THE IRREDUCIBILITY OF THE REGULAR SERIES ON AN ALGEBRAIC VARIETY

BY

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Let V be a projective algebraic variety over an algebraically closed field k which will serve as the field of definition for all that follows. There is canonically associated with V a rational mapping $f: V \to A$ of V into an abelian variety A, the Albanese variety of V. The Albanese variety may be defined by the universal mapping property: any rational map $g: V \to B$ of V into an abelian variety B factors as $g = h \circ f$, where h is a rational map of A into B. Classically, A is the torus formed from the period matrix of the q integrals of the first kind on V; when V is a curve, A is just its Jacobian.

It is convenient in what follows to assume that the canonical map f is single-valued; if this is not so to begin with, it will be if we replace V by the graph of the map f on $V \times A$, it being of course birationally equivalent to V. The map f then extends naturally to the set of positive zero-cycles on V by defining $F(x_1 + \cdots + x_n) = \sum f(x_i)$, where the addition on the right refers to the group law on A. We introduce now the n-fold symmetric product $V(n)$ of V with itself: it is definable as the Chow variety which parametrizes all positive zero-cycles of degree n on V. Then F may be viewed as a map $F: V(n) \to A$, which will be single-valued if f is. Such a single-valued, surjective map will be referred to in the sequel as a *foliation*, and the set-theoretic inverse images $F^{-1}(a)$ on V as the *leaves* of the foliation.

The leaves $F^{-1}(a)$ on $V(n)$ represent the equivalence systems of positive zero-cycles of degree n under the natural equivalence relation defined by the mapping F; Albanese called these the "regular series".[2] When V is a curve, the equivalence relation is just linear equivalence. For a study of equivalence relations on zero-cycles of V, it is important to know whether or not these leaves are irreducible varieties, and it is the purpose of this note to show that when n is sufficiently large, this is indeed so. The result we shall prove is the following.

THEOREM. *Let* $\dim V = r > 1$, *let* $q = \dim A$, *and let g be the genus of a generic curve on a normal model of V (so that $g \geqq q$).*

1. *When $n \geqq g$, the generic leaf of the (surjective) foliation $F: V(n) \to A$ is absolutely irreducible.*

If we let n_0 be the smallest value of n for which this occurs (so that certainly $n_0 \leqq g$),

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[2] G. ALBANESE, *Corrispondenze algebriche fra i punti di due superficie algebriche, I*, Ann. Scuola Norm. Sup. Pisa, ser. II, vol. 3 (1934), p. 1.

2. *When* $n \geqq n_0 + q$, *every leaf of the foliation* F *is absolutely irreducible and of dimension* $nr - q$.

When V is a curve, the theorem is a consequence of the Riemann-Roch and Abel-Jacobi theorems; we use this fact in the proof of statement 1 and are thus not offering a proof when $\dim V = 1$.

Proof of statement 1. Field-theoretically, the theorem is asserting that the function field $k(V(n))$ is a primary extension of $k(A)$, that is, the algebraic closure of $k(A)$ in $k(V(n))$ is a purely inseparable extension of $k(A)$. Since the theorem is therefore birational, we may suppose V is normal with a generic linear 1-section of genus g.

We recall (!) Chow's construction of the Albanese variety.[3] We take a generic 1-dimensional linear section C on V; it is defined therefore over a purely transcendental extension $K = k(u)$ of k. Let J be the Jacobian of C and $g: C \to J$ a canonical map, both defined over K. Then the Albanese variety A of V is the "k-image" of J, in other words, $k(A)$ is the maximal abelian subfield of $K(J)$ with k as ground field; Chow proves also that $K(J)$ is a primary extension of $K(A)$.[4] We have then two commutative diagrams:

$$\begin{array}{ccc} C & \xrightarrow{g} & J \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \lambda} \\ V & \xrightarrow{f} & A \end{array} \qquad\qquad \begin{array}{ccc} C(n) & \xrightarrow{G} & J \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \lambda} \\ V(n) & \xrightarrow{F} & A. \end{array}$$

Here on the left i is the inclusion map, λ the map resulting from the fact that $k(A) \subset K(J)$; all maps are defined over K (though actually, f is defined over k also). Roughly speaking, the reason why $K(J)$ is primary over $k(A)$ is that otherwise we could insert an abelian variety B between J and A, algebraic over A and defined (as it turns out) over k, but then B would be the Albanese variety of V. The diagram on the right is a consequence of the one on the left. If we view $C(n)$ as the Chow variety of all positive zero-cycles of degree n on V having C as carrier variety, it is a subvariety of $V(n)$. Of course J is still the Jacobian of $C(n)$ and G is to g as F is to f. Everything is still defined over K.

