A VARIATIONAL METHOD FOR TRIGONOMETRIC POLYNOMIALS[1]

BY

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1. Introduction

Let $f(x)$ be a trigonometric polynomial. We consider a linear functional $\mathfrak{L}$ defined by

$$\mathfrak{L}(f) = \sum_{\nu=1}^{m} \sum_{j=0}^{n_\nu} \alpha_\nu^{(j)} f^{(j)}(x_\nu),$$

where x_ν, α_ν are given real numbers, $0 \leqq x_\nu < 2\pi$, with the x_ν all different. We suppose that $\alpha_\nu^{(n_\nu)} \neq 0$ and that $n_\nu > 0$ for at least one ν. We call

$$l = n_1 + \cdots + n_m + m$$

the order of $\mathfrak{L}$; thus $f'(a) - f'(-a)$ is a functional of order 4. We are interested in the maximum of $|\mathfrak{L}(f)|$ when f runs through the class of trigonometric polynomials of type n which satisfy $|f(x)| \leqq 1$ for real x. (It is convenient to say that a trigonometric polynomial is of type n if it is of degree at most n; a trigonometric polynomial of type n is an entire function of exponential type n.) In looking for this maximum it is enough to consider the subclass $\mathfrak{I}_n$ whose members are in addition real for real x. For, if θ is real, we have $e^{i\theta}f(z) = f_1(z) + f_2(z)$, where f_1 and f_2 are elements of $\mathfrak{I}_n$. Since $\mathfrak{L}(e^{i\theta}f) = e^{i\theta}\mathfrak{L}(f)$, we can choose θ so that $\mathfrak{L}(e^{i\theta}f) = |\mathfrak{L}(f)|$, and so $\mathfrak{L}(f_1) = |\mathfrak{L}(f)|$. Hence the maximum of $|\mathfrak{L}(f)|$ is attained, if at all, for an f in $\mathfrak{I}_n$, and indeed for one for which $\mathfrak{L}(f) > 0$.

When $\mathfrak{L}(f) = f'(a)$, we have S. Bernstein's theorem that $|f'(a)| \leqq n$ when $|f'(x)| \leqq 1$ for all x. Here the bound for $|\mathfrak{L}(f)|$ is the same no matter which point a is selected; this is no longer true in the general case.

Bernstein's theorem on trigonometric polynomials is a special case of his theorem on entire functions of exponential type: if $f(z)$ is an entire function of exponential type τ (which we may suppose is real for real x), and $|f(x)| \leqq 1$ for all real x, then $|f'(x)| \leqq \tau$ for all real x. This does not happen for more general functionals $\mathfrak{L}$. In fact, Schaeffer and I [1] found that the maximum of $|\mathfrak{L}(f)|$ in this class $\mathfrak{F}_\tau$ of entire functions is not, in general, attained for a trigonometric polynomial f. However, methods similar to those used in [1] still work for the class $\mathfrak{I}_n$. The general result is stated in §3 below; in §4 it is applied to the special functional $\lambda n^2 f(0) + f''(0)$. As corollaries, we obtain two theorems for ordinary polynomials. Further applications will be given elsewhere.

The problem of maximizing the functional $f'(a) - f'(-a)$ is equivalent to the problem of maximizing $p_n'(x)$ for a given x on $(-1, 1)$ when the poly-

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nomial $p_n(x)$, of degree n, satisfies $| p_n(x) | \leqq 1$ throughout $(-1, 1)$. (Bernstein's well-known inequality $| p_n'(x) | \leqq n(1 - x^2)^{-1/2}$ is not sharp except at the points $x = \cos(k + \frac{1}{2})\pi/n$.) This problem was solved by A. Markov [2] by other methods; this is the paper in which he also showed that $| p_n'(x) | \leqq n^2$ throughout $(-1, 1)$. The latter result is well known, but the former seems to have been completely forgotten.

2. Lemmas

We require a series of lemmas on the element of $\mathfrak{I}_n$ which maximizes $| \mathfrak{L}(f) |$.

