A CLASS OF MULTIPLICATIVE LINEAR FUNCTIONALS ON THE MEASURE ALGEBRA OF A LOCALLY COMPACT ABELIAN GROUP

BY

EDWIN HEWITT AND SHIZUO KAKUTANI[1]

1. Introduction

1.1 *Group preliminaries.* Throughout this paper, "group" means "locally compact nondiscrete Abelian group"[2] unless the contrary is explicitly stated, and G, with elements x, y, u, $\cdots$, will denote such a group. The group of continuous characters of G (taken as mappings into the multiplicative group of complex numbers of absolute value 1) will be denoted by X, and elements of X will be denoted by χ, ψ, $\cdots$. The word "character" will mean "continuous character" unless the contrary is specified. For an integer $n > 1$, G^n will denote the Cartesian product of G with itself n times. Let R denote the additive group of real numbers, T the multiplicative group $\{\exp(2\pi i\theta)\}_{0 \leqq \theta < 1}$, Z the group of all integers, and K the field of complex numbers. The group operation in all groups considered will be written as addition, except for T and T^n. For an integer $b > 1$, the additive group of integers modulo b will be denoted by $Z(b)$, and the complete direct sum of groups $Z(b_\iota)$, $\iota \in I$, by $P_{\iota \in I} Z(b_\iota)$. In the special case where $I = \{1, 2, 3, \cdots\}$ and all b_ι have a single value a, we write D_a for this group.

For subsets A and B of G, let $A + B$ be the vector sum of A and B, that is, the set $\{x + y : x \in A, y \in B\}$. We write nA for $A + A + \cdots + A$ (n times), for $n = 2, 3, \cdots$. We write $-A$ for the set $\{-x : x \in A\}$. If $A = \{x\}$ for $x \in G$, we write $A + B$ as $x + B$.

1.2 *Measure-theoretic preliminaries.*[3] We shall be concerned with the algebra $\mathfrak{M}(G)$ of all complex-valued, bounded, countably additive, regular Borel measures on G, with setwise linear operations and multiplication of two measures λ and μ in $\mathfrak{M}(G)$ defined by convolution:

1.2.1
$$\lambda * \mu(E) = \int_G \lambda(E - x)\, d\mu(x)$$

for all Borel sets E in G.[4] The following evident fact will be useful. For a Borel set $E \subset G$ and an integer $n > 1$, let $E^{(n)}$ be the subset of G^n defined by $E^{(n)} = \{(x_1, \cdots, x_n) : x_1 + \cdots + x_n \in E\}$. Then for $\lambda_1, \cdots, \lambda_n \in \mathfrak{M}(G)$,

Received September 15, 1959.

[1] The work of the first named author was supported in part by the U. S. Air Force and in part by the National Science Foundation.

[2] For all group-theoretic notions and facts used without explanation, see [9].

[3] For all measure-theoretic notions and facts used without explanation, see [2].

[4] Karl R. Stromberg has recently shown that $\lambda * \mu$ is regular and hence is in $\mathfrak{M}(G)$ if λ and μ are in $\mathfrak{M}(G)$ [14].

we have

1.2.2 $$\lambda_1 * \cdots * \lambda_n(E) = \lambda_1 \times \cdots \times \lambda_n(E^{(n)}),$$

where $\lambda_1 \times \cdots \times \lambda_n$ is the direct product measure of $\lambda_1, \cdots, \lambda_n$ on G^n.

For $\lambda \in \mathfrak{M}(G)$, let $|\lambda|$ be defined for each Borel set $E \subset G$ by

1.2.3 $$|\lambda|(E) = \sup\{\textstyle\sum_{j=1}^{m} |\lambda(A_j)| : A_j \cap A_k = \emptyset \text{ for } j \neq k; \bigcup_{j=1}^{m} A_j = E; A_1, \cdots, A_m \text{ are Borel sets}\}.$$

Then $|\lambda|$ is in $\mathfrak{M}(G)$ and is the smallest nonnegative majorant of λ in $\mathfrak{M}(G)$. We define

1.2.4 $$\|\lambda\| = |\lambda|(G).$$

For $t \in G$, let ε_t be the Borel measure on G such that $\varepsilon_t(E) = 1$ if $t \in E$ and $\varepsilon_t(E) = 0$ if $t \notin E$. Plainly ε_t is an element of $\mathfrak{M}(G)$. With the algebraic operations defined above and the norm 1.2.4, $\mathfrak{M}(G)$ is a commutative Banach algebra, with unit element ε_0. It is easy to see that every λ in $\mathfrak{M}(G)$ can be uniquely written in the form $\lambda = \sum_{n=1}^{\infty} a_n \varepsilon_{t_n} + \lambda_c$, where the a_n are complex numbers, $\sum_{n=1}^{\infty} |a_n| < \infty$, and $\lambda_c(\{x\}) = 0$ for all $x \in G$. The measure λ_c is called the continuous part of λ, and if $\lambda = \lambda_c$, λ is called a continuous measure.

The carrier $C(\lambda)$ of a measure $\lambda \in \mathfrak{M}(G)$ is defined as the set $\{x : x \in G, |\lambda|(A) > 0$ for all neighborhoods A of $x\}$. For a closed subset F of G, we write $\mathfrak{M}(F)$ for the set of all measures $\lambda \in \mathfrak{M}(G)$ for which $C(\lambda) \subset F$, $\mathfrak{M}_c(F)$ for the set of all continuous measures in $\mathfrak{M}(F)$, and $\mathfrak{M}_d(F)$ for the set of all measures in $\mathfrak{M}(F)$ having zero continuous part. It is easy to see that $\mathfrak{M}(F)$, $\mathfrak{M}_c(F)$, and $\mathfrak{M}_d(F)$ are closed linear subspaces of $\mathfrak{M}(G)$, and that $\mathfrak{M}(F)$ is the direct sum of $\mathfrak{M}_c(F)$ and $\mathfrak{M}_d(F)$.

For $\lambda, \mu \in \mathfrak{M}(G)$, we write $\lambda \ll \mu$ to mean that $|\lambda|$ is absolutely continuous with respect to $|\mu|$, and $\lambda \perp \mu$ to mean that $|\lambda|$ and $|\mu|$ are mutually singular.

1.3. Let $\mathbf{S}$ be the compact Hausdorff space of all nonzero multiplicative linear functionals on $\mathfrak{M}(G)$, with the usual weak topology as linear functionals on $\mathfrak{M}(G)$. The structure of $\mathbf{S}$ is formidably complicated. For $\chi \in X$, the mapping

1.3.1 $$\lambda \to \hat{\lambda}(\chi) = \int_G \chi(x)\, d\lambda(x)$$

is obviously an element of $\mathbf{S}$, and if $\chi_1 \neq \chi_2$, then $\varepsilon_t(\chi_1) \neq \varepsilon_t(\chi_2)$ for some $t \in G$. Thus X is embedded in $\mathbf{S}$. The topology of X as a subspace of $\mathbf{S}$ agrees with its topology as the character group of G.[5] Yu. A. Šreĭder [13]

[5] One way to describe the usual topology of X is to define it as the weakest topology under which all functions $\hat{\alpha}$ are continuous, where the measures α in $\mathfrak{M}(G)$ are absolutely continuous with respect to Haar measure on G ([6], pp. 134–135). Since every function $\hat{\lambda}$ is continuous in this topology, we see that X retains its usual topology when embedded in $\mathfrak{S}$.

has given a concrete construction of the multiplicative linear functionals on $\mathfrak{M}(G)$ for the case in which G has a countable basis for open sets. His construction is valid for an arbitrary G. It is too general to yield by itself much specific information about $\mathbf{S}$.

1.4. Šreĭder has also produced a curious example of a multiplicative linear functional on $\mathfrak{M}(R)$ [12]. This multiplicative linear functional has the form $\gamma\mu(R)$ for every μ absolutely continuous with respect to Lebesgue's singular measure on Cantor's ternary set, where γ is a complex number such that $0 < |\gamma| < 1$. In fact, one has

1.4.1 $$\lim_{p\to\infty}\int_{-\infty}^{\infty}\exp\,(2\pi i 3^p x)\,d\mu(x) = \gamma\mu(R)$$

for all such μ. Then M can be taken as any point in $\bigcap_{l=1}^{\infty}(\{2\pi 3^p\}_{p=l}^{\infty})^{-}$, the closure being taken in $\mathbf{S}$ for the algebra $\mathfrak{M}(R)$.

In the present paper, we give two constructions of classes of multiplicative linear functionals on $\mathfrak{M}(G)$. The first of these generalizes the construction of asymmetric multiplicative linear functionals in $\mathfrak{M}(G)$, and the second displays in much stronger form the phenomenon produced by Šreĭder.

1.5 Definition. A subset A of G is said to be independent if, whenever $x_1, \cdots, x_n$ are distinct elements of A and $q_1, \cdots, q_n$ are integers, the equality $q_1 x_1 + \cdots + q_n x_n = 0$ implies that $q_1 = \cdots = q_n = 0$. Let a be an integer > 1. A subset A of G is said to be a-independent if all elements of A have order a, and if, whenever $x_1, \cdots, x_n$ are distinct elements of A and $q_1, \cdots, q_n$ are integers, the equality $q_1 x_1 + \cdots + q_n x_n = 0$ implies $q_1 \equiv q_2 \equiv \cdots \equiv q_n \equiv 0 \pmod{a}$.

Our first main result follows.

1.6 Theorem. *Let G be any group, and let P be any closed subset of G that is either independent or a-independent for some integer $a > 1$. Let L be any linear functional of norm 1 on the linear space $\mathfrak{M}(P \cup (-P))$ such that if $x_1, \cdots, x_n$ are elements of P (not necessarily distinct), $q_1, \cdots, q_n$ are integers, and $q_1 x_1 + \cdots + q_n x_n = 0$, then*

1.6.1 $$L(\varepsilon_{x_1})^{q_1}L(\varepsilon_{x_2})^{q_2}\cdots L(\varepsilon_{x_n})^{q_n} = 1.$$

Then there is a multiplicative linear functional M on $\mathfrak{M}(G)$ such that $L(\lambda) = M(\lambda)$ for all $\lambda \in \mathfrak{M}(P \cup (-P))$. If every neighborhood of 0 in G contains an element of infinite order, then every nonvoid open subset of G contains an independent set homeomorphic to Cantor's ternary set. If some neighborhood of 0 in G contains only elements of finite order, then every neighborhood of 0 in G contains an a-independent set A, for some integer $a > 1$, homeomorphic to Cantor's ternary set, and every nonvoid open subset of G contains a translate P of A for which $\mathfrak{M}(P \cup (-P))$ has the property stated above.

