TOPOLOGY OF MAPPINGS AND DIFFERENTIATION PROCESSES[1]

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Notations and terminology

The letters X and Y denote sets or topological spaces. We consider a topology τ of X as a family of subsets of X. With this in mind, set-theoretic notations should cause no confusion. For instance, $U \in \tau$ means "U is τ-open". If Σ_1 and Σ_2 are families of subsets of X, $\Sigma_1 \wedge \Sigma_2$ is defined as the family $\{B \mid B = A_1 \cap A_2, A_1 \in \Sigma_1 \text{ and } A_2 \in \Sigma_2\}$. Somewhat inaccurately, if $K \subset X$, $\tau \wedge K$ stands for $\tau \wedge \{K\}$, the relative topology of K induced by the topology τ of X.

The symbol R denotes the field of real numbers. The term *vector space*, except for an explicit statement to the contrary, should be interpreted to mean a *real* vector space. The letters E and F denote vector spaces or topological vector spaces; t.v.s. and l.c.t.v.s. abbreviate *topological vector space* and *locally convex t.v.s.*, respectively.

Introduction

The derivative of a mapping f of a subset H of a l.c.t.v.s. E into another l.c.t.v.s. F at $h_0 \in H$ is defined by regarding the expression $[f(h_0 + \lambda x) - f(h_0)]/\lambda$, where $\lambda \in R$ and x ranges over E, as a mapping $\delta_\lambda f(h_0): E \to F$. In case $h_0 + \lambda x \notin H$, we define $\{x, \delta_\lambda f(h_0)\}$, the image of x under $\delta_\lambda f(h_0)$, as 0. Now $\delta_\lambda f(h_0) \in \mathfrak{F}(E, F)$, the space of all (not necessarily continuous) mappings of E into F. Suppose, for some topology τ of $\mathfrak{F}(E, F)$, $\lim_{\lambda \to 0} \delta_\lambda f(h_0) = f'(h_0) \in \mathfrak{F}(E, F)$; then we call $f'(h_0)$ the τ-derivative of f at h_0. Most of the terminology and notations connected with differentiation of functions of one real variable can now be introduced naturally. Among them is the derivative function $f': H \to \mathfrak{F}(E, F)$.

Of interest are the Σ-derivatives, each defined as the τ-derivative, where τ is the topology of uniform convergence over the sets of a family Σ of subsets

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[1] This work contains material presented to the American Mathematical Society in the form of abstracts, April 23, 1955, February 25, 1956, and April 12, 1956. While this paper was in preparation, it came to our attention that some of the results in Section 5 were obtained independently by J. Sebastião e Silva, Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Nat. (8), vol. 20 (1956), pp. 743–750, and vol. 21 (1956), pp. 40–46 and pp. 172–178.

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of E. Among these are the point-open (Σ consists of all finite sets), and the bounded-open (Σ consists of all bounded sets) derivatives.

For $E = R$, f' is the ordinary derivative, if Σ covers E and consists of bounded sets and F is identified naturally with $\mathfrak{L}(R, F)$, the space of all continuous linear transformations of R into F. For general E, it is customary in the classical scheme of things to talk about the differential of f, referred to as the variation if it is not linear in the increment variable. The Σ-variation $df: H \times E \to F$ of f is defined as $df(h, x) = \{x, f'(h)\}$. The point-open variation is the classical Gâteaux variation, and the bounded-open variation is the classical Fréchet variation in normed spaces (Theorem 13). In analogy to the fundamental theorem of integral calculus, but without any assumption of integrability, Theorem 14 states that the increment $f(h + x) - f(h)$ is the limit of Riemann-type sums of $df(h + \lambda x, x)$, where λ is a real parameter.

Now we turn to the relation between compactness of the mapping f' and the continuity properties of f. We generalize to mappings $f: H \to F$, with $f'(H) \subset \mathfrak{L}(E, F)$, the work of E. H. Rothe [17] for the case $F = R$ and $f'(H) \subset E'$, where E' is the dual of E. Rothe proved that compactness of f' imposes two conditions on f. One is continuity in the weak bounded topology, the strongest topology of E coinciding with the weak topology of E on every bounded set. This is an example of a general topology $(\Sigma - \tau)$ associated with a topology τ of E and a family Σ of subsets of E, defined as the strongest topology that induces the topology $\tau \wedge A$ on every element $A \in \Sigma$, (see Section 1). This topology is, in general, not linear. The other condition on f is roughly that the increment ratio is near zero along directions in the vicinity of the polar of a suitable finite-dimensional subspace of E' (see condition **a**$'$ and Theorem 11 below).

In the general situation (see Sections 4 and 6) the weak topology of E is replaced by a point-open topology of E considered as a subspace of $\mathfrak{L}[\mathfrak{L}(E, F), F]$. It was discovered that, by slightly strengthening the formulation of the second condition above (or, rather, its generalization) in a natural way, a condition (condition **a**) was obtained that also includes the generalization of the first one. In a similar manner, a condition **b** is defined for mappings $g: H \to \mathfrak{L}(E, F)$. Our goal is to show that compactness of f' implies condition **a** on f. This is done in two steps. One consists in proving that compactness of $g: H \to \mathfrak{L}(E, F)$ implies condition **b** on g (Theorem 12). The second step consists in showing that condition **a** on f is equivalent to condition **b** on f' (Theorem 15). A special case of this last result is contained implicitly in the work of Rothe.

