AN AXIOMATIZATION OF THE HOMOTOPY GROUPS

BY

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1. Introduction

In the present paper an axiomatic characterization of homotopy groups will be given. The possibility of such a characterization was conjectured by S. Eilenberg and N. E. Steenrod in [1]. Another such axiomatization, which is essentially different from the present one, is due to J-P. Serre, J. W. Milnor [6], M. Kuranishi [5] and S. T. Hu [2]. The main difference is that we do *not* postulate that "π_0 is the set of the components". Nonessential is the fact that we consider only absolute groups. The results will be stated in terms of c.s.s. complexes. Free use will be made of the definitions and results of [3], [4], and [7].

In an appendix we discuss the influence of the first two axioms (homotopy and exactness) on $\pi_1(K(Z, 1))$, which plays a role similar to that of the coefficient group in homology theory.

2. The main result

Let $\mathcal{S}$ be the category of c.s.s. complexes with base point ([3], §2), and let $\mathcal{S}_c$ be the subcategory of the c.s.s. complexes which are of the weak homotopy type of a countable c.s.s. complex ([4], §6). We shall define the notion of a theory of homotopy groups on a subcategory $\mathcal{S}' \subset \mathcal{S}$ and state uniqueness theorems for theories of homotopy groups on the categories $\mathcal{S}_c$ and $\mathcal{S}$.

Let $\mathcal{G}$ be the category of groups, and let $\mathcal{G}_c$ be the subcategory of the countable groups. All groups will be written multiplicatively. A group consisting of one element will be denoted by 1.

DEFINITION 1. A *theory of homotopy groups* $\{\pi_i, \partial_i\}$ on a subcategory $\mathcal{S}' \subset \mathcal{S}$ is a collection which contains for every integer $n > 0$

(a) a functor $\pi_n : \mathcal{S}' \to \mathcal{G}$,

(b) a function ∂_{n+1} which assigns to every fibre sequence $F \xrightarrow{q} E \xrightarrow{p} B$ in $\mathcal{S}'$ ([3], §3) a homomorphism $\partial_{n+1}(q, p) : \pi_{n+1}(B) \to \pi_n(F)$ satisfying the naturality condition:

If commutativity holds in the diagram

$$\begin{array}{ccccc} F & \xrightarrow{q} & E & \xrightarrow{p} & B \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle b} \\ F' & \xrightarrow{q'} & E' & \xrightarrow{p'} & B' \end{array}$$

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where both horizontal sequences are fibre sequences, then the following diagram is also commutative

$$\begin{array}{ccc} \pi_{n+1}(B) & \xrightarrow{\partial_{n+1}(q,p)} & \pi_n(F) \\ \downarrow{\scriptstyle \pi_{n+1}(b)} & & \downarrow{\scriptstyle \pi_n(f)} \\ \pi_{n+1}(B') & \xrightarrow{\partial_{n+1}(q',p')} & \pi_n(F'). \end{array}$$

Unless stated otherwise the functions π_i and ∂_i are required to satisfy the following four axioms:

Axiom I. If $f:K \to L$ is a weak homotopy equivalence, then

$$\pi_n(f):\pi_n(K) \to \pi_n(L)$$

is an isomorphism for all n.

Axiom II. If $F \xrightarrow{q} E \xrightarrow{p} B$ is a fibre sequence, then the sequence

$$\to \pi_n(F) \xrightarrow{\pi_n(q)} \pi_n(E) \xrightarrow{\pi_n(p)} \pi_n(B) \xrightarrow{\partial_n(q,p)} \cdots \xrightarrow{\pi_1(q)} \pi_1(E) \xrightarrow{\pi_1(p)} \pi_1(B)$$

is exact. If F is connected, then $\pi_1(p)$ is onto.

Axiom III. If $K_1 \vee K_2$ is the union of K_1 and K_2 (with the base points identified) and if $k_i:K_i \to K_1 \vee K_2$ are the embedding maps ($i = 1, 2$), then $\pi_1(K_1 \vee K_2)$ is the free product of $\pi_1(K_1)$ and $\pi_1(K_2)$ under the maps $\pi_1(k_i)$.

Axiom IV. $\pi_n(K) \neq 1$ for some K and n.

Clearly the usual homotopy groups and boundary homomorphisms of a fibre sequence satisfy the above axioms. That this theory of homotopy groups is essentially the only one possible on $\mathcal{S}_c$ (the category of c.s.s. complexes which are of the weak homotopy type of a countable c.s.s. complex) is an immediate consequence of the following uniqueness theorem.

Theorem 1. *Let $\{\pi_i, \partial_i\}$ and $\{\bar{\pi}_i, \bar{\partial}_i\}$ be theories of homotopy groups on the category $\mathcal{S}_c$. Then there exist unique natural equivalences*

$$h_n:\pi_n \to \bar{\pi}_n \qquad (n = 1, 2, \cdots)$$

such that for every fibre sequence $F \xrightarrow{q} E \xrightarrow{p} B$ commutativity holds in the diagram

$$(1)\qquad \begin{array}{ccc} \pi_{n+1}(B) & \xrightarrow{\partial_{n+1}(q,p)} & \pi_n(F) \\ \downarrow{\scriptstyle h_{n+1}(B)} & & \downarrow{\scriptstyle h_n(F)} \\ \bar{\pi}_{n+1}(B) & \xrightarrow{\bar{\partial}_{n+1}(q,p)} & \bar{\pi}_n(F). \end{array}$$

The proof of Theorem 1 will be given in §4. In this proof an important role will be played by the following group-theoretical lemma.

