ON LINDELÖF'S CONJECTURE CONCERNING THE RIEMANN ZETA-FUNCTION

Dedicated to Hans Rademacher on the occasion of his seventieth birthday

BY

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1. The conjecture of Lindelöf asserts that (with $s = \sigma + it$) for $\sigma \geqq \frac{1}{2}$, $| s - 1 | \geqq 1/10$, with an arbitrary small $\varepsilon > 0$, the inequality

$$(1.1) \qquad | \zeta(s)| \leqq d_1(2 + | t |)^{\varepsilon}$$

holds, where $\zeta(s)$ stands for the Riemann zeta-function and d_1 (and later $d_2, \cdots$) depends only upon ε. As is well known this is unproved, just as is Riemann's conjecture. As to the latter I found in 1943 the following theorem.[1]

For the existence of a ϑ with $\frac{1}{2} \leqq \vartheta < 1$ such that for an arbitrarily small $\eta > 0$ the half-plane $\sigma \geqq \vartheta + \eta$ contains only a finite number of zeros of $\zeta(s)$, the existence of positive numerical α and β with the following property is *necessary and sufficient*: For ($t > 0$ and)

$$(1.2) \qquad c_1 \leqq t^{\alpha} \leqq N \leqq N_1 < N_2 \leqq 2N,$$

the inequality

$$(1.3) \qquad \left| \sum_{N_1 \leqq p \leqq N_2} e^{it \log p} \right| < c_2 \frac{N \log^3 N}{t^{\beta}}$$

holds. Here (and also later) $c_1, c_2, \cdots$ stand for positive numerical constants, p for primes.

Having this theorem I was interested by a communication of U. V. Linnik (without exact formulation and proof) in 1947 or 1948 that an equivalence theorem concerning Lindelöf's conjecture can be established in terms of the sums

$$(1.4) \qquad \textstyle\sum_{N_1 \leqq n \leqq N_2} d(n) e^{it \log n}.$$

When I discussed this communication with A. Selberg in Princeton in 1948, he remarked, again without exact formulation and proof, that such an equivalence theorem can be given in terms of sums

$$(1.5) \qquad \textstyle\sum_{N_1 \leqq n \leqq N_2} e^{it \log n}$$

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[1] *On Riemann's hypothesis*, Izv. Akad. Nauk SSSR, vol. 11 (1947), pp. 197–262. A shorter proof is contained in my book, *Eine neue Methode in der Analysis und deren Anwendungen*; a completely rewritten new English edition will appear among the Interscience Tracts.

too. Having once this idea it was not difficult to reconstruct the theorem (at least with the factor $(-1)^n$ in the summand) and its rather straightforward proof. I was convinced that this theorem is a commonplace among the "ζ-lists"; the study of the excellent book of Titchmarsh on the Riemann zeta-function and the subsequent pertinent literature makes me now, after more than ten years, wonder whether I was right. Thus it seems worthwhile to publish this theorem, which runs as follows.

For the truth of Lindelöf's conjecture the truth of the inequality

$$| \textstyle\sum_{n \leqq N} (-1)^n e^{-it\log n} | < d_2 N^{1/2+\varepsilon}(2 + |t|)^\varepsilon \tag{1.6}$$

with an arbitrarily small $\varepsilon > 0$ is *necessary and sufficient.*

2. The sufficiency of (1.6) follows at once, since, if we put

$$S_N = \textstyle\sum_{n \leqq N} (-1)^n e^{-it\log n},$$

for $\frac{1}{2} + 2\varepsilon \leqq \sigma \leqq 2$ we have

$$\left|\left(1 - \frac{2}{2^s}\right)\zeta(s)\right| = \left|\sum_{N=1}^{\infty} S_N\left(\frac{1}{N^\sigma} - \frac{1}{(N+1)^\sigma}\right)\right|$$

$$< 6d_2(2 + |t|)^\varepsilon \sum_{N=1}^{\infty} N^{\varepsilon-\sigma-1/2} < d_3(2 + |t|)^\varepsilon.$$

From this one gets (1.1) quite easily.