1.7. To state our second main result, we define sets of complex numbers Γ_0 and Γ_1 for every group G. If every neighborhood of 0 in G contains an element of infinite order, then

$$\Gamma_0 = \{z: z \in K, |z| = 1\} \quad \text{and} \quad \Gamma_1 = \{z: z \in K, |z| \leqq 1\}.$$

If there is a neighborhood of 0 in G containing only elements of finite order, then there is at least one integer $a > 1$ such that every neighborhood of 0 in G contains a replica of D_a (for the proof of this fact, see 2.2 *infra*). Select any such a, let $\Gamma_0 = \{1, \exp(2\pi i/a), \exp(4\pi i/a), \cdots, \exp(2(a - 1)\pi i/a)\}$, and let Γ_1 be the convex hull in K of Γ_0.

1.8 THEOREM. *Let Q be any subset of G homeomorphic to Cantor's ternary set such that every continuous function defined on Q with values in Γ_0 is arbitrarily uniformly approximable by characters of G. Let L be any linear functional on $\mathfrak{M}(Q)$ such that*

1.8.1 $$L(\lambda) \in \Gamma_1 \quad \text{if} \quad \lambda \in \mathfrak{M}(Q), \qquad \lambda \geqq 0, \text{ and } \lambda(G) \leqq 1$$

and

1.8.2 $$L(\varepsilon_x) \in \Gamma_0 \quad \text{if} \quad x \in Q.$$

Then there is a multiplicative linear functional $M \in X^-$ such that $M(\lambda) = L(\lambda)$ for all $\lambda \in \mathfrak{M}(Q)$. Furthermore, every nonvoid open subset of G contains a set Q of the sort described.

1.9. In §2, we show that every nonvoid open subset of an arbitrary group G contains a set P as described in Theorem 1.6. The proof of Theorem 1.6 is given in §3, and various inferences are drawn from Theorem 1.6 in §4. In §5, we construct sets Q as required in Theorem 1.8, and in §6 we give an analogue of Kronecker's approximation theorem for finite sets of measures on Q. This theorem is applied in §7 to prove Theorem 1.8. We are indebted to W. Rudin, K. R. Stromberg, and J. H. Williamson, respectively, for the privilege of reading [11], [14], and [16] in manuscript form.

2. Construction of sets for Theorem 1.6

2.1. Suppose that every neighborhood of 0 in G contains an element of infinite order. Rudin has shown [10] that every neighborhood U of 0 in G contains an independent perfect set homeomorphic to Cantor's ternary set,[6] which we denote by A.[7] Now let x be any element of G. Let $P = x + A$. If x has finite order, it is obvious that P is an independent set. If x has infinite order, let A_1 and A_2 be perfect complementary subsets of A. Assume

[6] The first construction of perfect independent sets in R is due to J. v. Neumann [8]. v. Neumann's set actually consists of *algebraically* independent elements.

[7] If G is nonmetrizable, then Rudin's construction can be modified in an obvious way to yield perfect independent sets not necessarily homeomorphic to Cantor's ternary set. This generalization is unimportant for our present purposes.

that neither $x + A_1$ nor $x + A_2$ is an independent set. Then we have $m_j x = q_1^{(j)} a_1^{(j)} + \cdots + q_{n_j}^{(j)} a_{n_j}^{(j)}$, where the $a_k^{(j)}$ are in A_j, the $q_k^{(j)}$ and m_j are integers, and $m_j \neq 0$ $(j = 1, 2)$. It follows that $m_1 m_2 x$ is a linear combination of elements from A_1 and also a linear combination of elements from A_2. This can occur only if $m_1 m_2 x = 0$, which is impossible. Hence one of the sets $x + A_1$, $x + A_2$ is independent. We choose P to be an independent set $x + A_j$ $(j = 1$ or $2)$. Since the neighborhood U of 0 is arbitrary, we find that every nonvoid open subset of G contains an independent set homeomorphic to Cantor's ternary set.

2.2. Suppose now that there is a neighborhood of 0 in G containing only elements of finite order. Here a little care is needed in constructing our sets P. Let U be any neighborhood of 0 in G with compact closure and let y be any element in G having finite order m. Let $V = U + \{0, y\}$, $W = V \cup (-V)$, and $G_0 = \bigcup_{n=1}^{\infty} nW$. Clearly G_0 is a compactly generated open and closed subgroup of G. The structure theorem of Pontryagin-van Kampen ([9], p. 274, Theorem 51) shows that G_0 is the direct sum $Z^l + G_1$, where G_1 is compact. Note that $y \in G_1$. Thus G_1 is an infinite compact group with a neighborhood of 0 containing only elements of finite order. Rudin ([10], p. 161, Lemma 3) has shown that the orders of all elements in G_1 are bounded. Hence the same is true of the character group X_1 of G_1. As a discrete Abelian group of bounded order, X_1 is the algebraic direct sum of cyclic groups of bounded order (see for example [1], p. 44, Theorem 11.2). Therefore G_1 is the complete direct sum of cyclic groups of bounded order, $G_1 = P_{\iota \in I} Z(b_\iota)$, where I is an infinite index class. The topological structure of G_1 as a Cartesian product of finite discrete spaces and the fact that there are only finitely many distinct integers b_ι show that every neighborhood of 0 in G_1 contains a replica of the group D_a, for some fixed integer $a > 1$.

2.3. Let D_a be represented as the group of all $Z(a)$-valued functions $x(\omega)$ defined on a countably infinite set Ω, with the usual addition and the Cartesian product topology. We may suppose that Ω is the set of all finite dyadic systems: $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$, where Ω_n consists of 2^n elements $\omega_{\eta_1, \dots, \eta_n}$, each η_i is 0 or 1 for $i = 1, \cdots, n$, and the sets Ω_n are pairwise disjoint. Let $Y = \{y\}$ be a realization of Cantor's ternary set as the set of all infinite dyadic sequences with the usual topology: $y = (\eta_1(y), \eta_2(y), \cdots)$, where $\eta_j(y)$ is 0 or 1 for $j = 1, 2, \cdots$. Put $x_y(\omega) = 1$ if $\omega = \omega_{\eta_1, \dots, \eta_n}$ for some n, and $x_y(\omega) = 0$ otherwise. Then it is easy to see that the mapping $y \rightarrow x_y$ is a homeomorphism of Y into D_a. Write the set $\{x_y\}_{y \in Y}$ as A. Suppose that $x_{y_1}, \cdots, x_{y_m}$ are distinct elements of A and that $q_1 x_{y_1} + \cdots + q_m x_{y_m} = 0$, where $q_1, \cdots, q_m$ are integers. There is obviously a positive integer n such that the elements $(\eta_1(y_j), \cdots, \eta_n(y_j)) \in \Omega_n$ are all distinct. Hence the only entry in the sum $q_1 x_{y_1} + \cdots + q_m x_{y_m}$ at $(\eta_1(y_j), \cdots, \eta_n(y_j))$ is q_j. Thus A is a-independent in D_a. (For a similar construction, see [7].)

2.4. Now let y be any element of finite order in G. We have already constructed G_1 so as to contain y. A neighborhood of y in G_1 (and hence in G, since G_1 is open and closed) consists of all $x \in G_1$ such that $x_\iota = y_\iota$ for all ι in a certain finite subset $\{\iota_1, \cdots, \iota_m\}$ of the index class I. Let y' be the element of G_1 equal to y on this finite set and equal to 0 for all other values of $\iota \in I$. Let D_a' be a replica of the group D_a contained in the subgroup of G_1 consisting of all x such that $x_{\iota_1} = \cdots = x_{\iota_m} = 0$. Let A' be any subset of D_a' of the sort constructed in 2.3. In this case, let $P = y' + A'$. The elements of P need not have order a, so that P need not be a-independent. However, P has the important property that no multiple of y' is in the group D_a' unless it is equal to 0.

Finally, let y be any element of G having infinite order. Plainly no multiple of y except $0 \cdot y$ lies in G_1. Let D_a' be any replica of D_a contained in a fixed neighborhood U of 0 in G_1, let A' be a subset of D_a' as constructed in 2.3, and let $P = y + A'$.

2.5. We summarize the constructions of 2.3 and 2.4. Suppose that there is a neighborhood of 0 in G containing only elements of finite order. Then every neighborhood of 0 in G contains a set P homeomorphic to Cantor's ternary set which is a-independent. Let H be an open subset of G not containing 0. Then H contains a compact set P of the form $w + A$, where no multiple of w different from 0 lies in the subgroup generated by A, and where A is a-independent and homeomorphic to Cantor's ternary set.

2.6. In 2.1–2.5, we have given rules for constructing a set P in an arbitrary nonvoid open subset of a group G. Throughout §3, and elsewhere where Theorem 1.6 is referred to, the set P will be taken to be one of the sets described in 2.1 or 2.5. If G has arbitrarily small elements of infinite order, we use the construction of 2.1; if not, we use 2.5. For all of the sets P constructed, we have $P \cap (-P) = \emptyset$ unless all elements of P have order 2, in which case it is obvious that $P = -P$.

3. Proof of Theorem 1.6

3.1. We break up the proof into several steps. The basic idea is simple. The elementary theory of commutative Banach algebras shows that to prove Theorem 1.6, we need only show that the set $\{\mu - L(\mu)\varepsilon_0\}$, $\mu \in \mathfrak{M}(P \cup (-P))$, is contained in some ideal of $\mathfrak{M}(G)$. That is, we must prove that the identity

3.1.1 $$\sum_{j=1}^{m} (\mu_j - L(\mu_j)\varepsilon_0) * \alpha_j = \varepsilon_0$$

can hold for no $\mu_1, \cdots, \mu_m$ in $\mathfrak{M}(P \cup (-P))$ and $\alpha_1, \cdots, \alpha_m$ in $\mathfrak{M}(G)$. We make several reductions to put the left side of 3.1.1 into tractable form, from which we will prove that 3.1.1 is impossible. Since μ_j is a linear combination of nonnegative measures and L is linear, we may obviously suppose that each μ_j in 3.1.1 is nonnegative and has total measure 1. Our second reduction is to the case in which the μ_j's have pairwise disjoint carriers. For this, we need a preliminary result.