LEMMA 1. *Let* $I:\mathcal{G}_c \to \mathcal{G}$ *be the inclusion functor, and let* $M:\mathcal{G}_c \to \mathcal{G}$ *be a functor such that*

(a) *M preserves short exact sequences, i.e. if a sequence*

$$1 \to A \xrightarrow{s} B \xrightarrow{t} C \to 1$$

is exact, then so is the sequence

$$1 \to M(A) \xrightarrow{M(s)} M(B) \xrightarrow{M(t)} M(C) \to 1,$$

(b) *M preserves (two-fold) free products, i.e. if B is the free product of A_i ($i = 1, 2$) under the maps $a_i:A_i \to B$, then $M(B)$ is the free product of $M(A_i)$ under the maps $M(a_i):M(A_i) \to M(B)$.*

Then either there exists exactly one natural equivalence $m:I \to M$, or $M(A) = 1$ for all $A \in \mathcal{G}_c$.

The proof will be given in §5.

It should be noted that the usual theory of homotopy groups on the whole category $\mathcal{S}$ satisfies the following axiom which is stronger than Axiom III.

AXIOM III′. If Q is any set, $K_\alpha \in \mathcal{S}$ for all $\alpha \in Q$, if L is the union of all K_α (with the base points identified) and if $k_\alpha:K_\alpha \to L$ is the embedding map for all $\alpha \in Q$, then $\pi_1(L)$ is the free product of the groups $\pi_1(K_\alpha)$ under the maps $\pi_1(k_\alpha):\pi_1(K_\alpha) \to \pi_1(L)$.

It then follows immediately from the following uniqueness theorem that the usual theory of homotopy groups is essentially the only theory of homotopy groups on the whole category $\mathcal{S}$ satisfying Axiom III′.

THEOREM 2. *Let $\{\pi_i, \partial_i\}$ and $\{\bar{\pi}_i, \bar{\partial}_i\}$ be theories of homotopy groups on the category $\mathcal{S}$ satisfying Axiom III′. Then there exist unique natural equivalences $h_n:\pi_n \to \bar{\pi}_n$ ($n = 1, 2, \cdots$) such that for every fibre sequence*

$$F \xrightarrow{q} E \xrightarrow{p} B$$

commutativity holds in diagram (1).

The proof of Theorem 2 is similar to that of Theorem 1 (see §4). The following lemma has to be used instead of Lemma 1.

LEMMA 2. *Let $I:\mathcal{G} \to \mathcal{G}$ be the identity functor and let $M:\mathcal{G} \to \mathcal{G}$ be a functor such that*

(a) *M preserves short exact sequences,*

(b) *M preserves all free products, i.e. if Q is any set, $A_\alpha \in \mathcal{G}$ for all $\alpha \in Q$, and if B is the free product of the groups A_α under the maps $a_\alpha:A_\alpha \to B$, then $M(B)$ is the free product of the groups $M(A_\alpha)$ under the maps $M(a_\alpha):M(A_\alpha) \to M(B)$.*

Then either there exists exactly one natural equivalence $m: I \to M$, *or* $M(A) = 1$ *for all* $A \in \mathcal{G}$.

The proof of Lemma 2 is similar to that of Lemma 1 (see §5).

3. An axiomatization which includes π_0

It is possible to give an axiomatization of the homotopy groups which includes π_0. This can be done as follows:

Let $\mathcal{R}$ be the category of sets with a distinguished element 1. A set consisting of one element will also be denoted by 1. A theory of homotopy groups $\{\pi_i, \partial_i\}$ on a subcategory $\mathcal{S}' \subset \mathcal{S}$ then contains in addition to the functions π_n and ∂_{n+1} $(n > 0)$

(a) a functor $\pi_0: \mathcal{S}' \to \mathcal{R}$,

(b) a function ∂_1 which assigns to every fibre sequence $F \xrightarrow{q} E \xrightarrow{p} B$ a function $\partial_1(q, p): \pi_1(B) \to \pi_0(F)$ such that $\partial_1(q, p)1 = 1$, and satisfying the obvious naturality condition.

Axioms I, III, and IV need not be changed, but Axiom II has to be replaced by

Axiom II′. If $F \xrightarrow{q} E \xrightarrow{p} B$ is a fibre sequence, then the sequence

$$\to \pi_n(F) \xrightarrow{\pi_n(q)} \pi_n(E) \xrightarrow{\pi_n(p)} \pi_n(B) \xrightarrow{\partial_n(q,p)} \cdots \xrightarrow{\pi_0(p)} \pi_0(B) \to 1$$

is exact in the sense that always "*the image of one map*" = "*the kernel of the next map*".

Clearly the usual theory of homotopy groups satisfies Axiom II′. That this theory of homotopy groups is essentially the only one possible on the category $\mathcal{S}_c$ is a consequence of the following uniqueness theorem.

Theorem 3. *Let* $\{\pi_i, \partial_i\}$ *and* $\{\bar{\pi}_i, \bar{\partial}_i\}$ *be theories of homotopy groups on the category* $\mathcal{S}_c$ *which satisfy Axiom II′. Then there exist unique natural equivalences* $h_n: \pi_n \to \bar{\pi}_n$ $(n = 0, 1, 2, \cdots)$ *such that for every fibre sequence*

$$F \xrightarrow{q} E \xrightarrow{p} B$$

and integer $n > 0$ *commutativity holds in diagram* (1), *while for* $n = 0$ *the diagram* (1) *is either always commutative or always anticommutative, i.e. for every element* $\beta \in \pi_1(\beta)$

$$(h_0(F) \circ \partial_1(q, p))\beta = (\bar{\partial}_1(q, p) \circ h_1(B))\beta^{-1}.$$

The proof will be given in §6.

4. Proof of Theorem 1

It clearly suffices to prove Theorem 1 only for the case that $\{\pi_i, \partial_i\}$ is the usual theory of homotopy groups on $\mathcal{S}_c$.