Somewhat off the main stream, the following material is presented. Since in differentiating f we place the emphasis on $f' \in \mathfrak{F}[H, \mathfrak{F}(E, F)]$, rather than on $df \in \mathfrak{F}(H \times E, F)$ as is done elsewhere, it would appear interesting to give an algebraic characterization of the passage from one to the other. This is done in a more general setting in Theorem 4. Theorem 6 states that, if F is complete, the space of compact transformations of H into F is closed in $\mathfrak{F}(H, F)$ under the bounded-open topology. From this it follows (Theorem 8) that if H is bounded and F complete, a mapping of H into F is compact if and only if it can be approximated arbitrarily closely and uniformly by finite-dimensional compact mappings.

PART I

1. The topologies $(\Sigma - \tau)$ and $(\Sigma - \tau)_c$

Let Σ be a family of subsets of X and τ a topology of X. The topology $(\Sigma - \tau)$ is defined as the strongest topology of X coinciding with $\tau \wedge A$ for every $A \in \Sigma$. A set $U \in (\Sigma - \tau)$ if and only if $U \cap A \in \tau \wedge A$ for every $A \in \Sigma$. If τ_1 and τ_2 are topologies of X, and $\tau_1 \subset \tau_2$, then $(\Sigma - \tau_1) \subset (\Sigma - \tau_2)$. If every set of Σ_1 is contained in a set of Σ_2, then $(\Sigma_1 - \tau) \supset (\Sigma_2 - \tau)$. Let τ be a topology of E. τ is said to be *semilinear* if $x + y$ and λx ($\lambda \in R$, x and $y \in E$) are continuous in each variable separately; *linear* if $x + y$ and λx are continuous in both variables jointly. Let τ_c denote the topology of E with a base consisting of all convex sets of τ. Then, if τ is semilinear, τ_c is linear. If τ is semilinear, $(\Sigma - \tau)$ is semilinear provided (i) every translate of every element of Σ is contained in some element of Σ, (ii) every scalar multiple of a set of Σ is contained in some element of Σ, (iii) every $x \in E$ is absorbed ([4], p. 6, Definition 3) by some element of Σ. The details of this material are given by Collins in [7], Part II.

THEOREM 1. *Let τ be a topology of X, Σ a family of subsets of X, and Y a topological space.* (a) *If for every $A \in \Sigma$, the τ-closure of A is contained in some element of Σ, then, for every $A \in \Sigma$, we have $cA \in (\Sigma - \tau)$ if and only if $cA \in \tau$.* (b) *A mapping $f:x \rightarrow Y$ is $(\Sigma - \tau)$-continuous if and only if, for every $A \in \Sigma$, $f \mid A$ (the restriction of f to A) is $\tau \wedge A$-continuous.*

Proof. (a) Let $A \in \Sigma$ and $cA \in (\Sigma - \tau)$. Denote by $\bar{A}$ the τ-closure of A. There exists $B \in \Sigma$ such that $\bar{A} \subset B$. Then there exists $K \subset X$, with $cK \in \tau$, such that $A = A \cap \bar{A} = A \cap B = K \cap B = K \cap \bar{A}$. Hence $cA = c(E \cap \bar{A})$. The converse is obvious.

(b) f is $(\Sigma - \tau)$-continuous if and only if, for every open set U of Y and every $A \in \Sigma$, $A \cap f^{-1}(U) \in \tau \wedge A$. But this is equivalent to $\tau \wedge A$-continuity of $f \mid A$ for every $A \in \Sigma$.

Let τ be a linear topology of a vector space E, Σ_n the family of all n-dimensional linear varieties of E, and $\Sigma_\infty = \bigcup_1^\infty \Sigma_n$. Clearly Σ_n satisfies conditions (i) through (iii) above if the dimension of E is at least n, and so does Σ_∞ if E is infinite-dimensional, so that for the corresponding dimensions $(\Sigma_n - \tau)$ and $(\Sigma_\infty - \tau)$ are semilinear. $(\Sigma_n - \tau)$ is called the *n-dimensional topology* of E. Since every linear topology of E induces the same topology on every finite-dimensional subspace of E, $(\Sigma_n - \tau)$ is independent of the choice of τ, and so is $(\Sigma_\infty - \tau)$ because $(\Sigma_\infty - \tau) = \bigcup_1^\infty (\Sigma_n - \tau)$. Consequently, we denote these topologies respectively by (Σ_n) and (Σ_∞). Let $n > m$. Since every m-dimensional variety is contained in an n-dimensional variety, $(\Sigma_n) \subset (\Sigma_m)$. This inclusion is actually strict. (Σ_∞) consists of all sets that are finitely open in the sense of E. Hille ([12], p. 71; see also [15]). (Σ_1) is the radial or core topology of V. L. Klee ([14], p. 446).

Let $K \subset E$; let τ be a locally convex linear topology of E, and Σ the family of all bounded subsets of K. Then $(\tau \wedge K)' = (\Sigma - \tau \wedge K)$ is called the *τ-bounded topology* of K. In case $K = E$ we have $\tau' = (\tau \wedge E)'$. τ_c' is called the *τ-bounded locally convex topology* of E.

If τ is the weak topology of a l.c.t.v.s. E, $(\tau \wedge K)'$ is called the *bounded weak topology* of K. If E' is the dual of E, γ the weak* topology of E', and $H \subset E'$, $(\gamma \wedge H)'$ is called the *bounded weak* topology* of H. The bounded weak* topology of E' was introduced by Alaoglu [1] for normed spaces, and proved by Dieudonné and Schwartz ([9], Theorem 5, p. 84) to be the compact open topology of E', provided E is an $\mathfrak{F}$-space.

