A FINE-CYCLIC ADDITIVITY THEOREM FOR A FUNCTIONAL[1]

BY

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Introduction

Let J be a closed finitely connected Jordan region, and let (T, J) be a continuous mapping from J into E_3. L. Cesari has introduced in his papers [2; 3] the concept of a fine-cyclic element of (T, J), and he has proven that the Lebesgue area is fine-cyclicly additive, thus extending a well-known cyclic additivity theorem for the Lebesgue area [8]. A fine-cyclic element is actually a decomposition of a proper cyclic element, and, in case J is a 2-cell, is equivalent to a proper cyclic element.

In [6] a B-set and a fine-cyclic element of a Peano space is introduced as a generalization of an A-set and a proper cyclic element. Specifically, a B-set of a Peano space P is a nondegenerate (more than one point) continuum of P such that either $B = P$ or else every component of $P - B$ has a finite frontier. A fine-cyclic element of P is a B-set of P whose connection is not destroyed by removing any finite set. It has been shown in [6] that in a Peano space P whose degree of multicoherence $r(P)$ is finite, B-sets and fine-cyclic elements possess essentially the same properties as A-sets and proper cyclic elements.

In this paper a generalization of Cesari's fine-cyclic additivity theorem for the Lebesgue area is studied. The generalization proceeds along lines similar to [4] by considering nonnegative functionals Φ defined for each continuous mapping T from a Peano space P into a metric space P^*. Let $T = sf$, $f:P \rightarrow M$, $s:M \rightarrow P^*$, $r(M) < \infty$, be an unrestricted factorization of T (§1), and let $\{\Delta\}$ be the collection of fine-cyclic elements of M. With each Δ there is associated a connected open set $G_\Delta \subset M$ containing Δ such that Δ is a (G_Δ, A)-set [7]. Denote by t_Δ the natural retraction [7] from G_Δ onto Δ, and let $A_\Delta = f^{-1}(G_\Delta)$. If Φ satisfies the conditions of §2, the main result of this paper states that $\Phi(T, P) = \sum \Phi(st_\Delta f, A_\Delta)$, $\Delta \subset M$.

1. Mappings

Let P be a Peano space, and let P^* be a metric space. Denote by $\mathfrak{A}$ the collection of all open subsets of P. Let $\mathfrak{T}^*$ be the class of all continuous mappings (T, A) from any $A \in \mathfrak{A}$ into P^*. The subclass of $\mathfrak{T}^*$ consisting of all mappings (T, P) from P into P^* will be designated by $\mathfrak{T}$. It is well-known that each $(T, P) \in \mathfrak{T}$ admits of a monotone-light factorization [10]. However, this paper is independent of this particular factorization of (T, P), and hence we will consider *unrestricted* factorizations [4].

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DEFINITION 1. An *unrestricted factorization* of a mapping $(T, A) \in \mathfrak{T}^*$ consists of a Peano space M, called *middle space*, and two continuous mappings s, f such that $f:A \to M$, $s:M \to P^*$, $(T, A) = sf$.

The definition below is a suitable generalization of the corresponding one in [4].

DEFINITION 2. Let $(T, P) \in \mathfrak{T}$ be given. Two mappings (T', A'), (T'', A'') of $\mathfrak{T}^*$ constitute a *partition* of (T, P) provided there are a finite set of points F in P^* and a pair of nonempty closed subsets E', E'' of P such that

(1) $P = E' \cup E''$, $E' \subset A'$, $E'' \subset A''$;
(2) $T'(x) = T(x)$ for $x \in E'$, and T' maps $A' - E'$ into F;
(3) $T''(x) = T(x)$ for $x \in E''$, and T'' maps $A'' - E''$ into F;
(4) $T(E' \cap E'') \subset F$.

2. Functional

In the sequel we will consider real-valued functionals Φ defined on $\mathfrak{T}^*$ satisfying the following conditions:

(a) $\Phi(T, A) \geqq 0$ for all $(T, A) \in \mathfrak{T}^*$. For some $(T, A) \in \mathfrak{T}^*$ we may have $\Phi(T, A) = +\infty$.