The proof consists of three parts. We first show, using only Axioms I and II, that $\bar{\pi}_n(K) = 1$ for certain n and K (Propositions 1–5) and that $\bar{\pi}_n(K) \approx \bar{\pi}_1(L)$ for all n and K and suitable L (Proposition 6). Then we define a functor $M:\mathcal{G}_c \to \mathcal{G}$ and prove, using again only Axioms I and II, that it preserves short exact sequences (Proposition 7) and, using Axioms I, II, and III, that it preserves (two-fold) free products (Proposition 8). Only at the last stage we need Axiom IV (in the proof of Proposition 9) in order to be able to define the natural equivalences $h_n:\pi_n \to \bar{\pi}_n$.

PROPOSITION 1. *Let P be a c.s.s. complex with one simplex in every dimension. Then $\bar{\pi}_n(P) = 1$ for all n.*

Proof. Let $i:P \to P$ be the identity map. Then

$$P \xrightarrow{i} P \xrightarrow{i} P$$

is a fibre sequence. Hence application of Axiom II yields the desired result.

PROPOSITION 2. *Let K be contractible. Then $\bar{\pi}_n(K) = 1$ for all n.*

Proof. This follows immediately from Proposition 1 and Axiom I.

PROPOSITION 3. *Let K be connected, and let $\pi_1(K) = 1$. Then*

$$\bar{\pi}_1(K) = 1.$$

Proof. Because K is connected there exists (see [4]) a fibre sequence

$$G(K) \to E(K) \to K$$

such that $E(K)$ is contractible. Application of Proposition 2 and Axiom II then yields that $\bar{\pi}_1(K) = 1$.

PROPOSITION 4. *Let S^0 be a c.s.s. complex with two simplices in every dimension. Then $\bar{\pi}_n(S^0) = 1$ for $n > 0$.*

Proof. Let $\phi \in S^0$ be the base point and let τ be the other 0-simplex. Define maps $p:S^0 \times S^0 \to S^0$, $t:S^0 \times S^0 \to S^0$ and $q:S^0 \to S^0 \times S^0$ by

$$p(\phi, \phi) = p(\tau, \phi) = \phi, \qquad p(\phi, \tau) = p(\tau, \tau) = \tau$$

$$t(\phi, \tau) = t(\tau, \phi) = t(\tau, \tau) = \tau,$$

$$q(\tau) = (\tau, \phi).$$

Application of Axiom II and Proposition 1 to the fibre map

$$P \to S^0 \times S^0 \xrightarrow{t} S^0$$

yields that $\bar{\pi}_n(t)$ is an isomorphism for all $n > 0$. As the composition

$$S^0 \xrightarrow{q} S^0 \times S^0 \xrightarrow{t} S^0$$

is the identity map, it follows that $\bar{\pi}_n(q)$ is also an isomorphism. Clearly $\bar{\pi}_n(p)$ is onto for all n. Hence application of Axiom II to the fibre sequence

$$S^0 \xrightarrow{q} S^0 \times S^0 \xrightarrow{p} S^0$$

yields that $\bar{\pi}_n(S^0) = 1$ for all $n > 0$.

PROPOSITION 5. *Let $A \in \mathcal{G}_c$. Then $\bar{\pi}_n(K(A, 1)) = 1$ for $n > 1$.*

Proof. There exists (see [7]) a fibre sequence

$$K(A, 0) \xrightarrow{q} E \xrightarrow{p} K(A, 1),$$

where E is contractible. Hence by Axiom II and Proposition 2

$$\bar{\partial}_n(q, p): \bar{\pi}_n(K(A, 1) \to \bar{\pi}_{n-1}(K(A, 0))$$

is an isomorphism for $n > 1$. Now consider the fibre sequence

$$P \to K(A, 0) \xrightarrow{a} S^0,$$

where $a(\sigma) = \tau$ for every 0-simplex $\sigma \in K(A, 0)$ different from the base point. Then it follows from Axiom II and Propositions 1 and 4 that

$$\bar{\pi}_{n-1}(K(A, 0)) = 1$$

for $n > 1$. Hence also $\bar{\pi}_n(K(A, 1)) = 1$ for $n > 1$.

Let K be a connected c.s.s. complex. Then there exists (see [3]) a fibre sequence

$$G(K) \xrightarrow{t} E(K) \xrightarrow{s} K,$$

where $E(K)$ is contractible and $G(K)$ satisfies the extension condition.

For every K and $n > 0$ consider the fibre sequence

$$K'_n \xrightarrow{q_n} K_n \xrightarrow{p_n} K(\pi_1(K_n), 1),$$

where

(a) K_1 is the subcomplex of $\mathrm{Ex}^\infty K$ consisting of the simplices of which all 0-dimensional faces are of the base point,

(b) K'_n is the subcomplex of K_n consisting of the simplices of which all 1-dimensional faces are of the base point (it is simply connected),

(c) $K_{n+1} = G(K'_n)$,

(d) $q_n: K'_n \to K_n$ is the inclusion map,

(e) p_n is the canonical map $p_n: K_n \to K(\pi_1(K_n), 1)$ (see [7]).

The complexes K_{n+1} and K'_n are connected by a fibre sequence

$$K_{n+1} \xrightarrow{t_n} E(K'_n) \xrightarrow{s_n} K'_n,$$

where $E(K'_n)$ is contractible.

Let $e^\infty K: K \to \mathrm{Ex}^\infty K$ be the embedding map and denote by $j: K_1 \to \mathrm{Ex}^\infty K$ the inclusion map. Then

PROPOSITION 6. *The maps $\bar{\pi}_1(p)$, $\bar{\pi}_2(q_{n-1})$, $\cdots$, $\bar{\pi}_n(q_1)$, $\bar{\partial}_2(t_{n-1}, s_{n-1})$, $\cdots$, $\bar{\partial}_n(t_1, s_1)$, $\bar{\pi}_n(j)$, and $\bar{\pi}_n(e^\infty K)$ (see Figure I) are isomorphisms.*

Proof. Because K_n' is simply connected it follows from Axiom II and Proposition 3 that $\bar{\pi}_1(p_n)$ is an isomorphism, while Axiom II and Proposition 5 imply that $\bar{\pi}_2(q_{n-1}), \cdots, \bar{\pi}_n(q_1)$ are isomorphisms.