2. Spaces of mappings

Let $X_1, X_2, \cdots, X_n$ be arbitrary sets and X_{n+1} a uniform space. We denote by $\mathfrak{F}(X_n, X_{n+1})$ the space of all mappings $f: X_n \to X_{n+1}$, and define $\mathfrak{F}(X_1, X_2, \cdots, X_n, X_{n+1})$ recursively as $\mathfrak{F}[X_1, \mathfrak{F}(X_2, X_3, \cdots, X_n, X_{n+1})]$. If $X_1 = X_2 = \cdots = X_n = X$, and $X_{n+1} = Y$, we denote this space by $\mathfrak{F}_n(X, Y)$.

Let E and F be t.v.s.'s. The space $\mathfrak{F}_n(E, F)$ is of particular interest in connection with differentiation of a mapping $f: E \to F$. In Section 5 we define the derivative of f as an element f' of $\mathfrak{F}_2(E, F)$. In general, the n^{th} derivative $f^{(n)} \in \mathfrak{F}_{n+1}(E, F)$.

Let Σ_i be a family of subsets of X_i, $i = 1, 2 \cdots, n$. We refer to the uniform structure of $\mathfrak{F}(X_n, X_{n+1})$ of uniform convergence over the sets of Σ_n ([3], p. 2) as the Σ_n-uniform structure of $\mathfrak{F}(X_n, X_{n+1})$ and define the $\Sigma_1 \Sigma_2 \cdots \Sigma_n$-uniform structure of $\mathfrak{F}(X_1, X_2 \cdots, X_n, X_{n+1})$ recursively as the Σ_1-uniform structure of $\mathfrak{F}[X_1, \mathfrak{F}(X_2, X_3, \cdots, X_n, X_{n+1})]$ if we give $\mathfrak{F}(X_2, X_3, \cdots X_n, X_{n+1})$ its $\Sigma_2 \Sigma_3 \cdots \Sigma_n$-uniform structure. The $\Sigma_1 \Sigma_2 \cdots \Sigma_n$-topology is the corresponding topology.

For any mapping f we may use the notation $\{x, f\}$ in the place of $f(x)$. For instance, if $f \in \mathfrak{F}(X_1, X_2, X_3)$ and $x_1 \in X_1$, $x_2 \in X_2$, then $f(x_1): X_2 \to X_3$, and $\{x_2, f(x_1)\}$ is the image of x_2 under $f(x_1)$. We now define $M_1: \mathfrak{F}(X_1, X_{n+1}) \to \mathfrak{F}(X_1, X_{n+1})$ as the identity mapping and $M_n: \mathfrak{F}(X_1, X_2, \cdots, X_n, X_{n+1}) \to \mathfrak{F}(X_1 \times X_2 \times \cdots \times X_n, X_{n+1})$ by recursion as $\{(x_1, x_2, \cdots, x_n), M_n f\} = \{(x_2, x_3, \cdots, x_n), M_{n-1}[f(x_1)]\}$.

If X and Y are uniform spaces, we say $f: X \to Y$ is a *uniform* mapping if f is one-to-one and onto, and both f and f^{-1} are uniformly continuous. If Σ_i is a family of subsets of X_i for $i = 1, 2, \cdots, n$, we denote by $\Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_n$ the family of subsets of $X_1 \times X_2 \times \cdots \times X_n$ of the form $A_1 \times A_2 \times \cdots \times A_n$, $A_i \in \Sigma_i$.

Theorem 2, given without proof, will permit the passage from $f^{(n)}$ to the n^{th} differential $d_n f$ by $d_n f = M_{n+1} f^{(n)}$.

THEOREM 2. *M_n is a uniform mapping of $\mathfrak{F}(X_1, X_2, \cdots, X_n, X_{n+1})$ with its $\Sigma_1 \Sigma_2 \cdots \Sigma_n$-uniform structure onto $\mathfrak{F}(X_1 \times X_2 \times \cdots \times X_n, X_{n+1})$ with its $\Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_n$-uniform structure. If X_{n+1} is a vector space, so are the range and domain of M_n, and M_n is linear.*

3. Vector spaces of mappings

Let τ be a topology of E. We say that E and τ form a *topological vector group* (t.v.g.) if $x + y$ is jointly continuous for x and $y \in E$, and $\lambda_0 x$ is continuous in $x \in X$ for every fixed $\lambda_0 \in R$. If E is a t.v.g. and Σ a family of subsets of X, then $\mathfrak{F}(X, E)$ is also a t.v.g. under the Σ-topology, but not necessarily a t.v.s. If E is a t.v.s., $\mathfrak{F}(X, E)$ is as t.v.s. if and only if $f(A)$ is bounded for $A \in \Sigma$ and $f \in \mathfrak{F}(X, E)$.

If E and F are two t.v.g.'s, we denote by $\mathfrak{L}(E, F)$ the vector space of all linear continuous mappings of E into F. The *point-open* topology of $\mathfrak{F}_n(E, F)$ is its $\Sigma_1 \Sigma_2 \cdots \Sigma_n$-topology, where each Σ_i is the family of all finite subsets of E. $\mathfrak{F}_n^P(E, F)$ denotes $\mathfrak{F}_n(E, F)$ with this topology, and similarly for $\mathfrak{L}^P(E, F)$.