LEMMA 1. *For a given functional* $\mathfrak{L}$, *if* $n \geqq [l/2]$, *the maximum* $M = \sup | \mathfrak{L}(f) |, f \in \mathfrak{I}_n$, *is finite, positive, and attained.*

By Bernstein's theorem M is finite; it is attained because $\mathfrak{I}_n$, a collection of trigonometric polynomials of bounded type, is sequentially compact. To show that M is positive we have to exhibit an element f of $\mathfrak{I}_n$ such that $\mathfrak{L}(f) \neq 0$. (This is impossible without some restriction on n since, for example, $f'''(a) + f'(a) = 0$ for all a and all elements of $\mathfrak{I}_1$.) To do this we appeal to the following simple lemma which will be used several times.

LEMMA 2. *If x_1 and x_2 are real points, not congruent* mod 2π, *there is an element g of $\mathfrak{I}_1$, not identically zero, such that $g(x_1) = g(x_2) = 0$. There is also an element g of $\mathfrak{I}_1$ with a double zero at x_1.*

For the first part put

$$2g(x) = 2\sin\tfrac{1}{2}(x - x_1)\sin\tfrac{1}{2}(x - x_2) = \cos\tfrac{1}{2}(x_1 - x_2) - \cos(x - \tfrac{1}{2}(x_1 + x_2));$$

for the second part put $2g(x) = 1 - \cos(x - x_1)$.

We now complete the proof of Lemma 1. We have some $n_\nu > 0$; for definiteness suppose that $n_1 > 0$. Then by taking products of functions g from Lemma 2 we can obtain a real trigonometric polynomial g such that $g^{(k)}(x_\nu) = 0$ for $\nu = 2, \cdots, m$ and $k = 0, 1, \cdots, n_\nu$, and for $\nu = 1$, $k = 0, 1, \cdots, n_1 - 1$; while $g^{(n_1)}(x_1) \neq 0$. Since this g has $l - 1$ zeros in a period, we require $\frac{1}{2}(l - 1)$ functions from Lemma 2 if $l - 1$ is even, $\frac{1}{2}l$ if $l - 1$ is odd, so that g is at worst of degree $[l/2]$; and $\mathfrak{L}(g) \neq 0$.

LEMMA 3. *Let f be an element of $\mathfrak{I}_n$, not a constant, maximizing $L(f)$, with $n \geqq [l/2]$. Then $| f(x) | = 1$ for some real x; and if $F \in \mathfrak{I}_n$ and F has a zero at each of the different points in a period where $f(x) = \pm 1$, then $\mathfrak{L}(F) = 0$.*

The first statement is immediate since if $| f(x) | < 1$ we can choose $b > 1$ so that $bf(x)$ takes one of the values ± 1 and still belongs to $\mathfrak{I}_n$, while $\mathfrak{L}(f)$ is increased.

We begin the proof of the second statement by showing that if $F(z)$ is any element of $\mathfrak{I}_n$ and

$$\sup_x | f(x) + \varepsilon F(x) | = 1 + o(\varepsilon), \qquad \varepsilon \to 0, \tag{2.1}$$

where ε is real, then $\mathfrak{L}(F) = 0$. Let $0 < \rho < 1$. According to (2.1), if $|\varepsilon| < \varepsilon_0(\rho)$, the function

$$\psi_\varepsilon(z) = \frac{f(z) + \varepsilon F(z)}{1 + |\varepsilon|\rho}$$

belongs to $\mathfrak{Z}_n$. If $\mathfrak{L}(F) \neq 0$, then

$$\mathfrak{L}(\psi_\varepsilon) = (1 + |\varepsilon|\rho)^{-1}\{\mathfrak{L}(f) + \varepsilon\mathfrak{L}(F)\}.$$

Choose ρ so that $\rho\mathfrak{L}(f) < |\mathfrak{L}(F)|$, then ε so that $|\varepsilon| < \varepsilon_0(\rho)$ and so that $\varepsilon\mathfrak{L}(F) > 0$. We then have $\mathfrak{L}(\psi_\varepsilon) > \mathfrak{L}(f)$, contradicting the maximizing property assumed for f. Hence $\mathfrak{L}(F) = 0$.