(b) Φ is lower semicontinuous on $\mathfrak{T}$, i.e., if (T_n, P), $n = 1, 2, \cdots$, is a sequence of mappings in $\mathfrak{T}$ converging on P uniformly to a mapping (T, P), then $\Phi(T, P) \leqq \liminf \Phi(T_n, P)$ as $n \to \infty$.

(c) Φ is *additive under partition*, i.e., if (T', A'), (T'', A'') constitute a partition of a mapping (T, P) in $\mathfrak{T}$, then $\Phi(T, P) = \Phi(T', A') + \Phi(T'', A'')$.

(d) If $(T, A) \in \mathfrak{T}^*$ admits of an unrestricted factorization $(T, A) = sf$, $f:A \to M$, $s:M \to P^*$, where M is a dendrite, then $\Phi(T, A) = 0$.

Remark. In view of (d) if (T, A) is constant, then $\Phi(T, A) = 0$. Moreover, $\Phi(T, A)$ will be defined to be zero in case $A = \emptyset$.

3. Preliminary results

In this paragraph we will make use of the theory of B-sets, local A-sets, and the concept of retraction [6; 7].

Let $(T, P) = sf, f:P \to M, s:M \to P^*$ be an unrestricted factorization of a mapping $(T, P) \in \mathfrak{T}$. Let B be a local A-set of M, and let t be a retraction from M onto B, i.e., (1) there exists a connected open set G of M containing B such that B is a (G, A)-set of M; (2) $t \mid G$ is the identity on B and sends every component of $G - B$ into its frontier relative to G; (3) $t(M - G)$ is a subset of a dendrite $D \subset B$.

Let $\{Q\}$ be the collection of components of $M - B$ for which $Q - G \neq \emptyset$. By [7, §5] the collection $\{Q\}$ is finite, and since B is a (G, A)-set of M, the set $C = \bigcup \mathrm{Fr}\,(Q)$, where the union is extended over all $Q \in \{Q\}$, is finite, say $C = \{x_1, \cdots, x_n\}$. Let $2\eta = \min\,[\rho(x_i, x_j), i \neq j, i, j = 1, \cdots, n]$, where ρ is the distance function of M. Let for each i, O_i be a connected open set with diameter less than η containing x_i. Denote by K the union of all $Q \in \{Q\}$, and let $O = O_1 \cup \cdots \cup O_n$. The set $G' = K \cup O$ is clearly open in M. In the lemma $c(K)$ stands for the closure of K.

LEMMA. *The mapping t' from G' onto $c(K)$ defined by $t'(x) = x$, if $x \in c(K)$, and $t'(x) = x_i$, if $x \in O_i - c(K)$, $i = 1, \cdots, n$, is continuous.*

Proof. It suffices to show that t' restricted to $c(K) \cup O_i$ is continuous. Clearly, t' is continuous on $c(K)$ and on $O_i - c(K)$. Since the frontier of $O_i - c(K)$ relative to G' is the point x_i, it follows that t' is continuous on $c(K) \cup O_i$.

(i) THEOREM. *Under the above conditions, let Φ be a real-valued, nonnegative functional on $\mathfrak{T}^*$ satisfying* (c), (d) *of* §2. *Let $A = f^{-1}(G)$. Then $\Phi(stf, P) = \Phi(stf, A)$.*

Proof. We may assume that $f(P) \cap B \neq \emptyset$. For, if $f(P) \cap B = \emptyset$, we have in view of (d), $\Phi(stf, P) = \Phi(stf, A) = 0$ (see also [7, §8]). We may also assume that $f(P) \cap (M - G) \neq \emptyset$. Otherwise, $f(P) \subset G$ and consequently $A = P$. Since $f(P) \cap (M - G) \neq \emptyset$, it follows that $f(P) \cap K \neq \emptyset$. Let now $E' = f^{-1}[c(K)]$ and let $E = f^{-1}(M - K)$. Then E', E are two nonempty closed sets of P whose union is P. Let $A' = f^{-1}(G')$. Then A', A are open subsets of P such that $A' \supset E'$, $A \supset E$. Let t' be the mapping of the lemma.