We now define the mappings $N: E \to \mathfrak{L}[\mathfrak{L}^P(E, F), F]$ and $S_n: (E)^n \to \mathfrak{L}[\mathfrak{F}_n^P(E, F), F]$ as follows: $\{u, N(x)\} = u(x)$ for $x \in E$ and $u \in \mathfrak{L}(E, F)$; $\{f, S_1(x)\} = f(x)$ for $f \in \mathfrak{F}_1^P(E, F)$ and $x \in E$. For $n > 1$, we define $P_n: (E)^n \to \mathfrak{L}[\mathfrak{F}^P((E)^n, F), F]$ as $\{g, P_n(x_1, x_2, \cdots, x_n)\} = g(x_1, x_2, \cdots, x_n)$, for $g \in \mathfrak{F}^P[(E)^n, F]$. Then we set

$$\{f, S_n(x_1, x_2, \cdots, x_n)\} = \{M_n f, P_n(x_1, x_2, \cdots, x_n)\} \text{ for } f \in \mathfrak{F}_n^P(E, F).$$

Theorem 3 is stated without proof.

THEOREM 3. *If the dimension of F is at least one, N is an algebraic embedding, and S_n is a set-theoretic embedding.*

Let Σ be a family of subsets of $\mathfrak{L}(E, F)$. Under the embedding N, the Σ-topology of $\mathfrak{L}[\mathfrak{L}^P(E, F), F]$ induces a relative topology on E, which we call the *F-Σ-topology of E*. In particular we will use the F-point-open topology of E, obtained by letting Σ be the family of all finite subsets of $\mathfrak{L}(E, F)$.

Let now E and F be two l.c.t.v.s's and τ the topology of E. The *bounded-open* topology of $\mathfrak{L}(E, F)$ is defined to be the Σ-topology, when Σ is the family of all bounded subsets of E. We denote $\mathfrak{L}(E, F)$ with this topology by $\mathfrak{L}^B(E, F)$. The *F-bounded-open* topology of E is the relative topology induced on E, under the embedding N, by the bounded-open topology of $\mathfrak{L}[\mathfrak{L}^B(E, F), F]$.

THEOREM 4. *The F-point-open topology of E is weaker than its original topology τ. If both E and F are l.c.t.v.s's, then τ is weaker than the F-bounded-open topology, and τ-bounded sets coincide with F-point-open-bounded sets.*

We conclude this section by describing certain closed (under various topologies) vector subgroups of $\mathfrak{F}(H, F)$, where $H \subset E$ and E and F are t.v.g.'s. Let Σ be a family of subsets of H and τ the original topology of E. Denote by $\mathcal{C}_\Sigma(H, F)$ the vector space of all $(\Sigma - \tau \wedge H)$-continuous mappings of H into F. If $A \subset H$ and $V \subset F$, we denote by $T(A, V)$ the set of all mappings $f:H \to F$ such that $f(A) \subset V$. Theorem 5 is given without proof.

THEOREM 5. *$\mathcal{C}_\Sigma(M, F)$ is a closed vector subgroup of $\mathfrak{F}(H, F)$ under the Σ-topology.*

Let $f:H \to E$; then $f(\Sigma)$ denotes the family of all $f(A)$ for $A \in \Sigma$, and if Γ is a family of subsets of F, $f^{-1}(\Gamma)$ is defined similarly. f is said to be *Σ-compact*, or *Σ-finite-dimensional* if for every $A \in \Sigma$, $f(A)$ is, respectively, relatively compact or finite-dimensional (is contained in a finite-dimensional vector subspace of F). f is *Σ-completely continuous* if it is Σ-compact and $(\Sigma - \tau \wedge H)$-continuous. If $H \in \Sigma$, we simply say f is H-compact, finite-dimensional, or H-completely continuous, as the case may be. If, on the other hand, E and F are l.c.t.v.s.'s and Σ is the family of all bounded subsets of H, we withhold Σ from the terms compact and completely continuous.

LEMMA 1. *Let X be a regular topological space and let $H \subset X$. A necessary and sufficient condition that H be relatively compact is that every filter of H be contained in a filter of H which converges to a point of X.*

We denote by $\mathcal{K}_\Sigma(H, F)$ the vector space of all Σ-compact mappings of H into F.

THEOREM 6. *If F is complete, $\mathcal{K}_\Sigma(H, F)$ is a closed subgroup of $\mathfrak{F}(H, F)$ under the Σ-topology.*

Proof. Let f_0 be in the Σ-topology-closure of $\mathcal{K}_\Sigma(H, F)$, $A \in \Sigma$, $B = f_0(A)$, Φ_1 a filter of B, and $g = f_0 \mid A$, the restriction of f_0 to A. $g^{-1}(\Phi_1)$ is a base of filter of A, which is contained in a maximal filter Ψ_1 of A. $g(\Psi_1)$ is a base of a filter Φ_2 of B finer than Φ_1. Φ_2 is a Cauchy (and therefore convergent) filter. To see this, let V and U be neighborhoods of $0 \in F$, such that $U + U + U \subset V$, and let $f \in \mathcal{K}_\Sigma(H, F)$ be such that $f - f_0 \in T(A, U)$. Now $f(\Psi_1)$ is a convergent (and therefore Cauchy) filter. Let $P \in \Psi_1$ be such that $f(P) - f(P) \subset U$. Then if x and $y \in P$,

$$f_0(x) - f_0(y) = f_0(x) - f(x) + f(x) - f(y) + f(y) - f_0(y) \in V,$$

so that $f_0(P) - f_0(P) \subset V$. Thus Φ_2 is a Cauchy filter. By Lemma 1, B is relatively compact and f_0 is compact.

A special case of Theorem 6 was given independently by J. W. Brace ([6], p. 172].

We denote by $\mathcal{E}(H, F)$ the vector space of H-compact finite-dimensional mappings of H into F, and by $\mathcal{K}(H, F)$ the vector space of H-compact mappings. The uniform topology of $\mathfrak{F}(H, F)$ is the Σ-topology where Σ consists of H alone.

