ON THE PURITY OF BRANCH LOCI IN REGULAR LOCAL RINGS[1]

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The purity of branch loci, proved by Zariski [5], is as follows:

Let V be an algebraic variety of a function field K over a ground field k, and let L be a finite separable algebraic extension of K. Then the branch locus of V with respect to the derived normal variety $N(V; L)$ of V in L is purely divisorial locally at every simple branch point.

The purpose of the present paper is to prove the generalization of the above result to general regular local rings, which can be stated as follows:

THEOREM. *Let P be a regular local ring, and let Q be a normal local ring which dominates P and which is a ring of quotients of a finite separable integral extension of P. If, for every prime ideal $\mathfrak{p}$ of rank 1 in P, $\mathfrak{p}$ is unramified in Q, then Q is unramified over P.*

Here, unramifiedness should be understood as follows:[2]

A quasi-local ring[3] Q' dominating another quasi-local ring Q'' is said to be unramified over Q'' if (i) the maximal ideal of Q' is generated by that of Q'', and (ii) the residue class field of Q' is separable over that of Q''. A prime ideal $\mathfrak{p}'$ in Q' is said to be unramified over Q'' if $Q'_{\mathfrak{p}'}$ is unramified over $Q''_{(\mathfrak{p}' \cap Q'')}$. A prime ideal $\mathfrak{p}''$ in Q'' is said to be unramified in Q' if every prime ideal of Q' lying over $\mathfrak{p}''$ is unramified over Q''.

We say that a ring R is of finite type over another ring S if R is a ring of quotients of a ring R' which is a finite module over S.

In §1, we shall prove a criterion of unramifiedness. In §2, we shall reduce the theorem to the case where P is complete, and in §3, we shall prove the theorem by induction on rank P, while the case where rank $P = 2$ is assumed to be known, because the case was proved by Serre and Auslander-Buchsbaum independently, and, though Serre is not publishing his proof, Auslander and Buchsbaum are publishing their proof.

1. A criterion of unramified extensions

We shall recall a well known easy lemma, which we shall call Krull-Azumaya's lemma:[4]

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[2] There are some other notions of "unramifiedness".

[3] A ring (commutative and having the identity) is said to be quasi-local if it has only one maximal ideal.

[4] Once the writer called this "Azumaya's lemma". But since this is an immediate corollary of Krull's lemma (which is the case where $N = 0$), and since Azumaya formulated this lemma and called attention to its convenience in the first instance, the writer now wants to call this "Krull-Azumaya's lemma".

Let M be a module over a ring R with Jacobson radical $\mathfrak{m}$, and let N be its submodule. If $M = N + \mathfrak{m}M$, and if M is finite (or, more generally, if M/N is finite), then $M = N$.

The criterion of unramifiedness which we shall use in §2 is as follows:

Let P be a Noetherian normal ring, and let Q be the integral closure of P in a finite separable algebraic extension L of the field of quotients K of P. Then a prime ideal $\mathfrak{q}$ in Q is unramified over P if and only if there exists an element c of Q such that (i) $L = K(c)$, and (ii) if $f'(x)$ denotes the derivative of the irreducible monic polynomial $f(x)$ for c over P, $f'(c) \notin \mathfrak{q}$ (Zariski [4]).[5]

We shall give a proof of this criterion in a more general form with a simpler proof than that given by Zariski; we note that our proof is substantially the same as that given by Chevalley [2].[6]

Proposition. *Assume that a quasi-local ring Q with maximal ideal $\mathfrak{M}$ dominates a quasi-local ring P with maximal ideal $\mathfrak{m}$ and that Q is of finite type over P. Then*

(1) *If Q is a ring of quotients of $P[u]$, with an element u of Q such that there exists a polynomial $f(x)$ over P with the properties* (i) $f(u) = 0$, *and* (ii) *if $f'(x)$ denotes the derivative of $f(x)$, $f'(u)$ is not in the maximal ideal $\mathfrak{M}$ of Q, then Q is unramified over P.*