Now suppose that F has a zero at each of the distinct points in $[0, 2\pi)$ where $f(x) = \pm 1$. Let λ be one of these points. Then there is an interval I with center at λ such that

$$|f(x)| \leqq 1 - h(x - \lambda)^{2\mu}$$

in I, where $h > 0$ and μ is a positive integer. Since $F(\lambda) = 0$, Bernstein's theorem shows that

$$|F(x)| = \left|\int_\lambda^x F'(t)\,dt\right| \leqq n|x - \lambda|,$$

and so

$$|f(x) + \varepsilon F(x)| \leqq 1 - h(x - \lambda)^{2\mu} + \varepsilon n|x - \lambda|$$

in I. The maximum of the right-hand side does not exceed $1 + B\varepsilon^{1+1/(2\mu-1)}$, where B depends only on h, τ and μ. Thus $|f(x) + \varepsilon F(x)| \leqq 1 + o(\varepsilon)$ in the intervals I, while $|f(x) + \varepsilon F(x)| \leqq 1$ for small ε when x is in the rest of $[0, 2\pi)$. This establishes (2.1) and so $\mathfrak{L}(F) = 0$.

LEMMA 4. *Given a set of $2r \leqq 2n - 2[l/2]$ distinct real points λ in $[0, 2\pi)$, there is an element g of $\mathfrak{Z}_n$ such that $g(\lambda) = 0$ for each λ and $\mathfrak{L}(g) \neq 0$.*

Let g_1 be a real trigonometric polynomial of degree at most r with zeros precisely at the λ which are not in $\{x_\nu\}$; if the number of such λ is odd, let g_1 have also a zero at some point which is neither a λ nor an x_ν. As in the proof of Lemma 1, let g_2 be a real trigonometric polynomial of degree $[l/2]$ with $g_2^{(k)}(x_\nu) = 0$ for $\nu = 2, \cdots, m$ and $k = 0, 1, \cdots, n_\nu$, and for $\nu = 1$, $k = 0, 1, \cdots, n_1 - 1$, while $g_2^{(n_1)}(x_1) \neq 0$. Then $g_3 = g_1g_2$ is of degree at most $r + [l/2] \leqq n$ and $g_3(\lambda) = 0$ for every λ; also $g_3^{(k)}(x_\nu) = 0$ for $\nu = 2, \cdots, m$; $k = 0, 1, \cdots, n_\nu$; $\nu = 1$, $k = 0, 1, \cdots, n_1 - 1$; and

$$g_3^{(n_1)}(x_1) = g_2^{(n_1)}(x_1)g_1(x_1) \neq 0.$$

Finally, $g = \varepsilon g_3$ belongs to $\mathfrak{Z}_n$ if ε is a sufficiently small positive number.

LEMMA 5. *If f is not a constant and is an extremal function for $\mathfrak{L}$ in $\mathfrak{Z}_n$, where $n \geqq [l/2]$, then f is of degree greater than $n - [l/2]$.*

Suppose that f is of degree $m \leqq n - [l/2]$. Let λ_j be the points in $[0, 2\pi)$ where $f(x) = \pm 1$ (each counted once), and suppose for definiteness that $n_1 > 0$. Then $f'(x)$ vanishes at each λ_j. By Lemma 4 there is an element g of $\mathfrak{J}_n$ vanishing at each λ_j and with $\mathfrak{L}(g) \neq 0$; but this contradicts Lemma 3.

3. The main theorem

We are now in a position to establish the following theorem.

THEOREM 1. *Let $n \geqq [l/2]$. An extremal element f of $\mathfrak{J}_n$ for $\mathfrak{L}$ is either constant, or a trigonometric polynomial whose degree is $m > n - [l/2]$. If f is not a constant, $1 - f^2$ has at most $2k$ imaginary zeros in a period strip, where $k/2 < m - n + [l/2]$, and f satisfies a differential equation*

$$\frac{f'(z)^2}{1 - f(z)^2} = R(z)/S(z), \tag{3.1}$$

where $R(z)$ and $S(z)$ are real trigonometric polynomials of degree $q \leqq k$; $S(z)$ has the form

$$S(z) = \prod_{j=1}^{k} P(b_j, z), \tag{3.2}$$

where $P(b, z) = \cosh \mathfrak{I}(b) - \cos(z - \mathfrak{R}(b)) = 2 \sin\frac{1}{2}(z - \bar{b}) \sin\frac{1}{2}(z - b)$, and the b_j are not real; $R(z)$ has zeros of even multiplicities not occurring at any b_j.

If $n \geqq 2[l/2] + 1$, the extremal function, if not constant, is unique.

