak{m}):t \geqq 0]$ which is independent of $\mathsf{B}_{\mathfrak{m}+}$ conditional on the *constant* $\mathfrak{x}(\mathfrak{m}) = \varepsilon$); as such, it is constant, and inserting this information back into 3 and letting $\varepsilon \downarrow 0$ establishes

4. $$(G_\alpha^\bullet f)(l) = (G_\alpha^- f)(l) + e^{-(2\alpha)^{1/2} l} \times \text{constant},$$

which implies the existence of $(G_\alpha^\bullet f)(0+)$ and leads at once to 1.

Given a bounded function f on $[0, +\infty)$, continuous apart from a possible jump at $l = 0$, define a new function $\hat{f}$ on $(-1) \cup [0, +\infty)$ as

5. $$\begin{aligned} \hat{f}(l) &= f(0) && \text{if } l = -1,\\ &= f(0+) && \text{if } l = 0,\\ &= f(l) && \text{if } l > 0, \end{aligned}$$

and introduce the new Green operators

6. $$\hat{G}_\alpha \hat{f} = (G_\alpha^\bullet f)^\wedge$$

mapping $C((-1) \cup [0, +\infty))$ into itself.

$\hat{G}_\alpha$ is the Green operator of a *strict* Markov motion on $(-1) \cup [0, +\infty)$ with sample paths $t \to \hat{\mathfrak{x}}(t) = \hat{\mathfrak{x}}(t+) \in (-1) \cup [0, +\infty) \cup \infty$, and $\mathfrak{x}^\bullet$ is identical in law to the projection of $\hat{\mathfrak{x}}$ under the identification $-1 \to 0$, as the reader can check for himself or deduce from the general embedding of D. Ray [1].

One now computes the domain $D(\hat{\mathfrak{G}})$ of the generator $\hat{\mathfrak{G}}$ of this *covering motion* and finds that it is the class of functions

$$u \in C((-1) \cap [0, +\infty)) \cup C^2[0, +\infty)$$

subject to

7a. $$-p_{+2}\, u^+(0) + p_{+3}(\hat{\mathfrak{G}} u)(0) = \int_{(-1)\cup(0,+\infty)\cup\infty} [u(l) - u(0)] p_{+4}(dl),$$

$$p_{+4}(0) = 0 \leqq p_{+2}\,, p_{+3}\,, p_{+4}(dl)$$

$$p_{+2} + p_{+3} + p_{+4}(-1) + \int_{0+} (l \wedge 1) p_{+4}(dl) + p_{+4}(\infty) = 1,$$

7b. $$p_{-3}(\hat{\mathfrak{G}} u)(-1) = \int_{[0,+\infty)\cup\infty} [u(l) - u(-1)] p_{-4}(dl),$$

$$p_{-4}(-1) = 0 \leqq p_{-3}\,, p_{-4}(dl),$$

$$p_{-3} + p_{-4}[0, +\infty) + p_{-4}(\infty) = 1,$$

where $u(\infty) \equiv 0$.

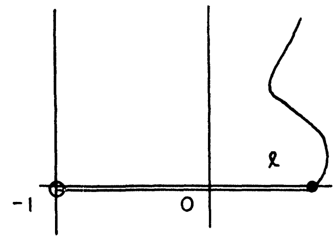


DIAGRAM 1

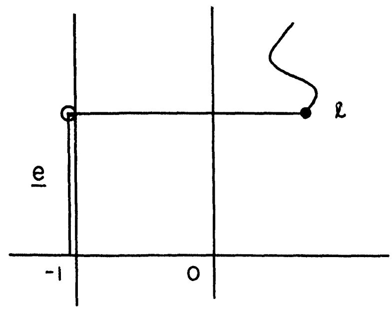


DIAGRAM 2

If $p_{-3} = 0$, the motion starting at -1 begins with a jump $l \in [0, +\infty) \cup \infty$ with law $p_{-4}(dl)$ as in Diagram 1, $u(-1) = \int_{[0,+\infty)} u(l)p_{-4}(dl)$, and 7a goes over into