(2) *Conversely, assume that Q is unramified over P, and let Q' be any finitely generated subring of Q over P such that* (i) *Q' is integral over P, and* (ii) *Q is a ring of quotients of Q'. Let $\mathfrak{m}_1, \cdots, \mathfrak{m}_r$ be the maximal ideals of Q', where $\mathfrak{m}_1$ is chosen to be $Q = Q'_{\mathfrak{m}_1}$. Let u be an element of Q' such that* (i) *u modulo $\mathfrak{m}_1$ generates $Q/\mathfrak{M}$ over $P/\mathfrak{m}$, and* (ii) *if $f_i(x)$ denotes a monic polynomial over P such that $f_i(x)$ modulo $\mathfrak{m}$ is the irreducible monic polynomial for u modulo $\mathfrak{m}_i$ over $P/\mathfrak{m}$ for each i, $f_1(u) \notin \mathfrak{m}_j$ for any $j \neq 1$. Then* (i) *Q is a ring of quotients of $P[u]$, and* (ii) *u is a root of a monic polynomial f such that $f - f_1 f_2^{n_2} \cdots f_r^{n_r} \in \mathfrak{m}P[x]$ for some natural numbers n_j; hence, in particular, if $f'(x)$ is the derivative of f above, then $f'(u)$ is not in $\mathfrak{M}$.*

Proof of (1). If $P[u] = P[x]/(f(x))$, then the assertion is obvious. The general case follows from the above special case and the following:

Lemma 1. *If a quasi-local ring Q is unramified over a quasi-local ring P, and if $\mathfrak{a}$ is an ideal of Q, then $Q/\mathfrak{a}$ is unramified over $P/(\mathfrak{a} \cap P)$.*

The proof of this lemma is easy by virtue of the definition.

Proof of (2). Set $Q'' = P[u]$, $\mathfrak{m}'' = \mathfrak{m}_1 \cap Q''$, and $Q^* = Q''_{\mathfrak{m}''}$. Since $f_1(u)$ is in $\mathfrak{m}_1$ but not in any other $\mathfrak{m}_j$, $\mathfrak{m}_1$ is the unique prime ideal of Q' which lies over $\mathfrak{m}''$. Therefore $Q = Q'_{\mathfrak{m}_1}$ is integral over Q^*. Since Q is of finite

[5] Though Zariski stated the criterion in a special case according to his purpose, he really proved this criterion.

[6] Though Chevalley treated a special case where P and Q are spots over a field, his proof can be applied to quasi-local rings. But we are modifying his proof so that we can prove a stronger assertion below.

type, it follows that Q is a finite module over Q^*. By the definition of unramifiedness, we have

LEMMA 2. *If a quasi-local ring Q is unramified over a quasi-local ring P, and if Q^* is a quasi-local ring such that $P \leqq Q^* \leqq Q$,*[7] *then Q is unramified over Q^*.*[8]

Therefore, by our choice of u, $Q/\mathfrak{M} = Q^*/\mathfrak{m}''Q^*$, and therefore $Q = Q^* + \mathfrak{m}''Q$, which shows that $Q^* = Q$ by Krull-Azumaya's lemma. Thus (i) is proved. Since $Q^* = Q$, there are natural numbers n_j such that, with $g(x) = f_1 f_2^{n_2} \cdots f_r^{n_r}$, $g(u) \in \mathfrak{m}P[u]$. Therefore u is a root of a polynomial $f^*(x)$ such that $f^* - g \in \mathfrak{m}P[x]$. In order to show that we can choose such an f^* to be monic, it is sufficient to show that u is a root of a monic polynomial f^{**} of degree at most degree of g. Set $d = \deg g$, and consider $M = P + Pu + \cdots + Pu^{d-1}$. Then $g(u) \in \mathfrak{m}P[u]$ shows that, since g is monic, $P[u] = M + \mathfrak{m}P[u]$. Therefore $P[u] = M$ by Krull-Azumaya's lemma. Thus u is a root of a monic polynomial of degree d, and the assertion is proved completely.