3.2 THEOREM. *Let B be any closed subset of an arbitrary group G, let $\mu_1, \cdots, \mu_m$ be any nonnegative measures in $\mathfrak{M}(B)$, and let η be any positive number. Then there are nonnegative measures $\lambda_1, \cdots, \lambda_n$ in $\mathfrak{M}(B)$ with pairwise disjoint carriers, and nonnegative real numbers $c_k^{(j)}$ ($j = 1, \cdots, m$; $k = 1, \cdots, n$), such that*[8]

3.2.1 $$\| \mu_j - \textstyle\sum_{k=1}^n c_k^{(j)} \lambda_k \| < \eta \qquad (j = 1, \cdots, m).$$

Proof. Let $\mu = \mu_1 + \cdots + \mu_m$. All of the μ_j's are absolutely continuous with respect to μ and have nonnegative finite-valued Radon-Nikodym derivatives ρ_j with respect to μ. For each ρ_j there is a simple Borel measurable function σ_j defined on B such that $0 \leqq \sigma_j \leqq \rho_j$ and

3.2.2 $$\int_B [\rho_j(x) - \sigma_j(x)]\, d\mu(x) < \eta/2 \qquad (j = 1, \cdots, m).$$

For every ordered m-tuple of real numbers $(a_1, \cdots, a_m)$, let $E(a_1, \cdots, a_m) = \{x : x \in B,\ \sigma_j(x) = a_j \text{ for } j = 1, \cdots, m\}$. There are only a finite number of nonvoid sets $E(a_1, \cdots, a_m)$, say $E_1, \cdots, E_n$. These sets are pairwise disjoint, and their union is B.

Let φ_k be the characteristic function of the set E_k ($k = 1, \cdots, n$). There are (obviously unique) nonnegative numbers $c_k^{(j)}$ such that

3.2.3 $$\sigma_j = \textstyle\sum_{k=1}^n c_k^{(j)} \varphi_k \qquad (j = 1, \cdots, m).$$

Let $c = \max \{c_1^{(j)} + \cdots + c_n^{(j)} : j = 1, \cdots, m\}$. Since μ is a regular measure, there are compact subsets F_k of E_k such that

3.2.4 $$\mu(E_k) < \mu(F_k) + \eta/2c \qquad (k = 1, \cdots, n).$$

Let ψ_k be the characteristic function of F_k, and λ_k the measure in $\mathfrak{M}(B)$ defined by

$$\lambda_k(Y) = \mu(F_k \cap Y) \qquad (k = 1, \cdots, n)$$

for Borel sets $Y \subset G$. Plainly the sets $C(\lambda_1), \cdots, C(\lambda_n)$ are pairwise disjoint. Relations 3.2.2, 3.2.3, and 3.2.4 imply that

$$\begin{aligned} \| \mu_j - \textstyle\sum_{k=1}^n c_k^{(j)} \lambda_k \| &= \mu_j(B) - \textstyle\sum_{k=1}^n c_k^{(j)} \lambda_k(B) \\ &= \int_B [\rho_j(x) - \sigma_j(x)]\, d\mu(x) + \int_B [\textstyle\sum_{k=1}^n c_k^{(j)} (\varphi_k(x) - \psi_k(x))]\, d\mu(x) \\ &< \eta/2 + c\eta/2c = \eta \quad (j = 1, \cdots, m). \end{aligned}$$

This is 3.2.1, which we wished to prove.

3.3 LEMMA. *If* 3.1.1 *holds, then there are nonnegative measures $\lambda_1, \cdots, \lambda_n$ in $\mathfrak{M}(P \cup (-P))$ and measures $\beta_1, \cdots, \beta_n$ in $\mathfrak{M}(G)$ such that*

3.3.1 $$\textstyle\sum_{k=1}^n (\lambda_k - L(\lambda_k)\varepsilon_0) * \beta_k = \varepsilon_0,$$

[8] The following result holds for measures on any locally compact Hausdorff space; we state it only for the case needed below.

and such that the sets $C(\lambda_1), \cdots, C(\lambda_n)$ *are pairwise disjoint and* $C(\lambda_k)$ *is contained in* P *or in* $-P$ $(k = 1, \cdots, n)$.

Proof. In Theorem 3.2, let $B = P \cup (-P)$, and let $\eta = (2\sum_{j=1}^{m} \| \alpha_j \|)^{-1}$. It is clear that in constructing the λ_k's of Theorem 3.2, we may suppose that $C(\lambda_k) \subset P$ or $C(\lambda_k) \subset -P$ for $k = 1, \cdots, n$ (recall that $P \cap (-P) = \emptyset$ or $P = -P$). A simple computation shows that

$$\begin{aligned} &\| \varepsilon_0 - [\textstyle\sum_{k=1}^{n} (\lambda_k - L(\lambda_k)\varepsilon_0) * (\sum_{j=1}^{m} c_k^{(j)}\alpha_j)] \| \\ &\quad = \| \textstyle\sum_{j=1}^{m} (\mu_j - L(\mu_j)\varepsilon_0) * \alpha_j - \sum_{j=1}^{m} [(\sum_{k=1}^{n} c_k^{(j)}\lambda_k) \\ &\qquad - L(\textstyle\sum_{k=1}^{n} c_k^{(j)}\lambda_k)\varepsilon_0] * \alpha_j \| < 2\eta(\sum_{j=1}^{m} \| \alpha_j \|) = 1 \end{aligned}$$

(recall that $\| L \| = 1$). Hence the measure

$$\textstyle\sum_{k=1}^{n} (\lambda_k - L(\lambda_k)\varepsilon_0) * (\sum_{j=1}^{m} c_k^{(j)}\alpha_j)$$

has an inverse, say δ, in $\mathfrak{M}(G)$. Consequently 3.3.1 holds with $\beta_k = (\sum_{j=1}^{m} c_k^{(j)}\alpha_j) * \delta$ $(k = 1, \cdots, n)$.

We now make a third reduction.

3.4 Lemma. *If* 3.3.1 *holds, then there are continuous nonnegative measures* $\gamma_1, \cdots, \gamma_n$ *in* $\mathfrak{M}(P \cup (-P))$, *points* $x_1, \cdots, x_n$ *in* $P \cup (-P)$, *and measures* $\alpha_1, \cdots, \alpha_m, \beta_1, \cdots, \beta_n$ *in* $\mathfrak{M}(G)$ *such that*

3.4.1 $\sum_{j=1}^{m} (\gamma_j - L(\gamma_j)\varepsilon_0) * \alpha_j + \sum_{k=1}^{n} (\varepsilon_{x_k} - L(\varepsilon_{x_k})\varepsilon_0) * \beta_k = \varepsilon_0$.

The sets $C(\gamma_1), \cdots, C(\gamma_m), \{x_1\}, \cdots, \{x_n\}$ *are pairwise disjoint, and each is contained in* P *or in* $-P$.

Proof. The measure λ_k in 3.3.1 has the form $\gamma_k + \sum_{l=1}^{\infty} t_l^{(k)}\varepsilon_{x_l^{(k)}}$, where γ_k is continuous, the $t_l^{(k)}$ are positive or zero, and $\sum_{l=1}^{\infty} t_l^{(k)} < \infty$ $(k = 1, \cdots, n)$. The norm $\| \lambda_k - (\gamma_k + \sum_{l=1}^{N_k} t_l^{(k)}\varepsilon_{x_l^{(k)}}) \| = \sum_{l=N_k+1}^{\infty} t_l^{(k)}$ can be made arbitrarily small by proper choice of the N_k $(k = 1, \cdots, n)$. The proof now follows that of Lemma 3.3. The disjointness and inclusion relations asserted follow from the inclusions

$$C(\gamma_k) \cup \{x_1^{(k)}, \cdots, x_{N_k}^{(k)}\} \subset C(\gamma_k) \qquad (k = 1, \cdots, n).$$

3.5 Lemma. *Let* γ *be any nonnegative continuous measure in* $\mathfrak{M}(G)$ *such that* $\gamma(G) = 1$, *and let* z *be any complex number such that* $|z| < 1$. *Let* η *be any positive number less than* 2. *Then there are complex numbers* u *and* v *such that* $|u| = |v| = 1$ *and nonnegative continuous measures* δ_1 *and* δ_2 *with* $\delta_1(G) = \delta_2(G) = 1$, $C(\delta_1) \cap C(\delta_2) = \emptyset$, *and* $C(\delta_1) \cup C(\delta_2) \subset C(\gamma)$, *such that*[9]

3.5.1 $\| \frac{1}{2}(\delta_1 - u\varepsilon_0) + \frac{1}{2}(\delta_2 - v\varepsilon_0) - (\gamma - z\varepsilon_0) \| < \eta.$

Proof. For $z \neq 0$, let u and v be the unique complex numbers such that $|u| = |v| = 1$ and $z = \frac{1}{2}(u + v)$. For $z = 0$, let $u = -1, v = 1$. Since

[9] See footnote 8.

γ is continuous, there is a measurable subset A of $C(\gamma)$ such that $\gamma(A) = \frac{1}{2}$. Since γ is regular, there are a compact subset F_1 of A and a compact subset F_2 of $C(\gamma) \cap A'$ such that $\gamma(A \cap F_1') < \frac{1}{4}\eta$, $\gamma(C(\gamma) \cap A' \cap F_2') < \frac{1}{4}\eta$. Let δ_j be the measure such that $\delta_j(E) = \gamma(F_j \cap E)(\gamma(F_j))^{-1}$ for Borel sets $E \subset G$. Then we have

$$\| \tfrac{1}{2}(\delta_1 - u\varepsilon_0) + \tfrac{1}{2}(\delta_2 - v\varepsilon_0) - (\gamma - z\varepsilon_0) \|$$
$$\leqq \| \gamma(F_1)\delta_1 + \gamma(F_2)\delta_2 - \gamma \| + \| \tfrac{1}{2}\delta_1 - \gamma(F_1)\delta_1 \|$$
$$+ \| \tfrac{1}{2}\delta_2 - \gamma(F_2)\delta_2 \| < \tfrac{1}{4}\eta + \tfrac{1}{4}\eta + \tfrac{1}{2}\eta = \eta,$$

which is 3.5.1.

3.6 LEMMA. *If* 3.4.1 *holds, there are continuous nonnegative measures* $\delta_1, \cdots, \delta_q$ *each of total measure* 1, *complex numbers* $a_1, \cdots, a_q$ *each of absolute value* 1, *and measures* $\pi_1, \cdots, \pi_q$ *in* $\mathfrak{M}(G)$, *such that*

3.6.1 $$\textstyle\sum_{l=1}^{q} (\delta_l - a_l\,\varepsilon_0) * \pi_l + \sum_{k=1}^{n} (\varepsilon_{x_k} - L(\varepsilon_{x_k})\varepsilon_0) * \beta_k = \varepsilon_0\,.$$

The sets $C(\delta_1), \cdots, C(\delta_q), \{x_1\}, \cdots, \{x_n\}$ *are pairwise disjoint, and each is contained in* P *or in* $-P$.