Let $F = s(x_1 \cup \cdots \cup x_n)$. Then the mappings (stf, A), $(stt'f, A')$ constitute a partition of (stf, P). Consequently, $\Phi(stf, P) = \Phi(stf, A) + \Phi(stt', A')$. We will show now that $(stt'f, A')$ admits of an unrestricted factorization whose middle space is the dendrite D. By [7, §8] we infer that $t[c(K)] \subset D$, and thus $(stt'f, A') = s(tt'f)$, $tt'f:A' \to D$, $s:D \to P^*$. By (d), $\Phi(stt'f, A') = 0$, and therefore $\Phi(stf, P) = \Phi(stf, A)$.

(ii) COROLLARY. *Under the conditions of* (i), *if B is a (G^*, A)-set of M and $A^* = f^{-1}(G^*)$, then $\Phi(st^*f, A^*) = \Phi(stf, A)$, where t^* is a retraction from M onto the (G^*, A)-set B.*

In the sequel the following observation will prove useful. Let P be a Peano space which can be written as the union of two B-sets B_1, B_2 with $B_1 \cap B_2$ finite. Then B_1, B_2 are local A-sets. To prove this, note that every component G of $P - B_i$, $i = 1, 2$, has its frontier in $B_1 \cap B_2$, and thus the number of components G of $P - B_i$, $i = 1, 2$, with a nondegenerate frontier is finite. From [7] the assertion follows.

Let $(T, P) = sf$, $f:P \to M$, $s:M \to P^*$ be an unrestricted factorization of a mapping $(T, P) \in \mathfrak{T}$. Assume that M can be written as the union of two B-sets B_1, B_2 with $B_1 \cap B_2$ finite. Then from the above remark, B_i is a (G_i, A)-set of M, $i = 1, 2$. Let t_i be a retraction from M onto the (G_i, A)-set B_i, $i = 1, 2$. Finally, let Φ be a real-valued nonnegative functional defined on $\mathfrak{T}^*$ satisfying (c), (d) of §2.

(iii) THEOREM. *Under the above conditions, the following formula subsists*: $\Phi(T, P) = \Phi(st_1 f, P) + \Phi(st_2 f, P)$.

Proof. For $i = 1, 2$, t_i maps each component of $M - B_i$ into either a single point or into a given dendrite $D_i \subset B_i$. We may assume that $B_i \cap f(P) \neq \emptyset$, $i = 1, 2$, as otherwise the theorem follows readily in view of (d).

Let $F = s(B_1 \cap B_2)$. Then F is a finite set of points in P^*. Define $A_1 = f^{-1}(G_1)$, $A_2 = f^{-1}(G_2)$, $E_1 = f^{-1}(B_1)$, $E_2 = f^{-1}(B_2)$. Then A_1, A_2 are open subsets of P such that $E_1 \subset A_1$, $E_2 \subset A_2$. Finally, define mappings T_1, T_2 from A_1, A_2 into P^* by $T_1(x) = st_1 f(x)$, $x \in A_1$ and $T_2(x) = st_2 f(x)$, $x \in A_2$.

We assert that (T_1, A_1), (T_2, A_2) constitute a partition of (T, P). We only have to verify that T_i maps $A_i - E_i$ into F, $i = 1, 2$. Let $p \in A_i - E_i$. Then $f(p) \in G_i - B_i$ and consequently $t_i f(p) \in B_1 \cap B_2$. Thus $st_i f(p) \in F$, which proves the assertion. Accordingly we have $\Phi(T, P) = \Phi(T_1, A_1) + \Phi(T_2, A_2)$. Application of (i) completes the proof.