The contractibility of $E(K_n')$ together with Axiom II and Proposition 2 yield that $\bar{\partial}_2(t_{n-1}, s_{n-1}), \cdots, \bar{\partial}_n(t_1, s_1)$ are isomorphisms.

The map $j: K_1 \to \mathrm{Ex}^\infty K$ may be factored

$$K_1 \xrightarrow{j_1} K_* \xrightarrow{j_2} \mathrm{Ex}^\infty K,$$

where K_* is the component of $\mathrm{Ex}^\infty K$ containing K_1. In view of Axiom I, $\bar{\pi}_n(j_1)$ is an isomorphism. If $K_* \neq \mathrm{Ex}^\infty K$, then define a map $k: \mathrm{Ex}^\infty K \to S^0$ in such a manner that

$$K_* \xrightarrow{j_2} \mathrm{Ex}^\infty K \xrightarrow{k} S^0$$

is a fibre sequence. (This clearly is possible in a unique way.) Proposition 4 and Axiom II then imply that $\bar{\pi}_n(j_2)$ is also an isomorphism, and hence $\bar{\pi}_n(j)$ is an isomorphism.

It follows from Axiom I that $\bar{\pi}_n(e^\infty K)$ is an isomorphism.

This completes the proof of Proposition 6.

$$\begin{tikzcd}
 & \pi_n(K) \arrow[dl, "\pi_n(e^\infty K)"] & \bar{\pi}(K) \arrow[dr, "\bar{\pi}_n(e^\infty K)"] & \\
\pi_n(\mathrm{Ex}^\infty K) & \pi_n(K_1) \arrow[l, "\pi_n(j)"'] & \bar{\pi}_n(K_1) \arrow[r, "\bar{\pi}_n(j)"] & \bar{\pi}_n(\mathrm{Ex}^\infty K) \\
\pi_n(K_1') \arrow[ur, "\pi_n(q_1)"] \arrow[r, "\partial_n(t_1, s_1)"] & \pi_{n-1}(K_2) & \bar{\pi}_{n-1}(K_2) & \bar{\pi}_n(K_1') \arrow[ul, "\bar{\pi}_n(q_1)"'] \arrow[l, "\bar{\partial}_n(t_1, s_1)"'] \\
 & \cdots \arrow[u] & \cdots \arrow[u] & \\
\pi_2(K_{n-1}') \arrow[ur] \arrow[r, "\partial_1(t_{n-1}, s_{n-1})"] & \pi_1(K_n) \arrow[d, "m(\pi_1(K_n))"'] & \bar{\pi}_1(K_n) \arrow[dl, "\bar{\pi}_1(p_n)"] & \bar{\pi}_2(K_{n-1}') \arrow[ul] \arrow[l, "\bar{\partial}_1(t_{n-1}, s_{n-1})"'] \\
 & \bar{\pi}_1(K(\pi_1(K_n), 1)) & &
\end{tikzcd}$$

FIGURE I

Let $M: \mathcal{G}_c \to \mathcal{G}$ denote the composite functor

$$\mathcal{G}_c \xrightarrow{K(\ ,1)} \mathcal{S}_c \xrightarrow{\bar{\pi}_1} \mathcal{G}.$$

PROPOSITION 7. *The functor $M:\mathcal{G}_c \to \mathcal{G}$ preserves short exact sequences.*

Proof. Let

$$1 \to A \xrightarrow{s} B \xrightarrow{t} C \to 1$$

be an exact sequence. Then it is readily verified that

$$K(A, 1) \xrightarrow{K(s, 1)} K(B, 1) \xrightarrow{K(t, 1)} K(C, 1)$$

is a fibre sequence, and because $K(A, 1)$ is connected, it now follows from Axiom II and Proposition 5 that the sequence

$$1 \to M(A) \xrightarrow{M(s)} M(B) \xrightarrow{M(t)} M(C) \to 1$$

is also exact.

PROPOSITION 8. *The functor $M:\mathcal{G}_c \to \mathcal{G}$ preserves (two-fold) free products.*

Proof. Let B be the free product of A_i ($i = 1, 2$) under the maps $a_i:A_i \to B$. Let $k_i:K(A_i, 1) \to K(A_1, 1) \vee K(A_2, 1)$ be the embedding maps and let $j:K(A_1, 1) \vee K(A_2, 1) \to K(B, 1)$ be the (unique) map such that the following diagram is commutative.

$$\begin{tikzcd}
K(A_1, 1) \arrow[r, "k_1"] \arrow[dr, "K(a_1, 1)"'] & K(A_1, 1) \vee K(A_2, 1) \arrow[d, "j"] & K(A_2, 1) \arrow[l, "k_2"'] \arrow[dl, "(Ka_2, 1)"] \\
 & K(B, 1) &
\end{tikzcd}$$

It is readily verified that $\pi_1(j):\pi_1(K(A_1, 1) \vee K(A_2, 1)) \to \pi_1(K(B, 1))$ is an isomorphism. Furthermore $\pi_i(K(B, 1)) = 1$ for $i > 1$ and it follows from a lemma of J. H. C. Whitehead (see [9]) and the fact that

$$\pi_i(K(A_1, 1)) = \pi_i(K(A_2, 1)) = 1$$

for $i > 1$, that $\pi_i((K(A_1, 1) \vee K(A_2, 1)) = 1$ for $i > 1$. Hence the map j is a weak homotopy equivalence.