That $m > n - [l/2]$ is the content of Lemma 5. The imaginary zeros of $1 - f^2$ (if any) occur in conjugate pairs. Let them be b_j, $\bar{b}_j$, with $0 \leqq \mathfrak{R}(b_j) < 2\pi$, $j = 1, 2, \cdots, k$. Defining $S(z)$ as in (3.2), we see that

$$Q(z) = \{1 - f(z)^2\}/S(z)$$

is entire, and hence is a real trigonometric polynomial of degree $2m - k$. Now $Q(z)$ has only real zeros, at the same points and with the same multiplicities as the real zeros of $1 - f^2$; and $f'(z)^2$ has at least these real zeros. Hence $f'(z)^2/Q(z) = R(z)$ is a real trigonometric polynomial of degree k, and its zeros are those zeros of $f'(z)^2$ which are not real zeros of $1 - f^2$; these (if there are any) are of even multiplicity and (if imaginary) occur in conjugate pairs. We thus have

$$\frac{f'(z)^2}{1 - f(z)^2} = R(z)/S(z). \tag{3.3}$$

Now since $1 - f^2$ has $2k$ imaginary zeros in a period strip, it has $4m - 2k$ real zeros in a period. These are of even multiplicity, and so occur at at most $2m - k$ distinct points. If k is even, Lemma 4 contradicts Lemma 3 unless $m - k/2 > n - [l/2]$, i.e. $k/2 < m - n + [l/2]$. If k is odd, the same thing happens unless $m - k/2 + \frac{1}{2} > n - [l/2]$, which leads to the same conclusion.

If R and S in (3.3) have zeros (necessarily imaginary) in common, we can divide them out and obtain (3.1) with a new R and S of degree $q < k$.

The remaining assertion of Theorem 1 is that there is only one extremal function f, other than a constant, if n is large enough. Suppose that there are two nonconstant extremal functions f_1 and f_2. Then $f_3 = \frac{1}{2}(f_1 + f_2)$ is also an extremal function, and at any real point where f_3 takes one of the values ± 1, f_1 and f_2 take the same value (since neither exceeds 1 in absolute value). Consider $g = \frac{1}{2}(f_1 - f_2)$, an element of $\mathfrak{Z}_n$, which has double zeros at the real points where $f_3 = \pm 1$. Let f_3 have degree m, which we know exceeds $n - [l/2]$; then $1 - f_3^2$ has $4m - 2k$ real zeros in a period, and hence g has at least this many. On the other hand, g is at most of degree n and so has at most $2n$ real zeros in a period. Hence $4m - 2k \leqq 2n$, or $k/2 \geqq m - n/2$. However, we know that $k/2 < m - n + [l/2]$. Combining these inequalities, we obtain $n < 2[l/2] + 1$. Hence a nonconstant extremal function must be unique if $n \geqq 2[l/2] + 1$.

4. A functional of order 3

When the order of the functional $\mathfrak{L}$ is 2, an extremal function, even in the larger class $\mathfrak{F}_n$, is of the form $\sin(nz + c)$, and the problem of identifying it is trivial.

The next case to consider is $l = 3$. In this case, if $n \geqq 1$, a nonconstant extremal function is of degree n, and it is unique if $n \geqq 3$. Since $[k/2] < [3/2] = 1$, we have $k = 0$ or 1.

We now consider the functional

$$\mathfrak{L}(f) = \lambda n^2 f(0) + f''(0),$$

which was studied in [1] for $f \in \mathfrak{F}_n$. It was shown in [1] that when $\lambda \leqq \frac{1}{3}$ the maximum of $\mathfrak{L}(f)$ for $f \in \mathfrak{F}_n$ is furnished by $\pm\cos nz$, and so this function also maximizes $\mathfrak{L}(f)$ in $\mathfrak{Z}_n$. If $\lambda > \frac{1}{2}$, the maximum of $\mathfrak{L}(f)$ cannot be provided by $\pm\cos nz$, since $\mathfrak{L}(1) > \mathfrak{L}(\pm\cos nz)$. If $\frac{1}{3} < \lambda < \frac{1}{2}$, $\mathfrak{L}(-\cos nz) > \mathfrak{L}(1)$. Hence we have to consider solutions of (3.1) with $q = 1$ and decide whether the maximum of $\mathfrak{L}(f)$ for these exceeds $\lambda n^2 = \mathfrak{L}(1)$ if $\lambda \geqq \frac{1}{2}$, and whether it exceeds $(1 - \lambda)n^2 = \mathfrak{L}(-\cos nz)$ if $\frac{1}{3} < \lambda < \frac{1}{2}$.