For later reference we will state here a cyclic additivity theorem due to E. J. Mickle and T. Radó [4]. Let Φ be a real-valued nonnegative functional satisfying the conditions of §2. On $\mathfrak{T}$, the class of all continuous mappings (T, P) from P into P^*, Φ satisfies the conditions of [4]. Consequently, we have by [4] the following theorem.

(iv) THEOREM. *Under the above conditions, we have for $(T, P) \in \mathfrak{T}$ the additivity formula*

$$\Phi(T, P) = \sum \Phi(sr_C f, P), \qquad C \subset M, \tag{1}$$

where $(T, P) = sf$, $f{:}P \to M$, $s{:}M \to P^*$ is an unrestricted factorization of (T, P), r_C is the monotone retraction from M onto a proper cyclic element C of M, and where the summation in (1) *is extended over all proper cyclic elements C of M.*

4. Some lemmas

Let $(T, P) = sf$, $f{:}P \to M$, $s{:}M \to P^*$ be an unrestricted factorization of a mapping $(T, P) \in \mathfrak{T}$. Assume there exists a finite number of B-sets $B_1, \cdots, B_n$ of M such that (1) $M = B_1 \cup \cdots \cup B_n$; (2) $(B_1 \cup \cdots \cup B_i) \cap B_{i+1}$ is finite, $i = 1, \cdots, n - 1$. Finally, let Φ be a real-valued nonnegative functional defined on $\mathfrak{T}^*$ satisfying (c), (d) of §2.

(i) LEMMA. *Under the above conditions, there exist retractions $t_1, \cdots, t_n$ from M onto $B_1, \cdots, B_n$, respectively, such that $\Phi(T, P) = \sum_{i=1}^{n} \Phi(st_i f, P)$.*

Proof. In view of (2) we have by [6] that $B_1 \cup \cdots \cup B_{n-1} = B^*_{n-1}$ is a B-set of M, and $B^*_{n-1} \cap B_n$ reduces to a finite number of points. Let now t^*_{n-1}, t_n be retractions from M onto B^*_{n-1}, B_n, respectively. Then by §3(iii) there follows that $\Phi(T, P) = \Phi(st^*_{n-1} f, P) + \Phi(st_n f, P)$.

Proceeding inductively assume that retractions t_i from M onto B_i, $1 < k \leqq i \leqq n$, and a retraction t^*_{k-1} from M onto $B^*_{k-1} = B_1 \cup \cdots \cup B_{k-1}$ have been defined such that

$$\Phi(T, P) = \Phi(st^*_{k-1} f, P) + \textstyle\sum_{i=k}^{n} \Phi(st_i f, P). \tag{3}$$

The mapping $(st_{k-1}^* f, P)$ admits of an unrestricted factorization $t_{k-1}^* f: P \to B_{k-1}^*$, $s: B_{k-1}^* \to P^*$. Set $B_{k-2}^* = B_1 \cup \cdots \cup B_{k-2}$. Since B_{k-2}^*, B_{k-1} have a finite intersection, they are local A-sets. By [7] we have retractions τ_{k-2}, t_{k-1}' from B_{k-1}^* onto B_{k-2}^*, B_{k-1}, respectively, such that $t_{k-2}^* = \tau_{k-2} t_{k-1}^*$, $t_{k-1} = t_{k-1}' t_{k-1}^*$. Thus by §3(iii),

$$\Phi(st_{k-1}^* f, P) = \Phi(s\tau_{k-2} t_{k-1}^* f, P) + \Phi(st_{k-1}' t_{k-1}^* f, P) = \Phi(st_{k-2}^* f, P) + \Phi(st_{k-1} f, P),$$

and in view of (3), $\Phi(T, P) = \Phi(st_{k-2}^* f, P) + \sum_{i=k-1}^{n} \Phi(st_i f, P)$.

In the sequel we will have to restrict ourselves to factorizations whose middle space M is of finite degree of multicoherence. Let us note that every B-set of a Peano space P of finite degree of multicoherence $r(P)$ is a local A-set of P [7].