8a. $$p^\bullet_1\, u(0) - p^\bullet_2\, u^+(0) + p^\bullet_3(\mathfrak{G}u)(0) = \int_{0+} [u(l) - u(0)]p^\bullet_4(dl),$$

8b. $$p^\bullet_1 = p_{+4}(\infty) + p_{+4}(-1)p_{-4}(\infty),$$
$$p^\bullet_2 = p_{+2}\,, \qquad p^\bullet_3 = p_{+3}\,,$$
$$p^\bullet_4(dl) = p_{+4}(dl) + p_{+4}(-1)p_{-4}(dl), \qquad l > 0,$$

i.e., the covering motion *does not land at* -1 which is a superfluous state, and $\hat{\mathfrak{x}} = \mathfrak{x}^\bullet$ is a *strict Brownian motion on* $[0, +\infty)$ as in Sections 5–16.

If $p_{-3} > 0$, then $\widehat{\mathfrak{G}}$ is the contraction of $\mathfrak{G} = D^2/2$ to $D(\widehat{\mathfrak{G}})$ with the added specification

9. $$(\widehat{\mathfrak{G}}u)(-1) = \int_{[0,+\infty)} [u(l) - u(-1)]\, \frac{p_{-4}(dl)}{p_{-3}}\,, \qquad u(\infty) \equiv 0,$$

at -1, and the particle starting at -1 waits there for an exponential holding time $\mathfrak{e}$ with law $e^{-p_{-4}t/p_{-3}}$ ($p_{-4} = p_{-4}([0, +\infty) \cup \infty)$), and then jumps to $l \in [0, +\infty) \cup \infty$ with law $p_{-4}(dl)/p_{-4}$ as in Diagram 2.

If, in addition to $p_{-3} > 0$, one has $p_2 = 0$ and $p_4(0, +\infty) < +\infty$, then the motion starting at 0 is of the same kind, and it is clear that the projection of this motion down to $[0, +\infty)$ $(-1 \to 0)$ cannot even be *simple* Markov unless $p_{-3} = p_{+3}$ and $p_{-4}(dl) = p_{+4}(dl)$ $(l \neq 0)$ up to a common multiplicative constant, in which case the projection is the Brownian motion associated with

9a. $$p_1\, u(0) + p_{+3}(\mathfrak{G}u)(0) = \int_{0+} [u(l) - u(0)]p_{+4}(dl),$$

9b. $$p_1 = p_{+4}(\infty)$$

studied in Section 9.

If $p_{-3} > 0$ and either $p_{+2} > 0$ or $p_{+4}(0, +\infty) = +\infty$, the particle starting at -1 waits for an exponential holding time $\mathfrak{e}_1$ and then jumps as in Diagram

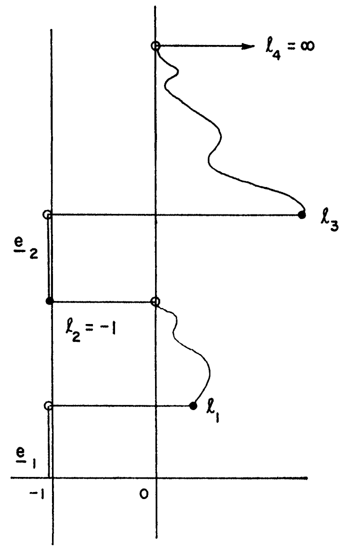


DIAGRAM 3

3 to $l_1 \in [0, +\infty) \cup \infty$ and starts afresh; if $0 \leqq l_1 < +\infty$, the particle performs the Brownian motion on $[0, +\infty)$ associated with

$$\text{10a.} \quad p_1 u(0) - p_{+2} u^+(0) + p_{+3}(\mathfrak{G}u)(0) = \int_{0+} [u(l) - u(0)] p_{+4}(dl),$$

$$\text{10b.} \quad p_1 = p_{+4}(-1 \cup \infty)$$

up to the killing time of that motion, at which instant it jumps to $l_2 = \infty$ or -1 with probabilities $p_{+4}(\infty) : p_{+4}(-1)$, and, if $l_2 = -1$, it starts afresh as in Diagram 3, while if $l_2 = \infty$, then the motion rests at that place at all later times.