THEOREM 2. *Let* $\mathfrak{L}(f) = \lambda n^2 f(0) + f''(0)$. *If* $n > 1$, *the largest value of* $|\mathfrak{L}(f)|$ *for* $f \in \mathfrak{Z}_n$ *is furnished by* $\pm\cos nz$ *if* $\lambda \leqq \frac{1}{3} + 1/(6n^2)$ *and by a function of the form*

$$\pm\cos\{n \cos^{-1}(\omega \cos z + \omega - 1)\}, \qquad 0 < \omega < 1, \tag{4.1}$$

if $\lambda > \frac{1}{3} + 1/(6n^2)$. *In the second case, the maximum of* $|\mathfrak{L}(f)|$ *is the maximum for* $0 < \theta < \pi$ *of* $|\lambda n^2 \cos n\theta - \frac{1}{2}n \cot \frac{1}{2}\theta \sin n\theta|$. *For* $n = 1$, *the maximum is furnished by* $\pm\cos nz$ *if* $\lambda \leqq \frac{1}{2}$ *and by* ± 1 *if* $\lambda \geqq \frac{1}{2}$.

(Thus when $\lambda = \frac{1}{2}$ there are two distinct extremal functions in $\mathfrak{Z}_1$.)

If $f(z)$ is an extremal function for $\mathfrak{L}$, so is $f(-z)$; for $n \geqq 3$ the extremal function is unique if not constant, and so it must be even. Again, if $f(z) \neq f(-z)$ and f is an extremal function, $\frac{1}{2}[f(z) + f(-z)]$ is an even extremal function in any case. Hence to determine the maximum of $\mathcal{L}(f)$ we need consider only even extremal functions f.

We require the following lemma.

LEMMA 6. *Under the hypotheses of Theorem* 1, *if f is a nonconstant extremal function and is either even or odd, and $k = 1$, then $f'(0) = f'(\pi) = 0$, and $1 - f^2$ vanishes at one of the points* 0, π *and not at the other.*

Let f be a nonconstant extremal function, either even or odd. If $k = 1$, the function $1 - f^2$ has, in a period strip, a single pair of conjugate imaginary zeros, and $f'(z)^2$ has a single pair of zeros which are not real zeros of $1 - f^2$. Since these zeros are also zeros of $f'(z)$, there must in fact be a single zero of $f'(z)^2$ of multiplicity 2. Since f'^2 is even, this zero can only be at 0 or π (otherwise its symmetric point with respect to 0 would be another zero of f'^2 which is not a real zero of $1 - f^2$). On the other hand, the zeros of f' (since it is even or odd) occur in symmetric pairs except perhaps for zeros at 0 or π. Since one of these points is a simple zero of f' and f' has an even number of zeros in a period, there must be an odd number of zeros besides the one at 0 or π, hence one at the other of these points. Since there is only one zero of f' which is not at a real zero of $1 - f'^2$, we must have $1 - f^2 = 0$ at 0 or at π but not at both of these points.

We now return to extremal functions for $\mathcal{L}(f)$. If $k = 1$ and f is an even extremal function, $1 - f^2$ has (in a period strip) a single pair of conjugate imaginary zeros. Since $1 - f^2$ is even, these zeros must have real part either 0 or π. Hence we may take $S(z)$ in (3.1) as $c \pm \cos z$, where $c > 1$. By Lemma 6, $f'(z)$ must have a zero at 0 or π (but not both) which is not a zero of $1 - f^2$. Now if $f(0) = 1$, $f''(0) \leqq 0$ and so $\mathcal{L}(f) \leqq \lambda n^2$. If $\lambda \geqq \frac{1}{2}$, then, $\mathcal{L}(f) \leqq \mathcal{L}(1)$, so either f is not an extremal function, or f makes $\mathcal{L}(f)$ no larger than $\mathcal{L}(1)$, in which case we do not need to consider f any further. If $\lambda < \frac{1}{2}$, $\mathcal{L}(f) < \lambda n^2 < (1 - \lambda)n^2 = \mathcal{L}(-\cos nz)$, so f is not an extremal function. If $f(0) = -1$, $\mathcal{L}(f) = -\lambda n^2 + f''(0)$, and since $|f''(0)| < n^2$ by Bernstein's theorem (since $|f''(0)| = 1$ only for $\pm\cos nz$), we have $\mathcal{L}(f) < n^2(1 - \lambda)$, and again f is not an extremal function. Hence $f(0) \neq \pm 1$, while $f'(0) = 0$. Hence we may take $R(z) = A^2(1 - \cos z)$, where A is real. Thus we have

$$\frac{f'(w)}{\{1 - f(w)^2\}} = A\,\frac{(1 - \cos w)^{1/2}}{(c \pm \cos w)^{1/2}}.$$