Proof. There is no loss of generality in supposing that $\gamma_j \neq 0$ for $j = 1, \cdots, m$, in 3.4.1. Writing $(\gamma_j - L(\gamma_j)\varepsilon_0) * \alpha_j$ as

$$(\| \gamma_j \|^{-1}\gamma_j - L(\| \gamma_j \|^{-1}\gamma_j)\varepsilon_0) * \| \gamma_j \|\, \alpha_j\,,$$

we may also suppose that $\gamma_j(G) = 1$ $(j = 1, \cdots, m)$. Since $\| L \| = 1$, we then have $| L(\gamma_j) | \leqq 1$. If $| L(\gamma_j) | = 1$, we set γ_j equal to a single measure δ_l and write $L(\gamma_j) = a_l$. If $| L(\gamma_j) | < 1$, we apply Lemma 3.5, choosing complex numbers u_j and v_j such that $| u_j | = | v_j | = 1$, $L(\gamma_j) = \frac{1}{2}(u_j + v_j)$, and finding measures $\delta_j^{(1)}$ and $\delta_j^{(2)}$ such that

$$\| \tfrac{1}{2}(\delta_j^{(1)} - u_j\,\varepsilon_0) + \tfrac{1}{2}(\delta_j^{(2)} - v_j\,\varepsilon_0) - (\gamma_j - L(\gamma_j)\varepsilon_0) \| < \eta.$$

Choosing η sufficiently small, using the argument of Lemma 3.3, and renumbering the δ's, we obtain 3.6.1.

3.7. We summarize our present situation. If Theorem 1.6 fails, there exist continuous nonnegative measures $\lambda_1, \cdots, \lambda_m$ in $\mathfrak{M}(P \cup (-P))$, points $x_1, \cdots, x_n$ in $P \cup (-P)$, complex numbers $a_1, \cdots, a_m$ of absolute value 1, and measures $\alpha_1, \cdots, \alpha_m, \beta_1, \cdots, \beta_n$ in $\mathfrak{M}(G)$ such that

3.7.1 $$\textstyle\sum_{j=1}^{m} (\lambda_j - a_j\,\varepsilon_0) * \alpha_j + \sum_{k=1}^{n} (\varepsilon_{x_k} - L(\varepsilon_{x_k})\varepsilon_0) * \beta_k = \varepsilon_0\,.$$

The sets $C(\lambda_1), \cdots, C(\lambda_m), \{x_1\}, \cdots, \{x_n\}$ are pairwise disjoint and each is contained in P or in $-P$. We will prove that 3.7.1 cannot hold.

3.8 THEOREM. *Let* p *be any positive integer, and all notation as in* 3.7. *Then*[10]

[10] Products and powers of measures are convolution products, here and below.

3.8.1 $$\Big\|\prod_{s=1}^{m} (\lambda_s^{p-1} + a_s \lambda_s^{p-2} + \cdots + a_s^{p-1}\varepsilon_0) * \prod_{t=1}^{n} (\varepsilon_{x_t}^{p-1} + L(\varepsilon_{x_t})\varepsilon_{x_t}^{p-2} + \cdots + L(\varepsilon_{x_t})^{p-1}\varepsilon_0)\Big\| = p^{m+n}.$$

Proof. Let π denote the measure written on the left side of 3.8.1. The general term of π is

3.8.2 $$A(l_1, \cdots, l_m)B(k_1, \cdots, k_n)\lambda_1^{l_1} * \cdots * \lambda_m^{l_m} * \varepsilon_{k_1x_1+\cdots+k_nx_n},$$

where $0 \leqq l_s \leqq p-1$ $(s = 1, \cdots, m)$, $0 \leqq k_t \leqq p-1$ $(t = 1, \cdots, n)$,

3.8.3 $$A(l_1, \cdots, l_m) = a_1^{p-1-l_1} \cdots a_m^{p-1-l_m},$$

and

3.8.4 $$B(k_1, \cdots, k_n) = L(\varepsilon_{x_1})^{p-1-k_1} \cdots L(\varepsilon_{x_n})^{p-1-k_n}.$$

We first show that

$$\lambda_1^{l_1} * \cdots * \lambda_m^{l_m} * \varepsilon_{k_1x_1+\cdots+k_nx_n} \perp \lambda_1^{l_1'} * \cdots * \lambda_m^{l_m'} * \varepsilon_{k_1'x_1+\cdots+k_n'x_n}$$

unless $l_1 = l_1', \cdots, l_m = l_m'$. With no loss of generality, we suppose that $l_1 > l_1'$ and will prove that

$$\lambda_1^{l_1} * \cdots * \lambda_m^{l_m} * \varepsilon_u(C(\lambda_1^{l_1'} * \cdots * \lambda_m^{l_m'} * \varepsilon_{u'})) = 0,$$[11]

where for brevity we have written $k_1 x_1 + \cdots + k_n x_n = u$, $k_1' x_1 + \cdots + k_1' x_n = u'$. This of course will prove that

$$\lambda_1^{l_1} * \cdots * \lambda_m^{l_m} * \varepsilon_u \perp \lambda_1^{l_1'} * \cdots * \lambda_m^{l_m'} * \varepsilon_{u'}.$$

Write $C(\lambda_j) = P_j$ $(j = 1, \cdots, m)$. It is easy to see that

$$C(\lambda_1^{l_1'} * \cdots * \lambda_m^{l_m'} * \varepsilon_{u'}) = l_1' P_1 + \cdots + l_m' P_m + u'.$$

As pointed out in 1.2.2, we have

3.8.5 $$\lambda_1^{l_1} * \cdots * \lambda_m^{l_m} * \varepsilon_u(l_1' P_1 + \cdots + l_m' P_m + u') = \lambda_1 \times \cdots \times \lambda_{1(l_1)} \times \cdots \times \lambda_m \times \cdots \times \lambda_{m(l_m)} \times \varepsilon_u(E),$$

where E is the set of all points $(x_1^{(1)}, \cdots, x_{l_1}^{(1)}, \cdots, x_1^{(m)}, \cdots, x_{l_m}^{(m)}, u)$ in $G^{l_1+\cdots+l_m+1}$ such that $x_s^{(j)} \in P_j$ $(s = 1, \cdots, l_j; j = 1, \cdots, m)$ and

$$x_1^{(1)} + \cdots + x_{l_1}^{(1)} + \cdots + x_1^{(m)} + \cdots + x_{l_m}^{(m)} + u \in l_1' P_1 + \cdots + l_m' P_m + u'.$$

For every point in E, therefore, we have

3.8.6 $$x_1^{(1)} + \cdots + x_{l_1}^{(1)} + \cdots + x_1^{(m)} + \cdots + x_{l_m}^{(m)} + u = y_1^{(1)} + \cdots + y_{l_1'}^{(1)} + \cdots + y_1^{(m)} + \cdots + y_{l_m'}^{(m)} + u',$$

[11] This does not assert that

$$\lambda_1^{l_1} * \cdots * \lambda_m^{l_m} * \varepsilon_u \quad\text{and}\quad \lambda_1^{l_1'} * \cdots * \lambda_m^{l_m'} * \varepsilon_{u'}$$

have disjoint carriers, which is a much stronger condition than mutual singularity.

or

3.8.7 $$\sum x_s^{(1)} - \sum y_{s'}^{(1)} + \cdots + \sum x_s^{(m)} - \sum y_{s'}^{(m)} + u - u' = 0,$$

where the $x_s^{(j)}$ and $y_{s'}^{(j)}$ are in P_j ($j = 1, \cdots, m$). Suppose first that P is independent or a-independent in the sense of 1.5. Then, since $P_1, \cdots, P_m$, $\{x_1\}, \cdots, \{x_n\}$ are pairwise disjoint and each is contained in P or in $-P$, it is clear that

3.8.8 $$\sum_{s=1}^{l_1} x_s^{(1)} - \sum_{s'=1}^{l_1'} y_{s'}^{(1)} = 0$$

if 3.8.7 is to hold. If the equality $x_s^{(1)} = \pm x_t^{(1)}$ holds for no distinct s and t, $1 \leqq s \leqq l_1$, $1 \leqq t \leqq l_1$, then each $x_s^{(1)}$ must combine with a distinct $y_{s'}^{(1)}$ in order for 3.8.8 to hold. Since $l_1 > l_1'$, this is impossible, and we have $x_s^{(1)} = x_t^{(1)}$ or $x_s^{(1)} = -x_t^{(1)}$ for some distinct s and t. Fubini's theorem and the continuity of λ_1 now show that the right side of 3.8.5 is zero, which we wished to prove.

If P has the form $w + A$ as in 2.5, then the equality 3.8.6 leads to the equality

$$Nw + \sum x_s^{(1)} + \cdots + \sum x_s^{(m)} + u = N'w + \sum y_{s'}^{(1)} + \cdots + \sum y_{s'}^{(m)} + u',$$

where N and N' are integers, the x's and y's lie in A, and u and u' are now linear combinations of elements of A. It follows that $Nw = N'w$, and then we argue as before to prove that the right side of 3.8.5 is zero.

Now look at the measures

$$\lambda_1^{l_1} * \cdots * \lambda_m^{l_m} * \varepsilon_{k_1x_1+\cdots+k_nx_n} \quad \text{and} \quad \lambda_1^{l_1} * \cdots * \lambda_m^{l_m} * \varepsilon_{k_1'x_1+\cdots+k_n'x_n},$$

which have carriers

$$C = l_1 P_1 + \cdots + l_m P_m + (k_1 x_1 + \cdots + k_n x_n)$$

and

$$D = l_1 P_1 + \cdots + l_m P_m + (k_1' x_1 + \cdots + k_n' x_n),$$

respectively, and suppose that $k_1 x_1 + \cdots + k_n x_n \neq k_1' x_1 + \cdots + k_n' x_n$ Assume that $C \cap D \neq \emptyset$. If P is independent or a-independent, then the disjointness of $P_1, \cdots, P_m$, $\{x_1\}, \cdots, \{x_n\}$, and the fact that each P_j is contained in P or in $-P$, give an immediate contradiction. Suppose that P has the form $w + A$ as in 2.5 and that $C \cap D \neq \emptyset$. Write $x_k^* = x_k - w$ ($k = 1, \cdots, n$). Then there are points $y_s^{(j)}$ and $z_s^{(j)}$ in $P_j - w \subset A$ such that

3.8.9 $$\begin{aligned} (l_1 + \cdots + l_m)w + y_1^{(1)} + \cdots + y_{l_1}^{(1)} + \cdots + y_1^{(m)} + \cdots + y_{l_m}^{(m)} \\ + (k_1 + \cdots + k_n)w + k_1 x_1^* + \cdots + k_n x_n^* \\ = (l_1 + \cdots + l_m)w + z_1^{(1)} + \cdots + z_{l_1}^{(1)} + \cdots + z_1^{(m)} + \cdots + z_{l_m}^{(m)} \\ + (k_1' + \cdots + k_n')w + k_1' x_1^* + \cdots + k_n' x_n^*. \end{aligned}$$

The sets $P_1 - w, \cdots, P_m - w$, $\{x_1^*\}, \cdots, \{x_n^*\}$ are obviously pairwise disjoint, and each is contained in A or in $-A$. No multiple of w except for 0 is in the group generated by A. Equality 3.8.9 therefore implies that $(k_1 + \cdots + k_n)w = (k_1' + \cdots + k_n')w$, and the properties of A imply that $k_1 x_1^* = k_1' x_1^*, \cdots, k_n x_n^* = k_n' x_n^*$. Hence $k_1 x_1 + \cdots + k_n x_n = k_1' x_1 + \cdots + k_n' x_n$, which is a contradiction. The measures

$$\lambda_1^{l_1} * \cdots * \lambda_m^{l_m} * \varepsilon_{k_1 x_1 + \cdots + k_n x_n} \quad \text{and} \quad \lambda_1^{l_1} * \cdots * \lambda_m^{l_m} * \varepsilon_{k_1' x_1 + \cdots + k_n' x_n}$$

therefore have disjoint carriers if $k_1 x_1 + \cdots + k_n x_n \neq k_1' x_1 + \cdots + k_n' x_n$ and are certainly singular with respect to each other.