Now the projection $\mathfrak{x}^\bullet$ of this motion onto $[0, +\infty)$ $(-1 \to 0)$ is simple Markov if the Brownian motion attached to 10 does not spend positive (Lebesgue) time at $l = 0$; otherwise the knowledge that $\mathfrak{x}^\bullet(s) = 0$ is not sufficient to discriminate between the two possible coverings, and the law of $\mathfrak{x}^\bullet(t) : t \geqq s$ is moot. But if e is the indicator of $l = 0$, and if

$$\text{11.} \quad \mathfrak{x}^\bullet(\mathfrak{f}^{-1}) \quad (t < \mathfrak{m}^\bullet_\infty), \qquad \infty \quad (t \geqq \mathfrak{m}^\bullet_\infty)$$

$$\text{12a.} \quad \mathfrak{f} = t + p_3 \mathfrak{p}^{-1} \mathfrak{t}^+,$$

$$\text{12b.} \quad \mathfrak{x}^\bullet = \mathfrak{p}\mathfrak{p}^{-1}\mathfrak{t}^+ - \mathfrak{t}^+ + \mathfrak{x}^+$$

is the motion attached to 10, then, in the notation of Section 14,

13. $$\text{measure}\,(s:\mathfrak{x}^\bullet(\mathfrak{f}^{-1}) = 0, s \leqq t) = \int_0^t e[\mathfrak{x}^\bullet(\mathfrak{f}^{-1})]\,ds$$
$$= \int_0^{\mathfrak{f}^{-1}(t)} e(\mathfrak{x}^\bullet)\mathfrak{f}\,(ds)$$
$$= p_{+3}\int_{\mathfrak{Z}^+\cap\mathfrak{Q}^+\cap[0,\mathfrak{f}^{-1}(t))} \mathfrak{p}^{-1}\mathfrak{t}^+\,(dt)$$
$$= p_{+3}\mathfrak{p}^{-1}\mathfrak{t}^+[\mathfrak{Q}^+ \cap [0, \mathfrak{f}^{-1}(t))], \qquad t \leqq \mathfrak{m}_\infty^\bullet,$$

and this cannot be positive unless $p_{+3} > 0$ and $0 < \mathfrak{t}^+(\mathfrak{Q}^+) = |\mathfrak{Q}|$, i.e., unless $p_{+2} > 0$ also; in short, *the projection is simple Markov unless* $p_{+2}\,p_{+3} > 0$, and now the classification is complete.

N. Ikeda had conjectured part of our classification (private communication); the case of a two-sided barrier on R^1 is similar except that three covering points lie over 0.

19. Feller's differential operators

Given a nonnegative mass distribution e on the open half line $(0, +\infty)$ with $0 < e(a, b]$ $(a < b)$, let $D(\mathfrak{G})$ be the class of functions $u \in C[0, +\infty)$ such that

1. $$u^+(b) - u^+(a) = \int_{(a,b]} f\,de, \qquad a < b,$$

for some $f \in C[0, +\infty)$, and introduce the differential operator $\mathfrak{G}u = f$:

2. $$(\mathfrak{G}u)(a) = \lim_{b \downarrow a} \frac{u^+(b) - u^+(a)}{e(a, b]}.$$

W. Feller [3] proved that if $e(0, 1] < +\infty$, and if $p_1, p_2, p_3, p_4(dl)$ are nonnegative with $p_4(0) = 0$ and $p_1 + p_2 + p_3 + \int_{0+} (l \wedge 1)p_4(dl) = 1$, then the contraction $\mathfrak{G}^\bullet$ of $\mathfrak{G}$ to

3. $$D(\mathfrak{G}^\bullet) = D(\mathfrak{G}) \cap \left(u: p_1 u(0) - p_2 u^+(0) + p_3(\mathfrak{G}u)(0) = \int_{0+} [u(l) - u(0)]p_4(dl)\right)$$

is the generator of a strict Markov motion (diffusion) on $[0, +\infty)$.