Application of the functor $\bar{\pi}_1$ yields the commutative diagram

$$\begin{tikzcd}
M(A_1) \arrow[r, "\bar{\pi}_1(k_1)"] \arrow[dr, "M(a_1)"'] & \bar{\pi}_1(K(A_1, 1) \vee K(A_2, 1)) \arrow[d, "\bar{\pi}_1(j)"] & M(A_2) \arrow[l, "\bar{\pi}_1(k_2)"'] \arrow[dl, "M(a_2)"] \\
 & M(B) &
\end{tikzcd}$$

By Axiom III (this is the only place where Axiom III is used)

$$\bar{\pi}_1(K(A_1, 1) \vee K(A_2, 1))$$

is the free product of $M(A_i)$ under the maps $\bar{\pi}_1(k_i)$, and by Axiom I, $\bar{\pi}_1(j)$ is an isomorphism. Hence $M(B)$ is the free product of $M(A_i)$ under the maps $M(a_i)$.

PROPOSITION 9. *There exists a unique natural equivalence* $m:I \to M$.

Proof. Suppose this is not the case. Then by Propositions 7 and 8 and Lemma 1, $M(A) = 1$ for all $A \in \mathcal{G}_c$, and in view of Proposition 6 this would imply that $\bar{\pi}_n(K) = 1$ for all n and K. By Axiom IV however this is impossible.

We now define for every $K \in \mathcal{S}_c$ and every integer $n > 0$ an isomorphism $h_n(K):\pi_n(K) \to \bar{\pi}_n(K)$ as the composition of all the isomorphisms in Figure I. Then it can readily be verified by "chasing diagram" that the functions h_n are natural equivalences $h_n:\pi_n \to \bar{\pi}_n$ such that for every fibre sequence $F \xrightarrow{q} E \xrightarrow{p} B$ commutativity holds in diagram (1) and that the uniqueness of the natural equivalence m implies the uniqueness of the natural equivalences $h_n:\pi_n \to \bar{\pi}_n$.

This completes the proof of Theorem 1.

5. Proof of Lemma 1[2]

The proof of Lemma 1 consists of two parts. First it is shown that $M(Z)$ is either infinite cyclic or trivial (Proposition 13), and then we prove that in the first case there exists exactly one natural equivalence $m:I \to M$ (Propositions 15 and 16), while in the other case $M(A) = 1$ for all $A \in \mathcal{G}_c$.

PROPOSITION 10. *Let D be the direct sum of A_i ($i = 1, 2$) under the maps $d_i:A_i \to D$. Then $M(D)$ is the direct sum of $M(A_i)$ under the maps $M(d_i)$.*

Proof. Let $j_i:A_i \to A_i$ be the identity maps, and let $p_i:D \to A_i$ be maps such that in the diagram

$$\begin{tikzcd}
1 \arrow[dr] & & & & 1 \arrow[dl] \\
& A_1 \arrow[dr, "d_1"] \arrow[dd, "j_2"'] & & A_2 \arrow[dl, "d_2"'] \arrow[dd, "j_2"] & \\
& & D \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \\
& A_1 \arrow[dl] & & A_2 \arrow[dr] & \\
1 & & & & 1
\end{tikzcd}$$

[2] A more straightforward proof of Lemma 1 will be given in a forthcoming paper in Bol. Soc. Mat. Mexicana, *On monoids and their dual.*

both sequences on the diagonal are exact. That D is the direct sum of A_i under the maps d_i then is equivalent to the statement that the above diagram is commutative. As application of the functor M yields a similar diagram, it follows that $M(D)$ is the direct sum of $M(A_i)$ under the maps $M(d_i)$.

PROPOSITION 11. *If A is abelian, then so is $M(A)$.*

Proof. Let D be the direct sum of A with itself under the maps

$$d_i: A \to D.$$

That A is abelian then is equivalent to the statement that there exists a map $g: D \to A$ such that the following diagram is commutative

$$\begin{tikzcd} & A \arrow[dl, "d_1"'] \arrow[dr, "i"] & \\ D \arrow[rr, "q"] & & A \\ & A \arrow[ul, "d_2"] \arrow[ur, "i"'] & \end{tikzcd}$$

where $i: A \to A$ is the identity map. By Proposition 10 application of the functor M yields a similar diagram. Hence $M(A)$ is also abelian.

PROPOSITION 12. *Let A be abelian, and let B be the free product of A with itself under the maps $a_i: A \to B$ ($i = 1, 2$). Let $p: B \to A$ be the (unique) map such that commutativity holds in the diagram*

$$\begin{tikzcd} & A \arrow[dl, "a_1"'] \arrow[dr, "i"] & \\ B \arrow[rr, "p"] & & A. \\ & A \arrow[ul, "a_2"] \arrow[ur, "i"'] & \end{tikzcd}$$

Then "kernel p" is the free group freely generated by the elements $a_1(\alpha) \cdot a_2(\alpha^{-1})$ where $\alpha \in A$ and $\alpha \neq 1$.

The proof of Proposition 12 is straightforward.

PROPOSITION 13. *Let Z be infinite cyclic; then $M(Z)$ is infinite cyclic or trivial.*

Proof. Let Z_2 be cyclic of order 2, let B be the free product of Z_2 with itself under the maps $a_i: Z_2 \to B$ ($i = 1, 2$), and let $p: B \to Z_2$ and $j: Z \to B$

be the maps such that commutativity holds and the horizontal sequence is exact in the diagram

$$\begin{tikzcd} & & Z_2 \arrow[dl, "a_1"'] \arrow[dr, "i"] & & \\ 1 \arrow[r, tail] & Z \arrow[r, "j"] & B \arrow[rr, "p"] & & Z_2 \arrow[r] & 1 \\ & & Z_2 \arrow[ul, "a_2"] \arrow[urr, "i"'] & & \end{tikzcd}$$

where $i:Z_2 \to Z_2$ is the identity. In view of Proposition 12 such a map $j:Z \to B$ exists. Application of the functor M yields a similar diagram. As $M(B)$ is the free product of $M(Z_2)$ with itself under the maps $M(a_i)$, and as in view of Proposition 11 $M(Z_2)$ is abelian, it follows from Proposition 12 that $M(Z)$ is free. However by Proposition 11, $M(Z)$ is also abelian. Hence $M(Z)$ is either infinite cyclic or trivial.