Condition 1.6.1 on the linear functional L shows that $B(k_1, \cdots, k_n) = B(k_1', \cdots, k_n')$ if $k_1 x_1 + \cdots + k_n x_n = k_1' x_1 + \cdots + k_n' x_n$.

We have thus proved that every pair of distinct measures appearing in the expansion of π as a sum of monomials $\lambda_1^{l_1} * \cdots * \lambda_m^{l_m} * \varepsilon_{k_1 x_1 + \cdots + k_n x_n}$ are mutually singular, and that equal measures appear with equal coefficients. The coefficients $A(l_1, \cdots, l_m)B(k_1, \cdots, k_n)$ all have absolute value 1. Since the norm of a sum of pairwise singular measures is the sum of the norms, the equality 3.8.1 is proved.

3.9 *Completion of the proof of Theorem* 1.6. Multiply both sides of the equality 3.7.1 by the measure π introduced in Theorem 3.8. An elementary computation then gives

$$\begin{aligned} &\textstyle\sum_{j=1}^m \alpha_j * (\lambda_j^p - a_j^p \varepsilon_0) \prod_{s=1, s\neq j}^m (\lambda_s^{p-1} + a_s \lambda_s^{p-2} + \cdots + a_s^{p-1}\varepsilon_0) \\ &\qquad * \textstyle\prod_{t=1}^n (\varepsilon_{x_t}^{p-1} + L(\varepsilon_{x_t})\varepsilon_{x_t}^{p-2} + \cdots + L(\varepsilon_{x_t})^{p-1}\varepsilon_0) \\ &+ \textstyle\sum_{k=1}^n \beta_k * (\varepsilon_{x_k}^p - L(\varepsilon_{x_k})^p) * \prod_{s=1}^m (\lambda_s^{p-1} + a_s \lambda_s^{p-2} + \cdots + a_s^{p-1}\varepsilon_0) \\ &\qquad * \textstyle\prod_{t=1, t\neq k}^n (\varepsilon_{x_t}^{p-1} + L(\varepsilon_{x_t})\varepsilon_{x_t}^{p-2} + \cdots + L(\varepsilon_{x_t})^{p-1}\varepsilon_0) = \pi. \end{aligned} \tag{3.9.1}$$

The usual norm inequalities show at once that the norm of the left side of 3.9.1 is less than or equal to $2p^{m+n-1}(\sum_{j=1}^m \| \alpha_j \| + \sum_{k=1}^n \| \beta_k \|)$, a contradiction if p is sufficiently large. This completes the proof of Theorem 1.6.

4. Some consequences of Theorem 1.6

4.1 THEOREM. *Let L be any linear functional on the linear space $\mathfrak{M}_c(P \cup (-P))$ of norm not exceeding* 1. *Then there is a multiplicative linear functional M on $\mathfrak{M}(G)$ that agrees with L on $\mathfrak{M}_c(P \cup (-P))$.*

Proof. Let ψ be any character of G, continuous or discontinuous. For $\lambda = \lambda_c + \sum_{l=1}^\infty a_l \varepsilon_{x_l} \in \mathfrak{M}(P \cup (-P))$, let $L_1(\lambda) = L(\lambda_c) + \sum_{l=1}^\infty a_l \psi(x_l)$. Since $\mathfrak{M}(P \cup (-P))$ is the direct sum of $\mathfrak{M}_c(P \cup (-P))$ and $\mathfrak{M}_d(P \cup (-P))$, L_1 is well-defined. Plainly, L_1 satisfies the hypotheses of Theorem 1.6.

4.2 THEOREM. *Let f be any Borel measurable complex-valued function of absolute value not exceeding* 1, *defined on $P \cup (-P)$. Then there is a multiplicative linear functional M on $\mathfrak{M}(G)$ such that*

$$M(\lambda) = \int_G f(x)\, d\lambda(x) \quad \textit{for all } \lambda \in \mathfrak{M}_c(P \cup (-P)).$$

This fact follows at once from Theorem 4.1.

4.3. Theorem 4.2 can be regarded as a partial generalization of the theorem of Yu. A. Šreĭder referred to in 1.4, except for the fact that the M of Theorem 4.2 need not lie in X^-. In §7, we shall show that a similar result can be obtained with $M \in X^-$ if $P \cup (-P)$ is replaced by a more special set Q.

The following fact is a slight improvement over previously obtained results ([3], [11], [16]).

4.4 THEOREM. *The algebra $\mathfrak{M}(G)$ is asymmetric. In fact, there is a multiplicative linear functional M on $\mathfrak{M}(G)$ such that $M(\tilde{\lambda}) = \overline{M(\lambda)}$ for $\lambda \in \mathfrak{M}_c(P \cup (-P))$ if and only if $\lambda(G) = 0$.*

Proof. In Theorem 4.2, let f be the function identically equal to i $(i^2 = -1)$. Then there is a multiplicative linear functional M on $\mathfrak{M}(G)$ such that $M(\lambda) = i\lambda(G)$ for all $\lambda \in \mathfrak{M}_c(P \cup (-P))$. Since $\tilde{\lambda}(G) = \overline{\lambda(G)}$ and $C(\lambda) \subset P \cup (-P)$, the present theorem will be proved as soon as we show that there are nonzero continuous measures on $P \cup (-P)$. Taking P homeomorphic to Cantor's ternary set, we see that $P \cup (-P)$ is also homeomorphic to Cantor's ternary set, and hence carries a large number of nonzero continuous positive measures.

The following theorem is also a slight generalization of known facts.

4.5 THEOREM. *There is a measure μ with carrier $P \cup (-P) \cup \{0\}$ such that $|\hat{\mu}|$ is bounded away from zero on X and μ has no inverse in $\mathfrak{M}(G)$.*[12]

Proof. Let P and λ be as in 4.4, and let $\mu = \lambda + \tilde{\lambda} - 2i\varepsilon_0$.

4.6 *Note.* Condition 1.6.1, which is evidently necessary for Theorem 1.6, imposes a severe restriction on $L(\varepsilon_x)$ for $x \in P \cup (-P)$. This is quite natural, since any multiplicative linear functional M is a character of G (continuous or discontinuous) when applied to the point measures ε_x $(x \in G)$.

4.7 THEOREM. *Let $\lambda_1, \cdots, \lambda_n$ be pairwise singular measures in $\mathfrak{M}_c(P \cup (-P))$, for which $\| \lambda_1 \| = \cdots = \| \lambda_n \| = 1$. Then the joint spectrum of $\lambda_1, \cdots, \lambda_n$ is the product of n unit disks $\{z: z \in K, |z| \leqq 1\}$. That is, for every n-tuple of complex numbers $(z_1, \cdots, z_n)$ for which $|z_1| \leqq 1, \cdots, |z_n| \leqq 1$, there is a multiplicative linear functional M on $\mathfrak{M}(G)$ such that $M(\lambda_1) = z_1, \cdots, M(\lambda_n) = z_n$.*

Proof. Consider the linear space $\mathfrak{M}_0$ spanned by $\lambda_1, \cdots, \lambda_n$ and the linear functional L_0 on $\mathfrak{M}_0$ defined by $L_0(a_1 \lambda_1 + \cdots + a_n \lambda_n) = a_1 z_1 + \cdots + a_n z_n$. The norm of L_0 is $\max(|z_1|, \cdots, |z_n|)$, and by the Hahn-Banach theorem, there is a linear extension L of L_0 over $\mathfrak{M}_c(P \cup (-P))$ with the same norm. Now apply Theorem 4.1.

[12] The last assertion is due to Wiener and Pitt [15] for the case $G = R$. The construction given by Wiener and Pitt is difficult to follow. The first satisfactory proof, for $G = R$, is due to Šreĭder [13].

4.8 THEOREM. *Let k and l be distinct positive integers, let $\lambda_1, \cdots, \lambda_k$ and $\mu_1, \cdots, \mu_l$ be nonnegative measures in $\mathfrak{M}_c(P \cup (-P))$, and let π and ρ be arbitrary elements of norm 1 in $\mathfrak{M}(G)$ that have inverses. Then the measures $\lambda = \lambda_1 * \cdots * \lambda_k * \pi$ and $\mu = \mu_1 * \cdots * \mu_l * \rho$ are mutually singular.*

Proof. Let t be a real number such that $0 < t < 1$, and let M be a multiplicative linear functional on $\mathfrak{M}(G)$ that is equal to $\sigma(G)$ for all $\sigma \in \mathfrak{M}_c(P \cup (-P))$. Theorem 4.1 shows that such an M exists. Šreĭder [13] has shown that M can be represented by integration with respect to a generalized character $\chi_\sigma(x)$ of G, which is defined as follows. For every $\sigma \in \mathfrak{M}(G)$, χ_σ is a Borel measurable function[13] defined on G such that

(1) $\sigma_1 \ll \sigma$ implies $\chi_{\sigma_1}(x) = \chi_\sigma(x)$ a. e. $(|\sigma_1|)$;

(2) $\sup_{\sigma \in \mathfrak{M}(G)} \{\text{ess sup}_{x \in G} |\chi_\sigma(x)|\} = 1$;

(3) $\chi_\sigma(x)\chi_\sigma(y) = \chi_\sigma(x + y)$ for almost all points $(x, y) \in G^2$ with respect to $\sigma \times \sigma$;

(4) $M(\sigma) = \int_G \chi_\sigma(x)\, d\sigma(x)$ for all $\sigma \in \mathfrak{M}(G)$.