2. Reduction to the complete case

We shall prove at first the following lemma on general (commutative) rings:

LEMMA 3. *Let R be a ring, and assume that $t, u \in R$ are such that* (i) *t is not a zero-divisor in R, and* (ii) *$tR:uR = tR$. If v is an element of the total quotient ring of R such that tv and uv are in R, then v is in R.*

Proof. Since $tuv \in tR$, we have $tv \in tR:uR = tR$. Therefore, there exists an element v' of R such that $tv = tv'$. Since t is not a zero-divisor, we have $v = v'$; hence $v \in R$.

Now we consider the pair (P, Q) in the theorem. We say that another pair (P', Q') satisfying the condition in the theorem is equivalent to (P, Q) if we have that Q is unramified over P if and only if Q' is unramified over P' (hence, if the theorem is proved, we see that all pairs are equivalent to each other).

We shall show in this section that for any given pair (P, Q), there exists a pair (P', Q') which is equivalent to (P, Q), and P' is equal to the completion of P.

We denote in general by c an integral element of Q over P which generates the field of quotients L of Q over the field of quotients K of P, and by $f(x; c)$ the irreducible monic polynomial in an indeterminate x over P which has c as a root. Furthermore, we denote by $g_i(x; c)$ $(i = 1, \cdots, n(c))$ the irreducible monic factors of $f(x; c)$ over the completion P^* of P. Let Q^* be

[7] The symbol $\leqq$ denotes domination.

[8] Q^* may not be unramified over P, as is easily seen.

the completion of Q and let $\mathfrak{q}_1^*, \cdots, \mathfrak{q}_m^*$ be the prime divisors of zero in Q^*. Observe that Q^* is analytically unramified, because Q is separable and of finite type over the regular local ring P. By the same reason, we see that, by a suitable renumbering of the g_i, $g_i(c;\ c) \in \mathfrak{q}_i^*$ for $i \leqq m$, and that $g_j(c;\ c) \notin \mathfrak{q}_i^*$ if $j \neq i$ for any c. We shall show that $m = 1$. Assume for a moment that $m > 1$. Set $\mathfrak{a}_i^* = \mathfrak{q}_i^* + (\bigcap_{j \neq i} \mathfrak{q}_j^*)$ and $\mathfrak{a}^* = \bigcap \mathfrak{a}_i^*$. Assume that there is a prime ideal $\mathfrak{p}^*$ of Q^* containing $\mathfrak{a}^*$ such that rank $(\mathfrak{p}^* \cap P) \leqq 1$. Since $\mathfrak{p}^*$ contains at least two of the $\mathfrak{q}_i^*$, $f(x;\ c)$ modulo $\mathfrak{p}^*$ has a multiple root (for any c). Therefore, if $f'(x;\ c)$ denotes the derivative of $f(x;\ c)$, we have $f'(c;\ c) \in \mathfrak{p}^*$; hence $f'(c;\ c) \in \mathfrak{p}^* \cap Q$. Therefore $\mathfrak{p}^* \cap Q \neq 0$, and rank $\mathfrak{p}^* \cap Q = 1$ by our assumption. By the criterion of unramifiedness, we see that $\mathfrak{p}^* \cap Q$ is ramified, which is a contradiction. Thus there is no such a $\mathfrak{p}^*$. Let d be the discriminant of $f(x;\ c)$ for a fixed c, and let S be the set of elements s of Q such that $dQ{:}sQ = dQ$. Since Q is normal, every prime divisor of dQ is of rank 1. Therefore the nonexistence of $\mathfrak{p}^*$ above shows that S meets every prime ideal of Q^* containing $\mathfrak{a}^*$. Hence Q_S^* contains an idempotent element e which is not the identity. Since e is integral over P^*, $de \in Q^*$. Since $e \in Q_S^*$, there is an element s of S such that $es \in Q^*$. Therefore Lemma 3 implies that $e \in Q^*$, which is a contradiction. Thus $m = 1$, and hence Q^* is an integral domain.