Given a reflecting Brownian motion $\mathfrak{x}^+$ on $[0, +\infty)$, the *local time*

4. $$\mathfrak{t}^+(t, l) = (\text{measure}\,(s:\mathfrak{x}^+(s) \in dl, s < t))/2\,dl$$

is continuous in the pair $(t, l) \in [0, +\infty)^2$ (see H. Trotter [1]), and the motion associated with $\mathfrak{G}^\bullet$ in the special case $p_1 = p_3 = p_4 = 0$ ($u^+(0) = 0$ is identical in law to $\mathfrak{x}^\bullet = \mathfrak{x}^+(\mathfrak{f}^{-1})$ where $\mathfrak{f} = \int_{0+} \mathfrak{t}^+(t, l)e(dl)$ (see V. A. Volkonskii [1] and K. Itô and H. P. McKean, Jr. [1]).

Because $\mathfrak{t}^+(dt, l) = 0$ outside $\mathfrak{Z} = (t: \mathfrak{x}^+ = l)$,

5. $$\int_0^t f(\mathfrak{x}^\bullet)\, ds = \int_0^{\mathfrak{f}^{-1}(t)} f(\mathfrak{x}^+) \int_{0+} \mathfrak{t}^+ (ds, l) e(dl)$$
$$= \int_{0+} \left(\int_0^{\mathfrak{f}^{-1}(t)} \mathfrak{t}^+ (ds, l) \right) f(l) e(dl)$$
$$= \int_{0+} \mathfrak{t}^+[\mathfrak{f}^{-1}(t), l] f(l) e(dl);$$

hence the *local time*

6. $$\mathfrak{t}^\bullet(t) = \lim_{\varepsilon \downarrow 0} e(0, \varepsilon]^{-1} \text{ measure } (s: \mathfrak{x}^\bullet(s) < \varepsilon, s \leqq t)$$
$$= \mathfrak{t}^+(\mathfrak{f}^{-1}, 0)$$

exists, and now it is clear that the discussion of the Brownian case can be adapted with little change.

20. Birth and death processes

Quite a general birth and death process on the nonnegative integers can be changed via a scale substitution into a motion on a discrete series $Q: 0 = l_0 < l_1 < l_2 < \cdots < 1$ having as its generator

1. $$\mathfrak{G}^\bullet u = (u^+ - u^-)/e,$$

2a. $$u^+(l_n) = u^-(l_{n+1}) = (l_{n+1} - l_n)^{-1} [u(l_{n+1}) - u(l_n)],$$

2b. $$e = e(l_n) > 0,$$

2c. $$e(l_0) + e(l_1) + \cdots < +\infty,$$

subject to

3a. $$u^+(0) = 0,$$

3b. $$p_1 u(1) + p_3(\mathfrak{G}^\bullet u)(1) = -p_2 u^-(1) + \int_Q [u(l) - u(1)] p_4(dl),$$
$$p_1 + p_2 + p_3 + \int_Q (1 - l) p_4(dl) = 1$$

(see W. Feller [4]). In the special case $p_1 = p_3 = p_4 = 0$ the corresponding motion is just the reflecting Brownian motion on [0, 1] run with the inverse function of $\mathfrak{f} = \int_Q \mathfrak{t}^+(t, l) e(dl)$, $\mathfrak{t}^+$ being the reflecting Brownian local time. Once this motion has been obtained, the general path can be built up using local times and differential processes as before.

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