In examining the measures λ and μ for singularity, we lose no generality in supposing that all λ_i and μ_j have total measure 1. Assume that there is a nonzero, nonnegative measure δ such that $\delta \ll \lambda$ and $\delta \ll \mu$. Then we have $\chi_\delta(x) = \chi_\lambda(x)$ a. e. (δ) and $\chi_\delta(x) = \chi_\mu(x)$ a. e. (δ), by condition (1). Hence there is a Borel set E such that $\lambda(E) > 0$, $\mu(E) > 0$, and $\chi_\lambda(x) = \chi_\mu(x)$ for all $x \in E$. We also have $M(\lambda) = t^k M(\pi)$ and $M(\mu) = t^l M(\rho)$. It is easy to see from this that $\chi_\lambda(x) = t^k M(\pi)$ a. e. $(|\lambda|)$ and $\chi_\mu(x) = t^l M(\rho)$ a. e. $(|\mu|)$, in view of condition (4). It follows that $t^k M(\pi) = t^l M(\rho)$. Since $0 < t < 1$ and $|M(\pi)| = |M(\rho)| = 1$, this is impossible.

4.9 THEOREM. *Let λ and μ be nonzero nonnegative measures in $\mathfrak{M}_c(P \cup (-P))$. Then $\lambda * \mu \notin \mathfrak{M}_c(P \cup (-P))$.*

Proof. Let M be the multiplicative linear functional used in the proof of Theorem 4.8. We have $M(\lambda * \mu) = M(\lambda) \cdot M(\mu) = t^2\lambda(G)\mu(G) = t^2 \lambda * \mu(G)$. If $\lambda * \mu$ were in $\mathfrak{M}_c(P \cup (-P))$, we would have $M(\lambda * \mu) = t\lambda * \mu(G)$, an impossibility.

5. Construction of the set Q for Theorem 1.8

The sets P that figure in Theorem 1.6 are pathological, to be sure, but they are constructible explicitly in groups such as R and D_a, and they are characterized essentially by the condition of independence or a-independence (barring the special case discussed in 2.5). If we construct more special sets, then we can expect even more bizarre results, like Theorem 1.8. We proceed to the construction of sets Q in arbitrary groups.

---

[13] Karl R. Stromberg has pointed out that the functions χ_σ can all be taken Borel measurable, and not merely measurable with respect to $|\sigma|$, as in Šreĭder's original construction.

5.1. Let G be a group, and let $V_1, \cdots, V_n$ be arbitrary nonvoid, pairwise disjoint open subsets of G. We wish to find points $x_j \in V_j$ $(j = 1, \cdots, n)$ with the following property. Let $z_1, \cdots, z_n$ be any complex numbers that are values of (continuous) characters of G, and let η be any positive number. Then a character $\chi \in X$ can be found such that $| \chi(x_j) - z_j | < \eta$ for $j = 1, \cdots, n$. If this can be done, we say that G is Kroneckerian. For the construction of the sets Q, we need to show that certain groups are Kroneckerian, as follows.

5.2. If $G = R$, then in V_j we can choose x_j such that $x_1, \cdots, x_n$ are rationally independent. Applying Kronecker's approximation theorem, we see that R is Kroneckerian.

5.3. Suppose next that G is compact and that every neighborhood of 0 in G contains an element of infinite order. Let $p_1, \cdots, p_n$ be integers not all zero, and let f be the function with domain G^n and range contained in G such that $f(y_1, \cdots, y_n) = p_1 y_1 + \cdots + p_n y_n$. Plainly f is a continuous homomorphism. If $f^{-1}(0)$ contains a nonvoid open subset of G^n, then $f^{-1}(0)$ contains a neighborhood $W_1 \times \cdots \times W_n$ of $(0, 0, \cdots, 0)$ in G^n, so that $p_j x_j = 0$ for all $x_j \in W_j$ $(j = 1, \cdots, n)$. This contradicts our hypothesis on G. The set $E(p_1, \cdots, p_n) = \{(y_1, \cdots, y_n) : p_1 y_1 + \cdots + p_n y_n \neq 0\}$ is thus an open dense subset of G^n. Since G^n is compact, the set $E = \bigcap E(p_1, \cdots, p_n)$, taken over all n-tuples $(p_1, \cdots, p_n)$ of integers not all zero, is dense in G^n. Hence $V_1 \times \cdots \times V_n$ contains a point $(x_1, \cdots, x_n)$ such that $p_1 x_1 + \cdots + p_n x_n = 0$ if and only if $p_1 = \cdots = p_n = 0$. Now look at the subset $B = \{\chi(x_1), \cdots, \chi(x_n)\}_{\chi \in X}$ of T^n. Plainly B is a subgroup of T^n. If B is not dense in T^n, there is a character ψ of T^n that is equal to 1 on B and is not identically 1 (see [9], p. 258, Theorem 42). That is, there is a sequence $(p_1, p_2, \cdots, p_n)$ of integers not all zero such that $\chi(x_1)^{p_1} \cdots \chi(x_n)^{p_n} = \chi(p_1 x_1 + \cdots + p_n x_n) = 1$ for all characters χ of G. Thus $p_1 x_1 + \cdots + p_n x_n = 0$, which is impossible, and therefore B is dense in T^n.[14] This means of course that G is Kroneckerian.

5.4. Suppose finally that $G = D_a$. For a sequence of integers $(r_1, \cdots, r_m)$, where $0 \leqq r_j < a$ $(j = 1, \cdots, m)$, let $F(r_1, \cdots, r_m)$ be the set of all $x \in D_a$ such that $x_j = r_j$ $(j = 1, \cdots, m)$. Pairwise disjoint open subsets $V_1, \cdots, V_n$ of D_a may be taken to be of the form $V_j = F(r_1^{(j)}, \cdots, r_m^{(j)})$ $(j = 1, \cdots, n)$. Let $x^{(j)}$ be the element of V_j such that $x_k^{(j)} = 1$ if $k = m + j$ and $x_k^{(j)} = 0$ if $k > m$ and $k \neq m + j$ $(j = 1, \cdots, n)$. Let b_j be any integers $0, 1, \cdots, a - 1$ $(j = 1, \cdots, n)$. Let χ be the function on D_a such that $\chi(y) = \exp [2\pi i a^{-1}(b_1 y_{m+1} + \cdots + b_n y_{m+n})]$. Plainly

[14] This fact can also be proved from a general approximation theorem of Hewitt and Zuckerman ([4], Theorem 2). The set $\{x_1, \cdots, x_n\}$ generates a free group, and there is a character of the *discrete* group G assuming arbitrary values of absolute value 1 at $x_1, \cdots, x_n$. This character is arbitrarily approximable at $x_1, \cdots, x_n$ by a continuous character of G.

χ is a continuous character of D_a, and χ assumes the value $\exp [2\pi i b_j/a]$ in the set V_j $(j = 1, \cdots, n)$. Since every character ψ of D_a has the property that $\psi^a = 1$, we have shown that D_a is Kroneckerian.

5.5. Suppose now that G is a metric group with metric d that is Kroneckerian. Let $\{\varepsilon_r\}_{r=1}^{\infty}$ be a sequence of positive real numbers with limit 0. Let $Q_1^{(0)}$ be any compact neighborhood in G. Suppose that for a nonnegative integer r, the pairwise disjoint compact neighborhoods $Q_1^{(r)}, \cdots, Q_{2^r}^{(r)}$ have been defined. We proceed inductively to define $Q_1^{(r+1)}, \cdots, Q_{2^{r+1}}^{(r+1)}$. First select nonvoid open subsets $W_{2j-1}^{(r+1)}$ and $W_{2j}^{(r+1)}$ of $Q_j^{(r)}$ that have disjoint closures $(j = 1, \cdots, 2^r)$. Let $x_k^{(r+1)}$ $(k = 1, \cdots, 2^{r+1})$ be points in $W_k^{(r+1)}$ such that the set $\{x_1^{(r+1)}, \cdots, x_{2^{r+1}}^{(r+1)}\}$ satisfies the Kroneckerian condition. It is clear from the Kroneckerian property that we can find a finite set Y_{r+1} of characters of G with the following property. Consider any sequence $\{u_1, \cdots, u_{2^{r+1}}\}$ of complex numbers each of which is a value of a character of G. Then there is a character $\chi \in Y_{r+1}$ such that $| \chi(x_k) - u_k | < \varepsilon_{r+1}/2$ $(k = 1, \cdots, 2^{r+1})$. Now let $Q_k^{(r+1)}$ be defined by

$$Q_k^{(r+1)} = \bigcap \{x : x \in G, | \chi(x) - \chi(x_k) | \leqq \varepsilon_{r+1}/2\} \cap \{x : x \in G, d(x, x_k^{(r+1)}) \leqq 1/(r+1)\} \cap \overline{W_k^{(r+1)}}, \tag{5.5.1}$$

where the first intersection is taken over all $\chi \in Y_{r+1}$ $(k = 1, \cdots, 2^{r+1})$.

We have thus defined by induction the sets $Q_1^{(r)}, \cdots, Q_{2^r}^{(r)}$ for every nonnegative integer r. The sets $Q_k^{(r)}$ are compact neighborhoods, are pairwise disjoint for each fixed r, and have the property that $Q_{2j-1}^{(r+1)} \cup Q_{2j}^{(r+1)} \subset Q_j^{(r)}$ $(r = 0, 1, 2, \cdots; j = 1, 2, \cdots, 2^r)$. They have a further vital property, to wit: if $\{u_1, \cdots, u_{2^r}\}$ is any sequence of complex numbers each of which is the value of some character of G, then there is a character χ of G such that $| \chi(x) - u_j | \leqq \varepsilon_r$ for all $x \in Q_j^{(r)}$ $(j = 1, \cdots, 2^r)$.

Finally we define Q as the set $\bigcap_{r=1}^{\infty} (\bigcup_{j=1}^{2^r} Q_j^{(r)})$.

5.6. It is easy to see that the set Q just defined is homeomorphic to Cantor's ternary set. It is also easy to see that continuous functions of absolute value 1 on Q can be approximated by characters as follows. If G is Kroneckerian and has arbitrarily small elements of infinite order, let f be any continuous complex-valued function on Q such that $|f| = 1$, and let η be any positive number. Then there is a character $\chi \in X$ such that $|f(x) - \chi(x)| < \eta$ for all $x \in Q$. If $G = D_a$, then any continuous function f on Q whose range is contained in the set $\{1, \exp(2\pi i/a), \cdots, \exp(2(a-1)\pi i/a)\}$ is actually equal to a character of G on Q. Note also that we have constructed sets Q in arbitrary nonvoid open subsets of R, D_a, and compact metric groups containing arbitrarily small elements of infinite order.