Let F_2 be the free product of Z with itself under the maps $a_i:Z \to F_2$, and let $h:Z \to M(Z)$ be a map. Then there clearly exists a unique map

$$h':F_2 \to M(F_2)$$

such that commutativity holds in the diagram

$$\begin{tikzcd} Z \arrow[r, "a_1"] \arrow[d, "h"] & F_2 \arrow[d, "h'"] & Z \arrow[l, "a_2"'] \arrow[d, "h"] \\ M(Z) \arrow[r, "M(a_1)"] & M(F_2) & M(Z). \arrow[l, "M(a_2)"'] \end{tikzcd}$$

PROPOSITION 14. *Suppose $M(Z)$ is infinite cyclic. Let $\zeta \in Z$ be a generator, and let $a:Z \to F_2$ be the map given by $a\zeta = a_1\zeta \cdot a_2\zeta$. Then there exists a unique isomorphism $h:Z \to M(Z)$ such that commutativity holds in the diagram*

$$\begin{tikzcd} Z \arrow[r, "a"] \arrow[d, "h"] & F_2 \arrow[d, "h'"] \\ M(Z) \arrow[r, "M(a)"] & M(F_2). \end{tikzcd}$$

In the proof of Proposition 14 use will be made of the following lemma (for a proof see [8]).

LEMMA 3. *Let $w(\alpha, \beta)$ be a reduced word in the free group on two generators such that*

$$w(\alpha, 1) = \alpha, \qquad w(1, \beta) = \beta,$$

$$w(\alpha, w(\beta, \gamma)) = w(w(\alpha, \beta), \gamma);$$

then either $w(\alpha, \beta) = \alpha\beta$ or $w(\alpha, \beta) = \beta\alpha$.

Proof of Proposition 14. Let $f:Z \to M(Z)$ be an isomorphism and let $\zeta' = f\zeta$. Clearly $M(F_2)$ is a free group on the generators $M(a_i)\zeta'$, and hence $M(a)\zeta'$ may be written as a reduced word $w(M(a_1)\zeta', M(a_2)\zeta')$ in these generators. Let $p_1:F_2 \to Z$ be the (unique) map such that $(p_1 \circ a_1)\zeta = \zeta$ and $(p_1 \circ a_2)\zeta = 1$. Then $(p_1 \circ a)\zeta = \zeta$, and hence $(M(p_1) \circ M(a))\zeta' = \zeta'$. Consequently the word w is such that

$$w(\alpha, 1) = \alpha.$$

Similarly it can be shown that

$$w(1, \beta) = \beta.$$

Let F_3 be the free product of three copies of Z under the maps

$$b_j:Z \to F_3 \qquad (j = 1, 2, 3).$$

Then it is readily verified that $M(F_3)$ is the free product of three copies of $M(Z)$ under the maps $M(b_j)$. Let $b:Z \to F_3$ be the map given by

$$b\zeta = b_1\zeta \cdot b_2\zeta \cdot b_3\zeta$$

and let $c:F_2 \to F_3$ be the map such that

$$(c \circ a_1)\zeta = b_1\zeta \quad \text{and} \quad (c \circ a_2)\zeta = b_2\zeta \cdot b_3\zeta.$$

Then $(c \circ a)\zeta = b\zeta$, and hence

$$\begin{aligned} M(b)\zeta' &= (M(c) \circ M(a))\zeta' \\ &= M(c)w(M(a_1)\zeta', M(a_2)\zeta') \\ &= w(M(b_1)\zeta', w(M(b_2)\zeta', M(b_3)\zeta')). \end{aligned}$$

Similarly it can be shown that

$$M(b)\zeta' = w(w(M(b_1)\zeta', M(b_2)\zeta'), M(b_3)\zeta').$$

Hence the word w is such that

$$w(\alpha, w(\beta, \gamma)) = w(w(\alpha, \beta), \gamma),$$

and it follows from Lemma 3 that either $w(\alpha, \beta) = \alpha\beta$ or $w(\alpha, \beta) = \beta\alpha$.

Let $g:Z \to M(Z)$ be the (only) other isomorphism, i.e. $g\zeta = f\zeta^{-1}$. If $w(\alpha, \beta) = \alpha\beta$, then f has the desired property, and it is readily verified that g has not. Conversely if $w(\alpha, \beta) = \beta\alpha$, then f does not have the desired property, but already g does. Hence there is exactly one isomorphism $h: Z \to M(Z)$ with the desired property.

PROPOSITION 15. *There exists at most one natural equivalence* $m: I \to M$.

Proof. Suppose $m: I \to M$ is a natural equivalence. Then it follows from Proposition 14 that $m(Z) = h$. The naturality of m implies that for every group A and map $f: Z \to A$ commutativity holds in the diagram

$$\begin{array}{ccc} Z & \xrightarrow{f} & A \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle m(A)} \\ M(A) & \xrightarrow{M(f)} & M(A). \end{array}$$

As every element $\alpha \in A$ is the image of the generator $\zeta \in Z$ under a unique map $f_\alpha: Z \to A$, it follows that

$$m(A)\alpha = (M(f_\alpha) \circ h)\zeta, \tag{2}$$

i.e. the natural equivalence m is completely determined by the functor M and the (unique) isomorphism h and hence is unique.