Integrating this, we find

$$f(z) = \cos\left\{A \int_0^z \frac{(1 - \cos w)^{1/2}}{(c \pm \cos w)^{1/2}}\,dw + B\right\}. \tag{4.2}$$

Consideration of the behavior of the two sides as $z \to \infty$ through pure imaginary values shows that the + sign can be excluded and that $A = \pm n$. The integral can be evaluated in terms of elementary functions:

$$\int \frac{(1 - \cos w)^{1/2}}{(c - \cos w)^{1/2}}\, dw = -\cos^{-1} \frac{2 \cos w - c + 1}{c + 1}.$$

Thus we have

$$\begin{aligned} f(z) &= \cos\left\{n \cos^{-1} \frac{2 \cos z - c + 1}{c + 1} + B_1\right\} \\ &= \cos\left\{n \cos^{-1} \frac{2 \cos z - c + 1}{c + 1}\right\} \cos B_1 \\ &\qquad - \sin\left\{n \cos^{-1} \frac{2 \cos z - c + 1}{c + 1}\right\} \sin B_1. \end{aligned}$$

Now $\cos n\theta$ is a polynomial in $\cos \theta$, so the first term on the right is a polynomial in $\cos z$. On the other hand, $\sin n\theta$ is of the form $\sin \theta\, P(\cos \theta)$, where P is a polynomial. Since $\sin(\cos^{-1} t) = \pm(1 - t^2)^{1/2}$, we have

$$\sin\left\{\cos^{-1} \frac{2 \cos z - c + 1}{c + 1}\right\} = \pm 2(c + 1)^{-1} \{(c - \cos z)(1 + \cos z)\}^{1/2},$$

and this has branch points at the zeros of $c - \cos z$. Hence $f(z)$ cannot be an entire function, and so not a trigonometric polynomial, unless $\sin B_1 = 0$. Thus we finally obtain

$$f(z) = \pm\cos\{n \cos^{-1}(\omega \cos z + \omega - 1)\}, \qquad \omega = 2/(c + 1), \tag{4.3}$$

as the form of an extremal function which is neither $\pm\cos nz$ nor a constant.

Calculating $\mathcal{L}(f)$ for (4.3), we find

$$\pm\mathcal{L}(f) = \lambda n^2 \cos n\theta - \tfrac{1}{2}n \cot \tfrac{1}{2}\theta \sin n\theta,$$

where $\cos \theta = 2\omega - 1$, $0 < \theta < \pi$. Since every θ in this range corresponds to some ω, our problem is reduced to the question of whether

$$| \lambda n^2 \cos n\theta - \tfrac{1}{2}n \cot \tfrac{1}{2}\theta \sin n\theta | > \begin{cases} \lambda n^2, & \lambda \geqq \tfrac{1}{2}; \\ (1 - \lambda)n^2, & \tfrac{1}{3} < \lambda < \tfrac{1}{2} \end{cases}, \tag{4.4}$$

for some θ in $0 < \theta < \pi$: for a given λ, if (4.4) holds for some θ, the extremal function for this value of λ is given by (4.3), and the maximum of $\mathcal{L}(f)$ is obtainable by calculating the maximum of the left-hand side of (4.4).

For $\lambda \geqq \tfrac{1}{2}$, (4.4) holds if there is a θ for which either

$$-\cot \tfrac{1}{2}\theta \sin n\theta > 2(1 - \cos n\theta)\lambda n \tag{4.5}$$

or

$$\cot \tfrac{1}{2}\theta \sin n\theta > 2(1 + \cos n\theta)\lambda n. \tag{4.6}$$

Both (4.5) and (4.6) fail when $n = 1$; hence an extremal function for $n = 1$ is never of the form (4.3), and the last statement of Theorem 2 follows.