3. As to the necessity we suppose the truth of (1.1). Putting

$$f(s) = \left(1 - \frac{2}{2^s}\right)\zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s},$$

we get for $\sigma \geqq \frac{1}{2}$ by (1.1)

$$|f(s)| \leqq d_4(2 + |t|)^{\varepsilon/2}. \tag{3.1}$$

Now we consider, with the integer $N \geqq 2$ and $w = u + iv$, the integral

$$J_N(t) = \frac{1}{2\pi i}\int_{1+1/\log N-iN}^{1+1/\log N+iN} \frac{(N+\frac{1}{2})^w}{w} f(w + it)\, dw. \tag{3.2}$$

Replacing $f(w + it)$ by its Dirichlet-series we get

$$J_N(t) = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-it} \frac{1}{2\pi i}\int_{1+1/\log N-iN}^{1+1/\log N+iN} \frac{((N+\frac{1}{2})/n)^w}{w}\, dw$$

$$= \sum_{n=1}^{N} (-1)^{n+1} e^{-it\log n} \tag{3.3}$$

$$- \sum_{n=1}^{\infty} (-1)^{n+1} n^{-it} \frac{1}{2\pi i}\left\{\int_{1+1/\log N-i\infty}^{1+1/\log N-iN} + \int_{1+1/\log N+iN}^{1+1/\log N+i\infty}\right\},$$

owing to the well-known formula

$$\frac{1}{2\pi i}\int_{(1+1/\log N)} \frac{x^w}{w}\,dw = \begin{cases} 1 & \text{for } x > 1, \\ 0 & \text{for } 0 < x < 1. \end{cases}$$

Hence partial integration gives from (3.3)

$$\left| \sum_{n=1}^{N} (-1)^{n+1} e^{-it\log n} - J_N(t) \right| \tag{3.4}$$
$$< c_3 \sum_{n=1}^{\infty} \left(n^{1+1/\log N} \left| \log \frac{N+\frac{1}{2}}{n} \right| \right)^{-1} < c_4 \log N.$$

(The inequality (3.4) could also be deduced from Lemma 3.12 of Titchmarsh's book.) Applying Cauchy's theorem to the rectangle with vertices

$$1 + 1/\log N \pm iN, \qquad \tfrac{1}{2} \pm iN,$$

we get

$$J_N(t) = (1/2\pi i)(J_N^{(1} + J_N^{(2)} + J_N^{(3)}),$$

where (with the same integrand as in (3.2))

$$J_N^{(1)} = \int_{1+1/\log N - iN}^{1/2-iN}, \qquad J_N^{(2)} = \int_{1/2-iN}^{1/2+iN}, \qquad J_N^{(3)} = \int_{1/2+iN}^{1+1/\log N+iN}.$$

Using (3.1) we get easily

$$| J_N^{(3)} | < \frac{d_5}{N}\int_{1/2}^{1+1/\log N} (N + \tfrac{1}{2})^u (2 + |t| + N)^{\varepsilon/2}\,du \tag{3.5}$$
$$< d_6(2 + |t| + N)^{\varepsilon/2} < d_6(2 + |t|)^{\varepsilon} N^{\varepsilon},$$

and the same for $J_N^{(1)}$. Finally from (3.1)

$$| J_N^{(2)} | < (N + \tfrac{1}{2})^{1/2} \int_{-N}^{N} \frac{1}{\sqrt{\frac{1}{4} + \vartheta^2}}\, d_4(2 + |\vartheta + t|)^{\varepsilon/2}\,d\vartheta$$
$$< d_7 N^{1/2+\varepsilon/2}(2 + |t|)^{\varepsilon/2} \log N < d_8 N^{1/2+\varepsilon}(2 + |t|)^{\varepsilon}.$$

In view of this, (3.4), and (3.5), the necessity of (1.6) is also proved.

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