Let Q^{**} be the derived normal ring of Q^*. Assume that there is a prime ideal $\mathfrak{p}^*$ of rank 1 in Q^{**} which is ramified over P^*; then $f'(c;\ c) \in \mathfrak{p}^*$ for any c by the criterion of unramifiedness, which leads to a contradiction as above. Thus, every prime ideal of rank 1 in Q^{**} is unramified over P^*. If Q is unramified over P, then obviously $Q^* = Q^{**}$, and Q^{**} is unramified over P^*. Assume that Q is ramified over P and that Q^{**} is unramified over P^*. Then $Q^{**}/\mathfrak{m}Q^{**} \neq Q/\mathfrak{M}$. Let $\bar{a}$ be an element of $Q^{**}/\mathfrak{m}Q^{**}$ which generates $Q^{**}/\mathfrak{m}Q^{**}$ over $P/\mathfrak{m}$, and let $h(x)$ be a monic polynomial over P such that h modulo $\mathfrak{m}$ is the irreducible monic polynomial for $\bar{a}$. Let $\mathfrak{M}''$ be the maximal ideal of $Q[x]/(h(x))$ which corresponds to the irreducible factor of $h(x)$ modulo $\mathfrak{M}$ over $Q/\mathfrak{M}$ whose root is $\bar{a}$. Then we have that the completion of $Q'' = (Q[x]/(h(x)))_{\mathfrak{M}''}$ is not an integral domain. Since the discriminant of $h(x)$ is unity, Q'' satisfies the condition in the theorem with respect to P; hence we have a contradiction by what we proved above. Therefore, if Q is ramified over P, Q^{**} must be ramified over P^*. Thus the pair (P^*, Q^{**}) is equivalent to the pair (P, Q).

3. The proof of the complete case

Now we assume that P is complete. Since any complete local ring is a Henselian ring, we can enlarge P so that $Q/\mathfrak{M}$ is purely inseparable over $P/\mathfrak{m}$ by the same method as in the last part of §2. Therefore we assume that $Q/\mathfrak{M}$ is purely inseparable over $P/\mathfrak{m}$. We shall make use of the following results without proof:

LEMMA 4. *Our theorem is true if rank* $P = 2$. (Serre, Auslander, Buchsbaum)

For the proof, see [1].

LEMMA 5. *Let* P *be a complete regular local ring of rank greater than* 2, *and let* $x_1, \cdots, x_n$ *be a regular system of parameters of* P. *Let* Q *be a normal ring which is a finite separable integral extension of* P. *Then there exist a finite number of elements* $a_1, \cdots, a_t$ *of* P *such that, for a transcendental element* y *over* P, *whenever an element* c *of* P *is not congruent to any of the* a_i *modulo the maximal ideal* $\mathfrak{m}$, $Q(y)/(x_1 y - x_2 - cx_3)$ *is analytically irreducible.*[9, 10] (Chow)

For the proof, see [3].

Now we shall prove the theorem under the assumption that P is complete. If rank $P \leqq 1$, then there is nothing to prove. If rank $P = 2$, then the assertion is contained in Lemma 4. Assume that rank $P = r > 2$, and we prove the assertion by induction on rank P.