We now suppose that $n \geqq 2$. Inequality (4.6) is equivalent to $\cot \frac{1}{2}\theta \tan \frac{1}{2}n\theta > 2\lambda n$, which is certainly satisfied when θ is slightly less than π/n. Hence for $\lambda \geqq \frac{1}{2}$ and $n > 1$ the maximum of $\mathfrak{L}(f)$ is attained for a function (4.3), and its value is the maximum of the left-hand side of (4.4).

When $\frac{1}{3} < \lambda < \frac{1}{2}$, (4.4) states that

$$(4.7) \qquad |\lambda n \cos n\theta - \tfrac{1}{2} \cot \tfrac{1}{2}\theta \sin n\theta| > (1 - \lambda)n.$$

As $\theta \to 0$ we have

$$\lambda n \cos n\theta - \tfrac{1}{2} \cot \tfrac{1}{2}\theta \sin n\theta = n(\lambda - 1) + \theta^2\{(\tfrac{1}{6} - \tfrac{1}{2}\lambda)n^3 + \tfrac{1}{12}n\} + O(\theta^4),$$

so that (4.7) holds if $\lambda > \frac{1}{3} + 1/(6n^2)$.

To complete the proof of Theorem 2 we now show that $-\cos nz$ is an extremal function when $\frac{1}{3} < \lambda \leqq \frac{1}{3} + 1/(6n^2)$. We have to show that, when λ is in this interval, and $0 < \theta < \pi$,

$$(4.8) \qquad |\lambda n \cos n\theta - \tfrac{1}{2} \cot \tfrac{1}{2}\theta \sin n\theta| < (1 - \lambda)n.$$

We shall consider separately the intervals (i) $\theta \geqq 3\pi/(2n)$, (ii) $\pi/n \leqq \theta < 3\pi/(2n)$, (iii) $0 < \theta < \pi/n$.

In (i), the left-hand side of (4.8) is at most

$$\tfrac{1}{2} \cot(\tfrac{3}{4}\pi/n) + \lambda n \leqq 2n/(3\pi) + \lambda n = n(\tfrac{2}{3}\pi^{-1} + \lambda),$$

and this is less than $(1 - \lambda)n$ since $\lambda \leqq \frac{1}{3} + 1/(6n^2) \leqq \frac{3}{8}$ $(n \geqq 2)$.

In (ii), $\cos n\theta$ and $\sin n\theta$ are both negative, so the absolute value on the left of (4.8) is at most the larger of $\frac{1}{2} \cot \frac{1}{2}\pi/n$, λn. Now $\lambda n < (1 - \lambda)n$ since $\lambda < \frac{1}{2}$; and $\frac{1}{2} \cot \frac{1}{2}\pi/n < (1 - \lambda)n$ follows from $\frac{1}{2} \cot \frac{1}{2}\pi/n < n/\pi < n/3 < n(1 - \lambda)$, which in turn follows from $\lambda \leqq \frac{1}{3} + 1/(6n^2) < \frac{3}{8}$.

In (iii), we have to show that

$$(4.9) \qquad \lambda n \cos n\theta - \tfrac{1}{2} \cot \tfrac{1}{2}\theta \sin n\theta < (1 - \lambda)n$$

and that

$$(4.10) \qquad \tfrac{1}{2} \cot \tfrac{1}{2}\theta \sin n\theta - \lambda n \cos n\theta < (1 - \lambda)n.$$

Since $\sin n\theta > 0$, we can rewrite (4.9) and (4.10) in the form

$$(4.11) \qquad \begin{aligned} &\lambda n \cot \tfrac{1}{2}n\theta - \tfrac{1}{2} \cot \tfrac{1}{2}\theta < n \csc n\theta, \\ &\lambda n \tan \tfrac{1}{2}n\theta + \tfrac{1}{2} \cot \tfrac{1}{2}\theta < n \csc n\theta. \end{aligned}$$

Now the second of these implies the first. In fact, is enough to show that

$$\lambda n \cot \tfrac{1}{2}n\theta - \tfrac{1}{2} \cot \tfrac{1}{2}\theta < \lambda n \tan \tfrac{1}{2}n\theta + \tfrac{1}{2} \cot \tfrac{1}{2}\theta,$$

i.e.