We are out to show that $k(V(n))$ is a primary extension of $k(A)$; for this, we may assume that the fields $K = k(u)$, $k(V)$, $k(V(n))$, and $k(A)$ have been chosen inside the universal domain so that the first is independent of the last three. Then since $k(A)$ is algebraically closed in $K(A)$, it will be enough to show that $K(V(n))$ is a primary extension of $K(A)$. So we work from now on over K.

What we have is a surjective map F whose restriction to the subvariety $C(n)$ is the same as λG. Let now x be a generic point of $V(n)$

[3] To appear.

[4] W.-L. Chow, *Abelian varieties over function fields*, Trans. Amer. Math. Soc., vol. 78 (1955), p. 268, Corollary 2.

and $\bar{x}$ one of $C(n)$. Then $F(x)$ and $F(\bar{x})$ are both generic points of A and $G(\bar{x})$ is a generic point of J (since $n \geqq g$). From the theory of curves, $K(G(\bar{x}))$ is algebraically closed in $K(\bar{x})$; since we know that $K(G(\bar{x}))$ is in turn a primary extension of $K(\lambda G(\bar{x})) = K(F(\bar{x}))$, it follows that $K(\bar{x})$ is a primary extension of $K(F(\bar{x}))$.

Extend now the specialization $x \rightarrow \bar{x}$ to a surjective place mapping $K(x) \rightarrow K(\bar{x})$. The place sends $F(x)$ onto $F(\bar{x})$, and since both are generic points of A, it sends $K(F(x))$ isomorphically onto $K(F(\bar{x}))$. This implies that it is also an isomorphism on any algebraic extension of $K(F(x))$—in particular the algebraic closure E of $K(F(x))$ in $K(x)$. But the image of E under the place is necessarily a purely inseparable extension of $K(F(\bar{x}))$ in $K(\bar{x})$ by what we have proved above, and so E is itself a purely inseparable extension of $K(F(x))$ in $K(x)$. Thus the extension is primary.

LEMMA. *Let $k(x)$ be a primary extension of $k(z)$, and suppose that the generic specialization $z \rightarrow z'$ over k extends separately to both the generic specialization $x \rightarrow x'$ and the arbitrary specialization $y \rightarrow y'$. Then it extends to $(x, y) \rightarrow (x', y')$, provided that x and y are independent over $k(z)$.*

Proof. Let E be the algebraic closure of $k(z)$ in $k(x)$, so that $k(x)/E$ is regular and $E/k(z)$ purely inseparable of degree p^e. Then $E(y)$ and $k(x)$ are linearly disjoint over E.

Let $F(Y)$ be a polynomial over $k[x]$ such that $F(y) = 0$. Write $F(Y) = \sum a_i G_i(Y)$, where the a_i are linearly independent over E and the $G_i(Y)$ have coefficients in E. Then $G_i(Y)^{p^e} = H_i(Y)$ has its coefficients in $k(z)$, and after applying the isomorphism $k(x, z) \rightarrow k(x', z')$ we get $H_i'(Y) = G_i'(Y)^{p^e}$.

Now $F(y) = 0 \Rightarrow \sum a_i G_i(y) = 0 \Rightarrow G_i(y) = 0 \Rightarrow H_i(y) = 0 \Rightarrow H_i'(y') = 0 \Rightarrow G_i'(y') = 0 \Rightarrow F'(y') = \sum a_i' G_i'(y') = 0$ also. Here the second implication follows from the linear disjointness, the fourth because $(z, y) \rightarrow (z', y')$ is a specialization.

Before proving the second statement—the "everywhere irreducible" part—we remark that if w is the Chow point of a zero-cycle $x_1 + \cdots + x_n$, then $k(x_1, \cdots, x_n)$ is algebraic over $k(w)$. For after dehomogenizing the coordinates, the associated form of the cycle, a polynomial over $k(w)$, factors into linear factors in $k(x_1, \cdots, x_n)$, therefore into linear factors over the algebraic closure of $k(w)$ in $k(x)$;[5] but the coefficients of these linear factors are just the coordinates of the x_1, which shows that all these coordinates are algebraic over $k(w)$.

Proof of statement 2. Consider the leaf $F^{-1}(a)$, defined over the field $k(a)$, and let U be an absolutely irreducible component, defined over $K = \overline{k(a)}$; we will show that U is the whole leaf. Since $\dim V = r$, certainly $\dim U \geqq nr - q$ by a general dimension result from the theory of algebraic

[5] C. CHEVALLEY, *Introduction to the theory of algebraic functions of one variable*, Amer. Math. Soc., Mathematical Surveys, no. 6, 1951, p. 82, Lemma 1.

correspondences. Let w be a generic point of U over K; since w is on $V(n)$, it represents some positive zero-cycle of degree n, say $x_1 + \cdots + x_n$, where $x_i \in V$. By the above remark, $\dim_K (x_1, \cdots x_n) = \dim_K (w) \geqq nr - q$. Since however $\dim_K x_i \leqq r$, it is easy to see that at least $n - q$ of the x_i, say $x_1, \cdots, x_{n-q}$, are independent generic points of V over K: just adjoin the x_i one at a time and observe that the dimension has to rise by r at least $n - q$ times to get up to $nr - q$.