5.7. We will now show that sets Q with the properties described in 5.6 can be constructed in every nonvoid open subset of an arbitrary group G. Suppose first that G contains arbitrarily small elements of infinite order. Let W be any open subset of G with compact closure, and let G_1 be an open and closed compactly generated subgroup of G that contains W. (One can take G_1 as $\bigcup_{n=1}^{\infty} nY$, where $Y = \overline{W} \cup \overline{V} \cup (-\overline{W}) \cup (-\overline{V})$, V being any neighborhood of 0 in G with compact closure.) A well-known structure theorem, already referred to in 2.2, asserts that $G_1 = Z^k + R^l + G_2$, where G_2 is compact. If l is positive, then every open subset of G_1, and hence in particular W, contains an open interval from the real line R, and this interval contains a set Q as constructed in 5.5. Since every continuous character of the closed subgroup R of G_1 admits an extension over G that is a character of G, we have our set Q in case l is positive. If $l = 0$, that is, if G_1 fails to admit R as a direct summand, then G_2 must be an infinite compact group containing arbitrarily small elements of infinite order.

Suppose that the open set W is contained in G_2 and that u is any point of W. We wish to show that there is a compact metric subgroup H of G_2 having arbitrarily small elements of infinite order such that $H \cap W \neq \emptyset$. Consider the discrete character group X_2 of G_2. Since G_2 as the character group of X_2 has the topology of pointwise convergence on X_2, we need to show that, given the character u of X_2, there exists a character x of X_2 that is arbitrarily close to u on a preassigned finite subset $\{\chi_1, \cdots, \chi_m\}$ of X_2 and also generates a metric subgroup of G_2 having arbitrarily small elements of infinite order. Since G_2 has arbitrarily small elements of infinite order, X_2 is not of bounded order (see [10]). Let Y be any countable subgroup of X_2 that contains $\{\chi_1, \cdots, \chi_m\}$ and is of unbounded order. Let Y' be a countable divisible group containing Y (see [1], p. 65, Theorem 20.1). The identity mapping of Y onto Y can be extended to a homomorphism carrying X_2 into Y' ([1], p. 59, Theorem 16.1). Let X_0 be the kernel of this homomorphism. Then distinct elements of Y lie in distinct cosets modulo X_0. Let H be the (compact) character group of X_2/X_0. Plainly H is a compact subgroup of G_1. Since X_2/X_0 is of unbounded order, H has arbitrarily small elements of infinite order. Since X_2/X_0 is countable, H is metric. Since no new relations are introduced among the elements of Y by the homomorphism carrying X_2 onto X_2/X_0, there is a character of X_2/X_0, that is, an element of H, that agrees with u on the set $\{\chi_1, \cdots, \chi_m\}$.

Thus H has nonvoid intersection with a preassigned neighborhood of u. Construct a set Q as in 5.5 lying in $W \cap H$, which is a nonvoid open subset of H. Then any continuous function of absolute value 1 on Q can be arbitrarily uniformly approximated on Q by a character of H. This character can be extended to a character of G.

Now suppose that W lies in some coset of G_1 modulo Z^k different from G_2. There is a character of $Z^k + G_2$ that is identically 1 on Z^k and is an arbitrary character on G_2, so that here there is no problem in translating a set Q with preservation of its required properties.

We have thus shown that sets Q with the properties given in 5.6 exist in every nonvoid open subset of every group G that contains arbitrarily small elements of infinite order.

5.8. We must now deal with the case in which G has a neighborhood of 0 containing only elements of finite order. In every neighborhood of 0, there is a replica of the group D_a for some $a > 1$, as was pointed out in 2.2, and for D_a we have the construction of 5.5. Just as in 2.4, we see that upon translating these groups D_a, we can always arrange to have this translating done by a direct summand of of D_a, so that the required properties of $Q \subset D_a$ can be preserved by translation of Q into an arbitrary open subset of G.

5.9. We summarize the constructions of the present section. Let G be a group containing arbitrarily small elements of infinite order. Then every nonvoid open subset of G contains a set Q that is homeomorphic to Cantor's ternary set and has the property that every continuous function of absolute value 1 on Q can be arbitrarily uniformly approximated on Q by a character of G. Let G be a group having a neighborhood of 0 consisting solely of elements of finite order. Then every neighborhood of 0 in G contains a replica of some group D_a. For every such a and every nonvoid open subset W of G, there is a set Q homeomorphic to Cantor's ternary set contained in W such that every continuous function on Q with range contained in

$$\{1, \exp(2\pi i/a), \cdots, \exp(2(a-1)\pi i/a)\}$$

is equal to a character of G on Q.

6. A property of measures on Q

6.1. Let Q be any subset of G of the sort described in 5.9. Let Γ_0 and Γ_1 be as in 1.7. The following result, which may be of independent interest, is an analogue of Kronecker's approximation theorem, for finite sets of measures on Q.

6.2 THEOREM. *Let $\lambda_1, \cdots, \lambda_m$ be nonnegative continuous measures in $\mathfrak{M}(Q)$, and $x_1, \cdots, x_n$ points of Q such that the sets $C(\lambda_1), \cdots, C(\lambda_m), \{x_1\}, \cdots, \{x_n\}$ are pairwise disjoint (either λ's or x's may be absent). Let $z_1, \cdots, z_m$, $w_1, \cdots, w_n$ be complex numbers such that $z_j \in \lambda_j(G)\Gamma_1$ $(j = 1, \cdots, m)$ and $w_k \in \Gamma_0$ $(k = 1, \cdots, n)$. Let η be a positive number. Then there is a character χ of G such that*

6.2.1 $$\left| \int_G \chi(y)\, d\lambda_j(y) - z_j \right| < \eta \qquad (j = 1, \cdots, m)$$

and

6.2.2 $$| \chi(x_k) - w_k | < \eta \qquad (k = 1, \cdots, n).$$

Proof. We may obviously suppose that $\lambda_j(G) = 1$ for $j = 1, \cdots, m$. Write $C_j = C(\lambda_j)$ $(j = 1, \cdots, m)$. The sets $C_1, \cdots, C_m, \{x_1\}, \cdots, \{x_n\}$

are pairwise disjoint, the measures λ_j are continuous, and Q is homeomorphic to Cantor's ternary set. Hence we can find a dissection of Q into pairwise disjoint open and closed sets, say $D_1, \cdots, D_r$, such that no D_l intersects more than one of the sets $C_1, \cdots, C_m, \{x_1\}, \cdots, \{x_n\}$ and such that $\lambda_j(D_l) < \frac{1}{4}\eta$ for $j = 1, \cdots, m$ and $l = 1, \cdots, r$. Consider now a fixed λ_j and let $E_1^{(j)}, \cdots, E_{m_j}^{(j)}$ be those sets D_l that intersect C_j, enumerated in some fixed order. Let $F^{(k)}$ be the set D_l that contains x_k $(k = 1, \cdots, n)$. Plainly no set D_l appears more than once among the E's and the F's.

We have $\sum_{u=1}^{m_j} \lambda_j(E_u^{(j)}) = 1$. Consider first the case in which G contains arbitrarily small elements of infinite order. We look for a character $\chi \in X$ for which 6.2.1 and 6.2.2. hold. We discuss 6.2.1 first. For the indices j such that $|z_j| = 1$, we require that

6.2.3 $$|\chi(x) - z_j| \leqq \tfrac{1}{2}\eta \quad \text{for all } x \in E_1^{(j)} \cup \cdots \cup E_{m_j}^{(j)}.$$

For the indices j such that $0 < |z_j| < 1$, let a_j and b_j be the complex numbers such that $|a_j| = |b_j| = 1$ and $z_j = \frac{1}{2}(a_j + b_j)$. For the indices j such that $z_j = 0$, let $a_j = -1$ and $b_j = 1$. For all indices j such that $|z_j| < 1$, let l_j be the greatest among the integers l for which $\sum_{u=1}^{l} \lambda_j(E_u^{(j)}) \leqq \frac{1}{2}$. Write $\rho_j = \sum_{u=1}^{l_j} \lambda_j(E_u^{(j)})$. For all indices j such that $|z_j| < 1$, we require further of the character χ that

6.2.4 $$|\chi(x) - a_j| < \tfrac{1}{4}\eta \qquad \text{for all } x \in E_1^{(j)} \cup \cdots \cup E_{l_j}^{(j)}$$

and

6.2.5 $$|\chi(x) - b_j| \leqq \tfrac{1}{4}\eta \qquad \text{for all } x \in E_{l_j+1}^{(j)} \cup \cdots \cup E_{m_j}^{(j)}.$$

We require finally of the character χ that

6.2.6 $$|\chi(x) - w_k| < \eta \qquad \text{for all } x \in F^{(k)} \ (k = 1, \cdots, n).$$

There is no inconsistency among the requirements 6.2.3–6.2.6, and they can all be satisfied by a single character χ of G, in view of the properties of Q (see 5.9).

For indices j such that $|z_j| = 1$, we have

6.2.7 $$\left| \int_G \chi(x)\, d\lambda_j(x) - z_j \right| \leqq \sum_{u=1}^{m_j} \int_{E_u^{(j)}} |\chi(x) - z_j|\, d\lambda_j(x) \leqq \tfrac{1}{2}\eta < \eta.$$

For indices j such that $|z_j| < 1$, we have

6.2.8 $$\begin{aligned} \left| \int_G \chi(x)\, d\lambda_j(x) - z_j \right| &\leqq \left| \sum_{u=1}^{l_j} \int_{E_u^{(j)}} \chi(x)\, d\lambda_j(x) - \tfrac{1}{2}a_j \right| \\ + \left| \sum_{u=l_j+1}^{m_j} \int_{E_u^{(j)}} \chi(x)\, d\lambda_j(x) - \tfrac{1}{2} b_j \right| &\leqq \sum_{u=1}^{l_j} \int_{E_u^{(j)}} |\chi(x) - a_j|\, d\lambda_j(x) \\ + \sum_{u=l_j+1}^{m_j} \int_{E_u^{(j)}} |\chi(x) - b_j|\, d\lambda_j(x) + 2\,|\rho_j - \tfrac{1}{2}| &\leqq \tfrac{1}{2}\eta + 2\,|\rho_j - \tfrac{1}{2}| < \eta. \end{aligned}$$

The inequalities 6.2.7 and 6.2.8 are just 6.2.1. Inequality 6.2.2 is obviously satisfied for the present choice of χ. This completes the proof for the case in which G contains arbitrarily small elements of infinite order.