In order to prove the first part of Lemma 1 it thus remains to show that

PROPOSITION 16. *Suppose $M(Z)$ is infinite cyclic. For every group A let $m(A): A \to M(A)$ be the function defined by* (2). *Then the function m is a natural equivalence $m: I \to M$.*

Proof. The naturality of m follows immediately from its definition. We first show that $m(A): A \to M(A)$ is a homomorphism for every A and then that it is actually an isomorphism.

Let $\alpha, \beta \in A$. Then we must show that $m(A)(\alpha \cdot \beta) = m(A)\alpha \cdot m(A)\beta$. Let $d: F_2 \to A$ be the map such that $(d \circ a_1)\zeta = \alpha$ and $(d \circ a_2)\zeta = \beta$. Then it follows from the naturality of m that

$$\begin{aligned} m(A)(\alpha \circ \beta) &= (m(A) \circ d \circ a)\zeta = (M(d) \circ M(a) \circ h)\zeta \\ &= M(d)\{(M(a_1) \circ h)\zeta \cdot (M(a_2) \circ h)\zeta\} \\ &= (M(d) \circ M(a_1) \circ h)\zeta \cdot (M(d) \circ M(a_2) \circ h)\zeta \\ &= m(A)\alpha \cdot m(A)\beta. \end{aligned}$$

Consider the commutative diagram

$$\begin{array}{ccccccccc} 1 \to & F_\infty & \xrightarrow{j} & F_2 & \xrightarrow{p} & Z & \to 1 \\ & \downarrow{\scriptstyle m(F_\infty)} & & \downarrow{\scriptstyle m(F_2)} & & \downarrow{\scriptstyle m(Z)} & \\ 1 \to & M(F_\infty) & \xrightarrow{M(j)} & M(F_2) & \xrightarrow{M(p)} & M(Z) & \to 1 \end{array} \tag{3}$$

where $p:F_2 \to Z$ is the map such that $p \circ a_1 = p \circ a_2 =$ identity and $j:F_\infty \to F_2$ is such that the upper sequence (and hence the lower sequence) is exact. It follows from Proposition 12 that F_∞ is an infinitely generated free group. As $m(Z)$ is an isomorphism, so is $m(F_2)$. Application of the "five lemma" (see [1]) now yields that $m(F_\infty)$ is also an isomorphism.

Every group $A \in \mathcal{G}_c$ can be embedded in a commutative diagram

$$\begin{array}{ccccccc} 1 \to F_\infty & \xrightarrow{s} & F_\infty & \xrightarrow{t} & A \to 1 \\ \downarrow{\scriptstyle m(F_\infty)} & & \downarrow{\scriptstyle m(F_\infty)} & & \downarrow{\scriptstyle m(A)} \\ 1 \to M(F_\infty) & \xrightarrow{M(s)} & M(F_\infty) & \xrightarrow{M(t)} & M(A) \to 1 \end{array} \tag{4}$$

where the maps s and t are such that the upper sequence (and hence the lower sequence) is exact. As $m(F_\infty)$ is an isomorphism, it follows from the "five lemma" that $M(A)$ is also an isomorphism.

This completes the proof of Proposition 16.

In order to prove the second part of Lemma 1 we need

PROPOSITION 17. *If $M(Z) = 1$, then $M(A) = 1$ for all $A \in \mathcal{G}_c$.*

Proof. Clearly $M(A) = 1$ implies $M(F_2) = 1$, and hence in view of the exactness of the lower sequence of diagram (3), $M(F_\infty) = 1$. Finally the exactness of the lower sequence of diagram (4) implies that $M(A) = 1$ for all $A \in \mathcal{G}_c$.

This completes the proof of Lemma 1.

6. Proof of Theorem 3

It clearly suffices to prove Theorem 3 for the case that $\{\pi_i, \partial_i\}$ is the usual theory of homotopy groups on $\mathcal{S}_c$.

The proof consists of two parts. We first show that Axioms I and II′ imply Axiom II (Proposition 21). Hence Theorem 1 may be applied and yields the existence and uniqueness of the natural equivalence

$$h_n : \pi_n \to \bar{\pi}_n$$

for $n > 0$. The second part of the proof consists of showing the existence and uniqueness of the natural equivalence $h_0 : \pi_0 \to \bar{\pi}_0$ (Propositions 24 and 25) and proving that this natural equivalence h_0 has the desired property (Proposition 26).

PROPOSITION 18. *Let P be a c.s.s. complex with one simplex in every dimension. Then $\bar{\pi}_n(P) = 1$ for all n.*

PROPOSITION 19. *Let K be contractible. Then $\bar{\pi}_n(K) = 1$ for all n.*

PROPOSITION 20. *Let K be connected. Then $\bar{\pi}_0(K) = 1$.*

The proofs of these propositions are similar to the proofs of Propositions 1, 2, and 3.

PROPOSITION 21. *Let* $F \xrightarrow{q} E \xrightarrow{p} B$ *be a fibre sequence, and let* F *be connected. Then* $\bar{\pi}_1(p):\bar{\pi}_1(E) \to \bar{\pi}_1(B)$ *is onto.*

Proof. This follows immediately from Axiom II′ and Proposition 20.

For every $X \in \mathfrak{R}$ let $K(X, 0)$ denote the c.s.s. complex defined as follows. An n-simplex is any pair (ξ, n) where $\xi \in X$; for every map $\alpha:[m] \to [n]$, $(\xi, n)\alpha = (\xi, m)$. The 0-simplex $(1, 0)$ is chosen as a base point. If X happens to be a group, then this definition reduces to the usual definition of $K(X, 0)$ (see [7]).

Clearly a map $u:X \to Y \in \mathfrak{R}$ induces a map $K(u, 0):K(X, 0) \to K(Y, 0)$.

PROPOSITION 22. *The set* $\bar{\pi}_0(K(Z_2, 0))$ *consists of two elements.*

Proof. There exists (see [7]) a fibre sequence

$$K(Z_2, 0) \to E \to K(Z_2, 1),$$

where E is contractible. By Theorem 1, $\bar{\pi}_1(K(Z_2, 1)) \approx Z_2$. Hence Axiom II′ and Proposition 20 imply that $\bar{\pi}_0(K(Z_2, 0))$ consists of two elements.