THEOREM 7. *Let E be a t.v.g., $H \subset E$, and F a l.c.t.v.s. Then $\mathcal{E}(H, F)$ is dense in $\mathcal{K}(H, F)$ under the uniform topology.*

Proof. Let $f_0 \in \mathcal{K}(H, F)$, $B = f_0(H)$, and V a neighborhood of $0 \in F$. $\bar{B}$ is compact, and by a theorem of Nagumo ([16], p. 500, Theorem 2) there is a completely continuous finite-dimensional mapping $\phi:\bar{B} \to F$, such that $f_0 - \phi \circ f_0 \in T(H, V)$, where $\phi \circ f_0$ is the composed mapping. Our theorem follows from the fact that $\phi \circ f_0 \in \mathcal{E}(H, F)$ and that V is arbitrary.

From Theorem 7 a slightly stronger result (Theorem 8) can be proved, with the help of Theorem 6.

THEOREM 8. *Let $H \subset E$, E a.t.v.g., Σ a family of subsets of H, and F a l.c.t.v.s. Then a mapping $f:H \to F$ is Σ-compact (Σ-completely continuous) if and only if for every $A \in \Sigma$, $f \mid A$ can be approximated arbitrarily closely, and uniformly over A, by finite-dimensional A-compact (finite-dimensional A-completely continuous) mappings of A into F.*

4. Conditions a and b

In this section we let E and F be two t.v.g.'s, H a subset of E which is radially open (i.e., (Σ_1)-open), and Σ a family of subsets of E satisfying conditions (i) through (iii) (see Section 1), and the additional condition that for every $A \in \Sigma$, $A - A \in \Sigma$ also. Several notions will be defined relative to this family, which is considered fixed, and in applications is often taken to be the family of all bounded subsets of E. Unlike in previous sections, the symbol Σ is not included as part of the term defined. We denote M_2 (Section 2) by M.

Let $f \in \mathfrak{F}(H,F)$ and $0 < |\lambda| \leqq 1$. We define $\delta_\lambda f \in \mathfrak{F}(H, E, F)$, by defining $M\delta_\lambda f$ (Theorem 8) as $\{(h, x), M\delta_\lambda f\} = (1/\lambda)[f(h + \lambda x) - f(h)]$, for $h + \lambda x \in H$, and $\{(h, x), M\delta_\lambda f\} = 0$ otherwise. Let $A \subset H$. $\delta f(A)$ denotes the union of all the sets of the form $\delta_\lambda f(A)$ for $0 < |\lambda| \leqq 1$. Let Γ be the family of all subsets of $\mathfrak{F}(E, F)$ of the form $\delta f(A)$ for $A \in \Sigma \wedge H$. The *δf-topology* of E is defined as the relative topology induced on E by the Γ-topology of $\mathfrak{L}[\mathfrak{F}(E, F), F]$ under the embedding S_1 (Section 3). If $g:H \to \mathfrak{L}(E, F)$ and $\Pi = g(\Sigma \wedge H)$, we define the g-topology of E as the relative topology induced on E by the Π-topology of $\mathfrak{L}[\mathfrak{L}^P(E, F), F]$ under the embedding N. Throughout this section ω denotes the F-point-open topology of E, $\omega_{\delta f}$ the δf-topology, and ω_g the g-topology.

For every $A \in \Sigma \wedge H$, $f \mid A$ is uniformly continuous with respect to $\omega_{\delta f} \wedge A$ (more accurately, with respect to the underlying uniform structure), for if h_1, $h_2 \in A$, V is a neighborhood of $0 \in F$, and $S_1(h_1 - h_2) \in T[\delta f(A), V]$, then $\{(h_2, h_1 - h_2), M\delta_1 f\} = f(h_1) - f(h_2) \in V$. It follows that f is $(\Sigma \wedge H - \omega_{\delta f} \wedge H)$-continuous.

We say $f:H \to F$ satisfies *condition* **a** if for every $\omega_{\delta f}$-neighborhood W of $0 \in E$ and $B \in \Sigma$, there exists an ω-neighborhood Z of $0 \in E$, so that $B \cap Z \subset W$;

and that $g:H \to \mathfrak{L}(E, F)$ satisfies *condition* **b** if the same statement holds if we replace $\omega_{\delta f}$ by ω_g. Condition **a** implies $(\Sigma - \omega) \supset (\Sigma - \omega_{\delta f})$, and condition **b** implies $(\Sigma - \omega) \supset (\Sigma - \omega_g)$.

THEOREM 9. *If $f:H \to F$ satisfies condition* **a**, *then for every $A \in \Sigma \wedge H$, $f \mid A$ is uniformly $\omega \wedge A$-continuous. Hence f is $(\Sigma \wedge H - \omega \wedge H)$-continuous.*

Proof. Let $A = A_1 \cap H$, $A_1 \in \Sigma$, and V a neighborhood of $0 \in F$. Since f is uniformly $\omega_{\delta f} \wedge A$-continuous, there is an $\omega_{\delta f}$-neighborhood W of $0 \in E$ so that $h_1 - h_2 \in (A - A) \cap W$ implies $f(h_1) - f(h_2) \in V$. But $A - A \subset A_1 - A_1$, and, since $(\Sigma - \omega) \supset (\Sigma - \omega_{\delta f})$, there is an ω-neighborhood Z of $0 \in E$ so that $(A_1 - A_1) \cap Z \subset (A_1 - A_1) \cap W$. From this the theorem follows.

THEOREM 10. *If $g:H \to \mathfrak{L}(E, F)$ is $\Sigma \wedge H$-compact when we give $\mathfrak{L}(E, F)$ the Σ-topology, then g satisfies condition* **b**.