If $P/\mathfrak{m}$ is a finite field, then taking a transcendental element z over P, we consider $P(z)$ and $Q(z)$. Since $Q(z) = P(z)[Q]$, we see that (P, Q) is equivalent to $(P(z), Q(z))$. Hence, reducing again to the complete case, we may assume that $P/\mathfrak{m}$ is not a finite field. As was remarked at the beginning of the present section, we assume furthermore that $Q/\mathfrak{M}$ is purely inseparable over $P/\mathfrak{m}$. Assuming that Q is ramified over P, we shall show a contradiction. Let x be an element of $\mathfrak{m}$ which is not in $\mathfrak{m}^2$, and let $\mathfrak{q}_1, \cdots, \mathfrak{q}_s$ be prime divisors of xQ. By the assumption on Q, $xQ = \bigcap \mathfrak{q}_i$. Set $Q_i = Q/\mathfrak{q}_i$, and let Q_i' be the derived normal ring of Q_i. Since Q_i is complete, Q_i' is a normal local ring. By the induction assumption, if $\mathfrak{q}$ is a prime ideal of Q different from $\mathfrak{M}$, then $Q_\mathfrak{q}$ is unramified over $P_{(\mathfrak{q} \cap P)}$. Applying this fact to those $\mathfrak{q}$ containing x, we have (i) $(Q_i)_{\mathfrak{q}/(x)}$ is unramified over $P_{(\mathfrak{q} \cap P)}/(x)$, and hence is a regular local ring, and consequently (ii) the conductor of Q_i' over Q_i contains a primary ideal to the maximal ideal $\mathfrak{M}/\mathfrak{q}_i$, and (iii) Q_i' is unramified over P/xP. If $Q_1 \neq Q_1'$, then by (iii) above, the residue class field of Q_1' is different from that of $Q/\mathfrak{M}$ and is separable, whence, extending the residue class fields of P and Q to that of Q_1' by the method used at the end of §2, we have that $\mathfrak{q}_1$ splits into several prime ideals. Since the total number s of $\mathfrak{q}_i$ is not greater than $[Q:P]$, we see that we have, after a finite number of steps, the case where $Q_1 = Q_1'$. Then, by the assumption that $Q/\mathfrak{M}$ is purely inseparable over $P/\mathfrak{m}$, and by the above observations, we have $Q/\mathfrak{M} = P/\mathfrak{m}$, and $Q_1 = P/xP$. Thus we assume that there are an element $x \in \mathfrak{m}, \notin \mathfrak{m}^2$ and a prime divisor $\mathfrak{q}_1$ of xQ such that $Q/\mathfrak{q}_1 = P/xP$.

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[9] The writer wants to express his thanks to Abhyankar for suggesting the use of this lemma.

[10] When R is a local ring with maximal ideal $\mathfrak{m}$ and when x is a transcendental element over R, the notation $R(x)$ denotes the ring $R[x]_{(\mathfrak{m}R[x])}$.

Now we apply Lemma 5 to our P and Q; let y, x_i, a_j be as in Lemma 5, and let $c \in P$ be such that $c - a_j \notin \mathfrak{m}$ for any j. Then

$$\bar{Q} = Q(y)/(x_1 y - x_2 - cx_3)$$

is analytically irreducible; hence the derived normal ring $\bar{Q}'$ of $\bar{Q}$ is a local ring and is a finite module over

$$\bar{P} = P(y)/(x_1 y - x_2 - cx_3).$$

Therefore the same observation applied above to Q_i and P/xP can be applied to $\bar{Q}$ and $\bar{P}$; namely, we have (i) $\bar{Q}'$ is unramified over $\bar{P}$, (ii) the conductor of $\bar{Q}'$ over $\bar{Q}$ is a primary ideal belonging to the maximal ideal, (iii) the residue class field of $\bar{Q}'$ is different from that of $\bar{Q}$, and therefore (by (ii) and (iii)) (iv) if $\mathfrak{q}$ is a prime ideal of $Q(y)$ containing $x_1 y - x_2 - cx_3$, and if $\mathfrak{q} \neq \mathfrak{M}Q(y)$, then the residue class field of the derived normal ring of $Q(y)/\mathfrak{q}$ is different from that of $Q(y)/\mathfrak{q}$. Now we apply this (iv) to the special case where $\mathfrak{q}$ is generated by $\mathfrak{q}_1$ and $x_1 y - x_2 - cx_3$; by the assumption that $Q/\mathfrak{q}_1 = P/xP$, $\mathfrak{q}$ is really a prime ideal, and $Q(y)/\mathfrak{q}$ is a regular local ring; hence there is no residue class field extension in the derived normal ring, which yields a contradiction to (iv) above. Thus Q is unramified over P, and we have proved the theorem completely.

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