Definition. A Peano space M will be termed a *generalized dendrite* provided M possesses no fine-cyclic elements.

(ii) Lemma. *Let Φ be a real-valued nonnegative functional defined on $\mathfrak{T}^*$ satisfying* (c), (d) *of* §2. *If (T, P) admits of an unrestricted factorization $(T, P) = sf$, $f: P \to M$, $s: M \to P^*$, $r(M) < \infty$, where M is a generalized dendrite, then $\Phi(T, P) = 0$.*

Proof. By [7], M can be written as a finite union of dendrites $D_1, \cdots, D_n$ which are B-sets of M, and $(D_1 \cup \cdots \cup D_i) \cap D_{i+1}$ is finite, $i = 1, \cdots, n-1$. By (i) we have retractions $t_1, \cdots, t_n$ from M onto $D_1, \cdots, D_n$ such that $\Phi(T, P) = \sum_{i=1}^{n} \Phi(st_i f, P)$. For each i, $st_i f$ admits of an unrestricted factorization $t_i f: P \to D_i$, $s: D_i \to P^*$, and consequently $\Phi(st_i f, P) = 0$, $i = 1, \cdots, n$. This completes the proof.

5. Fine-cyclic additivity theorem

Let $(T, P) = sf$, $f: P \to M$, $s: M \to P^*$, $r(M) < \infty$, be an unrestricted factorization of a mapping $(T, P) \in \mathfrak{T}$. Let $\{\Delta\}$ be the sequence of fine-cyclic elements of M [6]. Each $\Delta \in \{\Delta\}$ is also a local A-set [7], and consequently with each $\Delta \in \{\Delta\}$ there is associated a connected open set G_Δ containing Δ such that Δ is a (G_Δ, A)-set of M. Let now t_Δ be the retraction from G_Δ onto Δ [7], and set $A_\Delta = f^{-1}(G_\Delta)$, $\Delta \in \{\Delta\}$.

Theorem. *Let Φ be a functional defined on $\mathfrak{T}^*$ satisfying the conditions of* §2. *Then*

$$\Phi(T, P) = \sum \Phi(st_\Delta f, A_\Delta), \qquad \Delta \in \{\Delta\}. \tag{1}$$

Proof. If M contains no fine-cyclic elements, M is a generalized dendrite, and thus from §4(ii) the formula (1) follows. We may thus assume that M possesses fine-cyclic elements. By [7] we have a finite number of B-sets $B_1, \cdots, B_n$ of M satisfying the following properties: (a) $M = B_1 \cup \cdots \cup B_n$, (b) $(B_1 \cup \cdots \cup B_i) \cap B_{i+1}$ is a finite set of points, $i = 1, \cdots, n-1$, (c) each fine-cyclic element of M is a proper cyclic element of a unique B_i, (d) each proper cyclic element of B_i is a fine-cyclic element of M.

In view of (a) and (b) we have by §4(i) retractions $t_1, \cdots, t_n$ from M onto $B_1, \cdots, B_n$, respectively, such that

$$\Phi(T, P) = \sum_{i=1}^{n} \Phi(st_i f, P). \tag{2}$$

We may assume that none of the B_i are dendrites. Let then Δ be a fine-cyclic element of B_i. By (c), Δ is a proper cyclic element of B_i. Let r_Δ be the monotone retraction from B_i onto Δ. By §3(iv), in view of (c) and (d),

$$\Phi(st_i f, P) = \sum \Phi(sr_\Delta t_i f, P), \qquad \Delta \subset B_i. \tag{3}$$

By [7], $r_\Delta t_i = t_\Delta$ is a retraction from M onto Δ, and hence by (2) and (3),

$$\Phi(T, P) = \sum_{i=1}^{n} \sum_{\Delta \subset B_i} \Phi(st_\Delta f, P). \tag{4}$$

By applying §3(i), (ii) the desired formula (1) follows.

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