$$\lambda n(\cot \tfrac{1}{2}n\theta - \tan \tfrac{1}{2}n\theta) < \cot \tfrac{1}{2}\theta,$$

or

$$\lambda n \cot n\theta < \tfrac{1}{2} \cot \tfrac{1}{2}\theta.$$

This holds for $\theta < \pi/n$ since $\lambda < 1$ and $x \cot x$ decreases in $0 < x < \pi$. Hence it is enough to establish (4.11), or equivalently that

$$\lambda n \tan \tfrac{1}{2}n\theta < n \csc n\theta - \tfrac{1}{2} \cot \tfrac{1}{2}\theta.$$

Since $\lambda \leqq \frac{1}{3} + 1/(6n^2)$, it is enough to show that

$$(4.12) \qquad (\tfrac{1}{3} + 1/(6n^2))\, n \tan \tfrac{1}{2}n\theta < n \csc n\theta.$$

Now the Laurent expansions of all the functions in (4.12) (about $\theta = 0$) are valid in $0 < \theta < \pi/n$, and we have (for $n \geqq 2$)

$$(\tfrac{1}{3} + 1/(6n^2))n \tan \tfrac{1}{2}n\theta = \tfrac{1}{6}n^2\theta + \tfrac{1}{12}\theta + (\tfrac{1}{3} + 1/(6n^2))n \sum_{n=2}^{\infty} A_n\, n^{2n-1}\theta^{2n-1}$$

$$\leqq \tfrac{1}{6}n^2\theta + \tfrac{1}{12}\theta + \tfrac{3}{8} \sum_{n=2}^{\infty} A_n\, n^{2n}\theta^{2n-1}, \qquad A_n = \frac{2(2^{2n} - 1)}{(2n)!} B_{2n-1}$$

(the B's are Bernoulli numbers, the significant thing for our purposes being that they are positive). We also have

$$n \csc n\theta - \tfrac{1}{2} \cot \tfrac{1}{2}\theta = \tfrac{1}{6}n^2\theta + \tfrac{1}{12}\theta + \sum_{n=2}^{\infty} n^{2n} C_n\, \theta^{2n-1} + \phi(\theta),$$

where

$$c_n = 2(2^{2n-1} - 1)B_{2n-1}/(2n)!$$

(the coefficients in the expansion of $\csc x$), and $\phi(\theta) > 0$ (ϕ is the "tail" of the expansion of $-\frac{1}{2} \cot \frac{1}{2}\theta$). Hence (4.12) is implied by

$$\frac{3}{8} \sum_{n=2}^{\infty} A_n\, n^{2n}\theta^{2n-1} < \sum_{n=2}^{\infty} n^{2n} C_n\, \theta^{2n-1},$$

and hence by $\frac{3}{8}A_n < C_n$, i.e. $\frac{3}{8}(2^{2n} - 1) < 2^{2n-1} - 1$, which is true for $n \geqq 2$.

The point 0 plays no special role in Theorem 2 since $f(z + a) \in \mathfrak{Z}_n$ when $f(z) \in \mathfrak{Z}_n$. Hence we can replace 0 by any other point in Theorem 2. Applying the more general result to $p_n(\cos \theta)$ and to $p_n(e^{i\theta})$, we obtain the following corollaries.

COROLLARY 1. *If $p_n(z)$ is a polynomial of degree $n > 1$ and $| p_n(z) | \leqq 1$ for $-1 < x < 1$, then*

$$| \lambda n^2 p_n(x) - x p_n'(x) + (1 - x^2) p_n''(x) |$$

does not exceed $(1 - \lambda)n^2$ if $\lambda \leqq \frac{1}{3} + 1/(6n^2)$, and does not exceed the maximum of $| \lambda n^2 \cos n\theta - \frac{1}{2}n \cot \frac{1}{2}\theta \sin n\theta |$, $0 < \theta < \pi$, if $\lambda > \frac{1}{3} + 1/(6n^2)$.

The particular cases $x = 0$, $x = 1$ are of interest.

COROLLARY 2. *If $p_n(z)$ is a polynomial of degree $n > 1$ and $| p_n(z) | \leqq 1$ for $| z | < 1$, then $| \lambda n^2 p_n(z) - z p_n'(z) - z^2 p_n''(z) |$ does not exceed the bounds given in Corollary 1 when $| z | < 1$.*

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