Proof. We may take an ω_g-neighborhood of $0 \in E$ to be of the form $N^{-1}T[g(A), V]$, where $A \in \Sigma \wedge H$ and V is a neighborhood of $0 \in F$. Let $B \in \Sigma$, and U a neighborhood of $0 \in F$, such that $U + U \subset V$. Since $g(A)$ is relatively compact with respect to the Σ-topology, there is a finite subset Q of $\mathfrak{L}(E, F)$ so that for every $h \in A$, $g(h) - u \in T(B, U)$ for some $u \in Q$. $N^{-1}[T(Q, U)]$ is an ω-neighborhood of $0 \in E$. If $x \in B \cap N^{-1}[T(Q, U)]$, then $\{x, g(h)\} = \{x, g(h) - u\} + \{x, u\} \in U + U \subset V$, where u is some element of Q. This completes the proof.

We say $B \subset E$ is *bounded away from zero* if there is a neighborhood of $0 \in E$ disjoint from B. Then $f:H \to F$ is said to satisfy condition **a′** if for every $\omega_{\delta f}$-neighborhood W of $0 \in E$ and every $B \in \Sigma$, which is bounded away from zero, there is an ω-neighborhood Z of $0 \in E$, such that $B \cap Z \subset W$. *Condition* **b′**, for $g:H \to \mathfrak{L}(E, F)$ is obtained similarly by again restricting B to be bounded away from zero. We conclude this section by studying the form that these two weaker conditions assume in the case E is a l.c.t.v.s. (in particular a normed space) and $F = R$. The polar C^0 of a subset C of E' is the subset of E consisting of every x for which $\mid \langle x, x' \rangle \mid \leqq 1$ for every $x' \in C$, ([8], p. 499). It can be easily verified that $f:H \to R$ satisfies condition **a′** if and only if for every $\gamma > 0$, $B \in \Sigma$, B bounded away from zero, and $A \in \Sigma \wedge H$, there is a finite subset $Q \subset E'$, so that $S_1(B \cap Q^0) \subset T[\delta f(A), I_\gamma]$, where I_γ is the interval $[-\gamma, \gamma]$; and that $g:H \to E'$ satisfies condition **b′** if for every set $B \in \Sigma$, B bounded away from 0, and $A \in \Sigma \wedge H$, there is a finite set $Q \subset E'$, so that $N(B \cap Q^0) \subset [g(A)]^0$. Let $A \subset H$ and $B \subset E$. We denote by $A(B)$ the intersection of H with the convex hull of $A \cup (A + B)$. Clearly, if A and B are bounded, so is $A(B)$.

THEOREM 11. *Let E be a normed space and Σ the family of all bounded subsets of E. A mapping $f:H \to R$ satisfies condition* **a′** *if and only if for every* *bounded subset* D *of* H *and* $\varepsilon > 0$, *there exists a finite set* $Q_1 \subset E'$, *so that* $x \in E$, h *and* $h + x \in D$, *and* $|\langle x, x' \rangle| \leqq \varepsilon \| x \|$ *for every* $x' \in Q_1$ *imply* $|f(h + x) - f(h)| \leqq \varepsilon \| x \|$.

Proof of necessity. Let D be a bounded subset of H, and $\varepsilon > 0$. Since $D - D$ is bounded, let $\mu = \sup \| y \|$ for $y \in D - D$. Let Q be the finite subset provided by condition **a**$'$ for $A = D$, $\varepsilon\mu = \gamma$, and B the set of all $y \in E$ such that $\| y \| = \mu$, which is bounded away from zero. Then $Q_1 = \varepsilon\mu Q$ is the required set.

Proof of sufficiency. Suppose f satisfies the condition of the theorem. Let $\gamma > 0$, $A \in \Sigma \wedge H$, $B \in \Sigma$ bounded away from zero, μ and ν respectively $\sup \| y \|$ and $\inf \| y \|$ for $y \in B$, Q_1 the finite subset of E' given by the condition of the theorem for $D = A(B)$, $\varepsilon = \gamma/\mu$, $Q = (\mu/\nu\gamma)Q_1$, $x \in B \cap Q^0$, $h \in A$, and $0 < |\lambda| \leqq 1$. If $h + \lambda x \notin A(B)$, then $\{x, \delta_\lambda f(h)\} = 0 \in I_\gamma$. If $h + \lambda x \in A(B)$, let $y = \lambda x$. Then $|\langle y, x' \rangle| \leqq (\lambda\gamma/\mu)\| x \| = (\gamma/\mu)\| y \|$. Then $|f(h + \lambda x) - f(h)| \leqq (1/\mu)(\lambda\gamma \| x \|) \leqq \lambda\gamma$, from which follows $\{x, \delta_\lambda f(h)\} \in I_\gamma$, and from this, sufficiency.

A similar, but simpler, argument establishes Theorem 12 which follows.

THEOREM 12. *If* E *is a normed space and* Σ *is the family of all bounded subsets of* E, $g: H \to E'$ *satisfies condition* **b**$'$ *if and only if for every bounded subset* D *of* H, *and* $\varepsilon > 0$ *there exists a finite subset* Q_1 *of* E' *such that* $h \in D$ *and* $|\langle x, x' \rangle| \leqq \varepsilon \| x \|$ *for every* $x' \in Q_1$, *imply* $|\langle x, g(h) \rangle| \leqq \varepsilon \| x \|$.