Suppose finally that every neighborhood of 0 in G contains a replica of D_a for some $a = 2, 3, \cdots$. Here we have $z_j = \sum_{v=0}^{a-1} b_v^{(j)} \exp(2\pi iv/a)$, where the $b_v^{(j)}$ are nonnegative numbers such that $\sum_{v=0}^{a-1} b_v^{(j)} = 1$, uniquely determined by z_j $(j = 1, \cdots, m)$. Also each w_k is one of the numbers $1, \exp(2\pi i/a), \cdots, \exp(2(a-1)\pi i/a)$. The proof is a repetition of the preceding case, with a_j and b_j replaced by the set

$$\{1, \exp(2\pi i/a), \cdots, \exp(2(a-1)\pi i/a)\}.$$

We omit the details.

7. Proof of Theorem 1.8

7.1. Let G be any group, Γ_0 and Γ_1 as in 1.7, and Q as in 5.9.

7.2. Let L be a linear functional on $\mathfrak{M}(Q)$ satisfying the hypotheses of Theorem 1.8. Let $\{\mu_1, \cdots, \mu_m\}$ be any finite subset of $\mathfrak{M}(Q)$, and η any positive number. Let $\Delta(\mu_1, \cdots, \mu_m : \eta)$ be the set of all $\chi \in X$ such that

7.2.1 $$| \mu_j(\chi) - L(\mu_j) | < \eta \qquad \text{for } j = 1, \cdots, m.$$

If $\Delta(\mu_1, \cdots, \mu_m : \eta)$ is nonvoid for all choices of $\mu_1, \cdots, \mu_m$ and η, then the set

7.2.2 $$\bigcap \Delta(\mu_1, \cdots, \mu_m : \eta)^- = \mathbf{I}_L$$

is nonvoid, where the intersection is taken over all $\{\mu_1, \cdots, \mu_m\}$ and $\eta > 0$ (the closure is in the space $\mathbf{S}$). This follows at once from the compactness of $\mathbf{S}$ and the finite intersection property of the sets $\Delta(\mu_1, \cdots, \mu_m : \eta)$. Now let M be any multiplicative linear functional in the set $\mathbf{I}_L$. It is obvious that $M(\mu) = L(\mu)$ for all $\mu \in \mathfrak{M}(Q)$ and that $M \in X^-$.

We have thus only to prove that the set $\Delta(\mu_1, \cdots, \mu_m : \eta)$ is nonvoid for each $\{\mu_1, \cdots, \mu_m\} \subset \mathfrak{M}(Q)$ and $\eta > 0$. As in the proof of Theorem 1.6, we make a number of reductions. The first of these is the trivial reduction to the case in which all μ_j are nonnegative.

7.3. Our second reduction is to the case where the sets $C(\mu_1), \cdots, C(\mu_m)$ are pairwise disjoint. In fact, every set $\Delta(\mu_1, \cdots, \mu_m : \eta)$ contains a set $\Delta(\lambda_1, \cdots, \lambda_n : \zeta)$ such that the sets $C(\lambda_1), \cdots, C(\lambda_n)$ are pairwise disjoint, the λ_k's are nonnegative measures in $\mathfrak{M}(Q)$, and ζ is a positive number. This is proved from Theorem 3.2 and the linearity of L by a simple computation, which we omit.

7.4. Our third and last reduction is to the case where each λ_k is either a continuous measure of total measure 1 or a measure ε_k with $x \in Q$. This reduction is accomplished by an argument like that used in proving Lemma

3.4. We omit the details. Changing our notation, we thus have to prove that $\Delta(\lambda_1, \cdots, \lambda_m, \varepsilon_{x_1}, \cdots, \varepsilon_{x_n} : \eta) \neq \emptyset$, where the sets $C(\lambda_1), \cdots, C(\lambda_m)$, $\{x_1\}, \cdots, \{x_n\}$ are pairwise disjoint subsets of Q and $\lambda_1(G) = \cdots = \lambda_m(G) = 1$. This is just Theorem 6.2.

8. Some consequences of Theorem 1.8

We observe first that Theorems 4.1, 4.2, 4.8, and 4.9 remain true with $P \cup (-P)$ replaced by Q. Note too that Theorem 1.8 cannot be used to prove the asymmetry of $\mathfrak{M}(G)$, since the multiplicative linear functionals constructed in Theorem 1.8 lie in X^- and necessarily satisfy the relation $M(\tilde{\lambda}) = \overline{M(\lambda)}$ for all $\lambda \in \mathfrak{M}(G)$.

8.1 THEOREM. *Let G be a group containing arbitrarily small elements of infinite order, and let $\lambda_1, \cdots, \lambda_n$ be nonnegative, pairwise singular measures in $\mathfrak{M}_c(Q)$ such that $\lambda_k(G) = 1$ $(k = 1, \cdots, n)$. For every sequence of complex numbers $\{z_1, \cdots, z_n\}$, each of absolute value $\leqq 1$, there is a multiplicative linear functional M on $\mathfrak{M}(G)$ such that $M \in X^-$ and $M(\lambda_k) = z_k$ $(k = 1, \cdots, n)$.*

8.2 THEOREM. *Let G contain arbitrarily small replicas of D_a $(a = 2, 3, \cdots)$. Let Γ_1 be as in 1.7. Theorem 8.1 holds for G, if the numbers $z_1, \cdots, z_n$ lie in Γ_1.*

The proofs of Theorems 8.1 and 8.2 are very like the proof of Theorem 4.7. We omit the details.

8.3 THEOREM. *Let Γ_0 and Γ_1 be as in 1.7. Let φ be any function on Q with range contained in Γ_0, and let L_0 be any linear functional on $\mathfrak{M}_c(Q)$ such that $L_0(\lambda) \in \Gamma_1$ if $\lambda \in \mathfrak{M}_c(Q)$, $\lambda \geqq 0$, and $\lambda(G) \leqq 1$. Then there is a multiplicative linear functional M on $\mathfrak{M}(G)$ such that $M \in X^-$, $M(\varepsilon_x) = \varphi(x)$ for all $x \in Q$, and $M(\lambda) = L_0(\lambda)$ for all $\lambda \in \mathfrak{M}_c(Q)$.*

Proof. For $\mu \in \mathfrak{M}(Q)$, write $\mu = \mu_c + \sum_{l=1}^{\infty} t_l \varepsilon_{x_l}$, and define $L(\mu) = L_0(\mu_c) + \sum_{l=1}^{\infty} t_l \varphi(x_l)$. Then L is well-defined, is linear, and satisfies the hypotheses of Theorem 1.8.

8.4. *Other multiplicative extensions of L.* Let βX be the Stone-Čech compactification of the completely regular space X, and let X^- be the closure of X in the compact Hausdorff space $\mathbf{S}$. The identity map ι of X onto itself admits a continuous extension ι_0 mapping βX onto X^- (see for example [5], p. 153, Theorem 24). Let L and M be as in Theorem 1.8, and let p be any point of βX lying in $\iota_0^{-1}(M)$. It is easy to see that the evaluation $f(p)$ is a multiplicative linear extension of the linear functional L over the algebra $\mathfrak{C}(X)$ of all bounded continuous complex-valued functions on X. Using X with its discrete topology, denoted by X_d, we can similarly extend L to be a multiplicative linear functional on the algebra $\mathfrak{C}(X_d)$ of all bounded complex-valued functions defined on X. Thus we find infinite-dimensional linear subspaces $\mathfrak{F}$ of $\mathfrak{C}(X)$ and $\mathfrak{C}(X_d)$ such that all linear functionals on $\mathfrak{F}$ satisfying certain weak conditions are actually evaluation at points of βX or $\beta(X_d)$.

8.5. Theorem 1.8 is not strictly a generalization of Šreĭder's theorem quoted in 1.4, since Šreĭder's multiplicative linear functional is exhibited as a limit of a sequence of values of Fourier-Stieltjes transforms, while Theorem 1.8 exhibits the multiplicative linear functional M only as an element of X^{-}. If we limit ourselves to separable subspaces of $\mathfrak{M}(Q)$, we can produce similar representations for our M. Note too that Šreĭder's measures have carriers contained in Cantor's ternary set, while ours have carriers contained in the pathological set Q. Distinct improvements in Šreĭder's results for Cantor's ternary set can be obtained, however, and we hope to discuss these in another communication.

LITERATURE

1. L. FUCHS, *Abelian groups*, Budapest, Publishing House of the Hungarian Academy of Sciences, 1958.
2. PAUL R. HALMOS, *Measure theory*, New York, D. Van Nostrand Co., 1950.
3. EDWIN HEWITT, *The asymmetry of certain algebras of Fourier-Stieltjes transforms*, Michigan Math. J., vol. 5 (1958), pp. 149–158.
4. EDWIN HEWITT AND H. S. ZUCKERMAN, *A group-theoretic method in approximation theory*, Ann. of Math. (2), vol. 52 (1950), pp. 557–567.
5. JOHN L. KELLEY, *General topology*, New York, D. Van Nostrand Co., 1955.
6. LYNN H. LOOMIS, *An introduction to abstract harmonic analysis*, New York, D. Van Nostrand Co., 1953.
7. MASAHIRO NAKAMURA AND SHIZUO KAKUTANI, *Banach limits and the Čech compactification of a countable discrete set*, Proc. Imp. Acad. Tokyo, vol. 19 (1943), pp. 224–229.
8. JOHN v. NEUMANN, *Ein System algebraisch unabhängiger Zahlen*, Math. Ann., vol. 99 (1928), pp. 134–141.
9. L. S. PONTRYAGIN, *Nepreryvnye gruppy*, 2nd ed., Moscow, Gostehizdat, 1954.
10. WALTER RUDIN, *Independent perfect sets in groups*, Michigan Math. J., vol. 5 (1958), pp. 159–161.
11. ———, *Measure algebras on abelian groups*, Bull. Amer. Math. Soc., vol. 65 (1959), pp. 227–247.
12. YU. A. ŠREĬDER, *Ob odnom primere obobščennogo haraktera*, Mat. Sbornik, vol. 29(71) (1951), pp. 419–426.
13. ———, *Stroenie maksimal'nyh idealov v kol'cah mer co svertkoĭ*, Mat. Sbornik, vol. 27(69) (1950), pp. 297–318.
14. KARL R. STROMBERG, *A note on the convolution of regular measures*, Math. Scand., vol. 7 (1959), pp. 347–352.
15. N. WIENER AND H. R. PITT, *On absolutely convergent Fourier-Stieltjes transforms*, Duke Math. J., vol. 4 (1938), pp. 420–436.
16. JOHN H. WILLIAMSON, *A theorem on algebras of measures on topological groups*, Proc. Edinburgh Math. Soc., vol. 11 (1959), pp. 195–206.

THE UNIVERSITY OF WASHINGTON
SEATTLE, WASHINGTON
YALE UNIVERSITY
NEW HAVEN, CONNECTICUT