Consider then the rational map $\phi: V(n - q) \times V(q) \to A$ defined by $\phi(x, y) = F(x) + F(y)$. Let y_1 on $V(n - q)$ represent the cycle $x_1 + \cdots + x_{n-q}$ and y_2 on $V(q)$ represent $x_{n-q+1} + \cdots + x_n$. Then (y_1, y_2) is on the leaf $\phi^{-1}(a)$, and its locus over K is a subvariety W of $\phi^{-1}(a)$ which evidently covers U under the natural projection map $\pi: V(n - q) \times V(q) \to V(n)$, since $\pi(y_1, y_2) = w$. We are going to show now that W is independent of the choice of U, and that $\dim W = nr - q$; U will therefore be uniquely determined as the image $\pi(W)$, so that the leaf $F^{-1}(a)$ has only the one component U, of dimension $nr - q$.

We claim first that y_1 and y_2 are generic points of $V(n - q)$ and $V(q)$ respectively, and that $\dim_K (y_1, y_2) = nr - q$. Since $\dim_K y_1 = \dim_K (x_1, \cdots, x_{n-q}) = (n - q) r$, we see that y_1 is a generic point of $V(n - q)$ over K. The map F is surjective, so that the image $F(y_1)$ is thus a generic point of A over K; it follows from $K(z) = K(a - z)$ that $F(y_2) = a - z$ is also a generic point of A over K. Consequently, $K(y_1)$ and $K(y_2)$ both contain $K(z)$, and since surely $\dim_{K(z)} y_2 \leqq qr - q$, a simple dimension computation shows that $\dim_K (y_1, y_2) \leqq nr - q$. Confronting this with $\dim_K (y_1, y_2) = \dim_K (x_1, \cdots, x_n) \geqq nr - q$ shows that equality holds, and so $\dim W = nr - q$, as asserted. And this further implies that $\dim_{K(z)} y_2 = qr - q$, so that y_2 is a generic point of $V(q)$ over K.

To show now that W is independent of U, we characterize it invariantly as follows: let y_1' and y_2' be generic points of $V(n - q)$ and $V(q)$ respectively, such that $\phi(y_1', y_2') = a$ and with $\dim_K (y_1', y_2') = nr - q$; then W is, we claim, just the locus of (y_1', y_2') over K. In fact since the dimension of this would-be generic point is correct, it is enough to show that it lies on W, i.e., that $(y_1, y_2) \to (y_1', y_2')$ is a specialization over K. This however follows immediately from the lemma. For put $z' = F(y_1')$. Then the generic specialization $(z, a - z) \to (z', a - z')$ extends to both generic specializations $y_1 \to y_1'$ and $y_2 \to y_2'$. Dimension considerations show that y_1 and y_2 are independent over $K(z)$, and $K(y_1)$ is indeed a primary extension of $K(z)$ because we are assuming that $n - q \geqq n_0$. It is thus at this point that the first irreducibility statement is needed.

Examples. Andreotti[6] gives an example of a surface V with irregularity 2 which is a two-fold covering of its Albanese variety; taking the product of V

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[6] A. ANDREOTTI, *Recherches sur les surfaces algébrique irrégulières*, Acad Roy. Belg. Cl. Sci. Mém. Coll. in 8°, vol. 27 (1952), fasc. 7 (no. 1631), p. 16. The first equation should read $u^2 = z\, q(x, y)$ there.

with a projective space we can get varieties V' of any dimension for which the foliation $F: V' \to A$ has reducible generic leaf.

For an example relevant to statement 2, let A^q be an abelian variety and V^q its quadratic transform with O as center, so that O is blown up into a projective space P^{q-1} on V. Then A is the Albanese variety of V, and $n_0 = 1$. Consider the leaf $F^{-1}(O)$ of the foliation $F: V(n) \to A$; it consists of the Chow points of those cycles $x_1 + \cdots + x_n$ for which (speaking loosely) $\sum x_i = 0$. Those cycles for which all $x_i \in P$ make up an irreducible subvariety W of $F^{-1}(O)$ of dimension $n(q - 1)$. Now if $n < q$, a generic point of W cannot be a specialization of any point p on $F^{-1}(O)$ which doesn't lie on W, since such a point represents a cycle $x_1 + \cdots + x_n$ for which not all $x_i \in P$: thus $\dim P \leqq nq - q$, and so its dimension is too small: $n(q - 1) > (n - 1)q$. This is even true when $n = q$, though for a less crude reason. Therefore for this variety V, the leaf $F^{-1}(O)$ is reducible when $n < 1 + q = n_0 + q$.

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