Remarks. The converse of Theorem 10 can be deduced from the proof of Theorem 3.3, p. 428 of [17], in the special case in which E is a Hilbert space, Σ is the family of all bounded subsets of E, and $F = R$. Again condition **a** implies condition **a**$'$ and $(\Sigma \wedge H - \omega \wedge H)$-continuity (Theorem 9). The converse is an open question. If $F = R$, we see from Theorem 9 that condition **a** implies continuity in the bounded weak topology of H.

PART II

5. Derivatives

In this section we assume both E and F are l.c.t.v.s.'s, H a radially open subset of E, and Σ an arbitrary family of subsets of E. Let $h_0 \in H$, and $\mathfrak{F}(E, F)$ be given the Σ-topology. If $\lim_{\lambda \to 0} \delta_\lambda f(h_0)$ exists in this topology, we denote it by $f'(h_0)$ and call it the Σ-derivative of f at h_0. If f is derivable (has a derivative) everywhere in H, f' is a mapping of H into $\mathfrak{F}(E, F)$. The Σ-variation of f is defined as Mf' and denoted by $df: H \times E \to F$. In a similar way higher Σ-derivatives are defined, and $f^{(n)}: H \to F_n(E, F)$. The n^{th} Σ-variation $d_n f$ is defined as $d_n f = M_{n+1} f^{(n)}$.

Of particular interest are the point-open and the bounded-open derivatives and variations. These occur when Σ is assumed to be, respectively, the family of all finite subsets and the family of all bounded subsets of E.

We now define the Gâteaux and the Fréchet variations along classical lines.

DEFINITION 1. The mapping $df:H \times E \rightarrow F$ is called the *Gâteaux variation* of $f:H \rightarrow F$ if, for every $h \in H$ and $x \in E$,

$$df(h, x) = \lim_{\lambda \to 0} [f(h + \lambda x) - f(h)]/\lambda.$$

This limit is called the *directional derivative* of f at h along x, and denoted by $f_x(h)$.

DEFINITION 2. If E is a normed space and H is open, df is called the *Fréchet variation* of f if (i) $df(h, x)$ is homogeneous in x, i.e., $df(h, \lambda x) = \lambda\, df(h, x)$ for $h \in H$, $x \in E$ and $\lambda \in R$; and (ii) for every $h \in H$ and every neighborhood V of $0 \in F$, there exists an $\varepsilon > 0$ such that

$$[f(h + x) - f(h) - df(h, x)] \in \| x \| V,$$

for every $x \in E$ for which $0 < \| x \| \leqq \varepsilon$, and $h + x \in H$.

It follows from the definition of the Gâteaux variation that it is homogeneous in the sense of Definition 2.

We now give different (but equivalent) definitions of the Gâteaux and Fréchet variations.

DEFINITION 3. The point-open variation of f is called its *Gâteaux variation*, and the bounded-open variation of f is called its *Fréchet variation.*

It is easy to see that the Fréchet variation of f is its Gâteaux variation, both in the sense of Definition 3. It can be verified that df is the Gâteaux variation of f in the sense of Definition 1 if and only if it is its Gâteaux variation in the sense of Definition 3; i.e., our definition agrees with the classical one. We will use both definitions in the sequel. They are both applicable to general l.c.t.v.s.'s. The classical Fréchet variation, on the other hand, is defined only for E normed (Definition 2). For this case, the classical definition agrees with ours, as seen from Theorem 13.

THEOREM 13. *If E is a normed space and H is open, then $df:H \times E \rightarrow F$ is the Fréchet variation of $f:H \rightarrow F$ in the sense of Definition* 3 *if and only if it is the Fréchet variation of f in the sense of Definition* 2.

Proof of sufficiency. Let df be the Fréchet variation of f in the sense of Definition 2, $f' = M^{-1}(df)$, A bounded, $h \in H$, $\mu \geqq 1$, and $\mu \geqq \| x \|$ for every $x \in A$. Further, let U be a neighborhood of $0 \in F$. There exists an $\varepsilon' > 0$ such that, for $\| x \| \leqq \varepsilon'$, we have $h + x \in H$, and

$$[f(h + x) - f(h) - df(h, x)] \in (\| x \|/\mu) U.$$

Let $\varepsilon = \varepsilon'/\mu$. Then, for $0 < | \lambda | \leqq \varepsilon$, $[\delta_\lambda f(h) - f'(h)] \in T(A, U)$.

Proof of necessity. Let $df = Mf'$ be the Fréchet variation of f in the sense of Definition 3. It is also the Gâteaux variation of f and is, therefore, homogeneous in the sense of Definition 2. Let U be a neighborhood of $0 \in F$ and $h \in H$. The unit solid sphere S is bounded, and there exists $\varepsilon > 0$ such that, for $0 < |\lambda| \leqq \varepsilon$, $\delta_\lambda f(h) - f'(h) \in T(S, U)$. Then, if $x \in E$ and $\lambda = \|x\| \leqq \varepsilon$, we have $z = x/\lambda \in S$; and $f(h + x) - f(h) - df(h, x) = \lambda[\{z, \delta_\lambda f(h)\} - \{z, f'(h)\}] \in \|x\| U$.

Another generalization of the Fréchet variation to arbitrary l.c.t.v.s.'s is due to D. H. Hyers [13]. The relation between his and our variation is still an open question. In the sequel we use the terms *Gâteaux and Fréchet derivatives* to refer to the point-open and the bounded-open derivatives, respectively.

In a sense, the derivative coincides with the classical one for $E = R$. Classically, the derivative $f'(\lambda_0)$ of $f: H \to F$ is defined as

$$f'(\lambda_0) = \lim_{\lambda \to \lambda_0} [f(\lambda) - f(\lambda_0)]/(\lambda - \lambda_0) \in F.$$

Now, we can identify F with $\mathfrak{L}(R, F) \subset \mathfrak{F}(R, F)$ by means of a natural embedding $G: F \to \mathfrak{L}(R, F)$ defined by $\{\lambda, Gy\} = \lambda y$, for every $\lambda \in R$ and $y \in F$. It can be shown that f is differentiable (classically) if and only if it is Gâteaux derivable and if and only if it is Fréchet derivable, and that the Fréchet (and, therefore, the Gâteaux) derivative of f at λ_0 is $Gf'(\lambda_0)$.

6. Applications

In this section we study the connection between derivability and conditions **a** and **b** discussed in Section 4. We assume H to be a radially open convex subset of E. This makes H finitely open. E and F are still l.c.t.v.s.'s. Let π denote a subset of I, the unit interval, consisting of points $0 = \rho_0 < \rho_1 < \cdots < \rho_n = 1$, and points ζ_i, such that $\rho_{i-1} \leqq \zeta_i \leqq \rho_i$, for i $= 1, 2, \cdots, n$. For $x \in E$, h and $h + x \in H$, and f Gâteaux derivable, we define the symbol $J_\pi(f, h, x)$ as $\sum_{i=1}^{n} df(h + \zeta_i x, x)(\rho_i - \rho_{i-1})$ and $J(f, h, x)$ as the set $\{J_\pi(f, h, x)\}_\pi$. Theorem 14 is an analogue of the fundamental theorem of integral calculus (see also [10], p. 171, Theorem 4].

THEOREM 14. *If H is convex, f Gâteaux derivable, $x \in E$, and h and $h + x \in H$, then $\{x, \delta_1 f(h)\}$ is in the closure of $J(f, h, x)$.*

Proof. Let V be a symmetric convex neighborhood of $0 \in F$ and $\alpha \in I$. There is an open interval $N(\alpha)$ about α, such that

$$\{[f(h + \lambda x) - f(h + \alpha x)/(\lambda - \alpha)] - df(h + \alpha x, x)\} \in V$$

for every $\lambda \in N(\alpha)$. A finite number of these intervals $N(\alpha_i)$, corresponding to points $0 = \alpha_1 < \alpha_2 < \cdots < \alpha_n = 1$, cover I. Let $\beta_i \in N(\alpha_i) \cap N(\alpha_{i+1})$, and $\alpha_i < \beta_i < \alpha_{i+1}$. We denote by $0 = \rho_0 < \rho_1 < \cdots < \rho_n = 1$ the α's and the β's arranged in ascending order. For every i, either ρ_i or ρ_{i+1} is an α. We denote by ζ_i the one that is. Then if π consists of the ρ's and the ζ's, $[\{x, \delta_1 f(h)\} - J_\pi(f, h, x)] \in V$, and therefore, $\{x, \delta_1 f(h)\} \in \overline{J(f, h, x)}$.

THEOREM 15. *Let $f:H \to F$ be Gâteaux derivable, H convex, and $f':H \to \mathfrak{L}(E, F)$. Then, for Σ the family of all bounded sets of E, f satisfies condition* **a** *if and only if f' satisfies condition* **b** (Section 4).

Proof of necessity. Let f satisfy condition **a**; let B be a bounded subset of E, and W_1 an $\omega_{f'}$-neighborhood of $0 \in E$. We show that there is an ω-neighborhood Z of $0 \in E$, so that $B \cap Z \subset W_1$. We may assume without loss of generality that W_1 is determined by a bounded set $A \subset H$ and a closed symmetric neighborhood U of $0 \in F$ (Section 4). Let W_2 be the $\omega_{\delta f}$-neighborhood defined by A and U. Then there is an ω-neighborhood Z of $0 \in E$, so that $B \cap Z \subset W_2$, since f satisfies condition **a**. Let $x \in B \cap Z$, and $h \in A$. Then for $0 < |\lambda| \leqq 1$, $\{x, \delta_\lambda f(h)\} \in U$, and, since $\{x, f'(h)\} = \lim_{\lambda \to 0} \{x, \delta_\lambda f(h)\}$, and U is closed, we have $\{x, f'(h)\} \in U$, and $B \cap Z \subset W_1$.

Proof of sufficiency. Let f' satisfy condition **b**, and let W_1 be an $\omega_{\delta f}$-neighborhood of $0 \in E$ determined by a bounded set $A_1 \subset H$ and a neighborhood U of $0 \in F$. We again assume without loss of generality that U is symmetric, convex, and closed. Let $A_2 = A_1(B)$; let W_2 be the $\omega_{f'}$-neighborhood of $0 \in E$ defined by A_2 and U, and Z an ω-neighborhood of $0 \in E$ such that $B \cap Z \subset W_2$. Then $B \cap Z \subset W_1$.

A similar proof yields Theorem 16, connecting conditions **a′** and **b′**.

THEOREM 16. *Under the hypothesis of Theorem* 15, *f satisfies condition* **a′** *if and only if f' satisfies condition* **b′**.

E. H. Rothe has shown ([17], p. 427, Theorem 3.2) that if f is a Fréchet differentiable real valued function defined on a normed space, having a compact Fréchet derivative, then f is continuous in the bounded weak topology. The following generalization, which follows from Theorems 15, 10, and 9, assumes only Gâteaux derivability and that $f':H \to \mathfrak{L}(E, F)$. By the *bounded F-point-open* topology of H we mean the bounded topology of H, when we give E the F-point-open topology as original topology.

THEOREM 17. *Under the hypothesis of Theorem* 15, *if f' is compact, then f is continuous, if we give H its bounded F-point-open topology.*

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