Let L be a c.s.s. complex and let $r(L):L \to K(\pi_0(L), 0)$ be the map which assigns to every n-simplex $\sigma \in L$ the element (ξ, n) where $\xi \in \pi_0(L)$ is the component containing σ. Then

PROPOSITION 23. *The map* $\bar{\pi}_0(r(L)):\bar{\pi}_0(L) \to \bar{\pi}_0(K(\pi_0(L), 0))$ *is an isomorphism.*

Proof. Let L_1 be the component of L containing the base point. Then there exists (see [3]) a fibre sequence

$$G(L_1) \to E(L_1) \xrightarrow{p_1} L_1,$$

where $E(L_1)$ is contractible. Furthermore for every component L_ξ of L there exists a fibre map $p_\xi : E(L_\xi) \to L_\xi$ where $E(L_\xi)$ is also contractible. There results a fibre sequence

$$G(L_1) \to E \xrightarrow{p} L,$$

where E is the disjoint union of the $E(L_\xi)$. By Axiom II′ the map

$$\bar{\pi}_0(p):\bar{\pi}_0(E) \to \bar{\pi}_0(L)$$

is onto. The contractibility of the $E(L_\xi)$ implies that the composite map

$$E \xrightarrow{\quad p \quad} L \xrightarrow{\ r(L)\ } K(\pi_0(L), 0)$$

is a weak homotopy equivalence. Hence by Axiom I, $\bar{\pi}_0(r(L) \circ p)$ is an isomorphism, and because $\bar{\pi}_0(p)$ is onto it follows that $\bar{\pi}_0(r(L))$ is also an isomorphism.

PROPOSITION 24. *There is at most one natural equivalence* $h_0 : \pi_0 \to \bar{\pi}_0$.

Proof. Suppose $h_0 : \pi_0 \to \bar{\pi}_0$ is a natural equivalence. It follows from Proposition 22 that there exists only one isomorphism

$$\pi_0(K(Z_2, 0) \approx \bar{\pi}_0(K(Z_2, 0)).$$

Hence $h_0(K(Z_2, 0))$ is this unique isomorphism. The naturality of h_0 implies that for every map $f: K(Z_2, 0) \to K(\pi_0(L), 0)$ commutativity holds in the diagram

$$\tag{5} \begin{array}{ccccc} \pi_0(K(Z_2,0)) & \xrightarrow{\pi_0(f)} & \pi_0(K(\pi_0(L),0)) & \xleftarrow{\pi_0(r(L))} & \pi_0(L) \\ \downarrow{\scriptstyle h_0(K(Z_2,0))} & & \downarrow{\scriptstyle h_0(K(\pi_0(L),0))} & & \downarrow{\scriptstyle h_0(L)} \\ \bar{\pi}_0(K(Z_2,0)) & \xrightarrow{\bar{\pi}_0(f)} & \bar{\pi}_0(K(\pi_0(L),0)) & \xleftarrow{\bar{\pi}_0(r(L))} & \bar{\pi}_0(L). \end{array}$$

Let $\psi \in \pi_0(K(Z_2, 0))$ be the element different from 1. For every element $\xi \in \pi_0(L)$ there exists a unique map $f_\xi : K(Z_2, 0) \to K(\pi_0(L), 0)$ such that $(\pi_0(r(L))^{-1} \circ \pi_0(f_\xi))\psi = \xi$. Hence in view of Proposition 23

$$h_0(L)\xi = \{\bar{\pi}_0(r(L))^{-1} \circ \bar{\pi}_0(f_\xi) \circ h_0(K(Z_2, 0))\}\psi,$$

i.e., the natural equivalence h_0 is completely determined by the functor $\bar{\pi}_0$ and hence is unique.

PROPOSITION 25. *For every c.s.s. complex* $L \in \mathcal{S}_c$ *let* $h_0(L): \pi_0(L) \to \bar{\pi}_0(L)$ *be the function defined by* (5). *Then the function* h_0 *is a natural equivalence* $h_0 : \pi_0 \to \bar{\pi}_0$.

Proof. The naturality of h_0 follows immediately from its definition. In view of Proposition 23 it suffices to prove that $h_0(K(X, 0))$ is an isomorphism for every countable $X \in \mathcal{R}$. This is done in two steps. First it is shown that $h_0(K(X, 0))$ is an isomorphism into and then that it is onto.

Suppose $h_0(K(X, 0))$ is not an isomorphism into for some X, i.e., there exist two elements $\xi_1, \xi_2 \in \pi_0(K(X, 0))$ such that $h_0(K(X, 0))\xi_1 = h_0(K(X, 0))\xi_2$. Then there clearly exists a map $f: K(X, 0) \to K(Z_2, 0)$ such that $\pi_0(f)\xi_1 = 1$ and $\pi_0(f)\xi_2 \neq 1$. This however is in contradiction with the commutativity of the diagram

$$\begin{array}{ccc} \pi_0(K(X,0)) & \xrightarrow{\pi_0(f)} & \pi_0(K(Z_2,0)) \\ \downarrow{\scriptstyle h_0(K(X,0))} & & \downarrow{\scriptstyle h_0(K(Z_2,0))} \\ \bar{\pi}_0(K(X,0)) & \xrightarrow{\bar{\pi}_0(f)} & \bar{\pi}_0(K(Z_2,0)). \end{array}$$

Hence $h_0(K(X, 0))$ is an isomorphism into for all X.

Let Z_n be a cyclic group of order n. Then there exists (see [7]) a fibre sequence

$$K(Z_n, 0) \to E \to K(Z_n, 1),$$

where E is contractible. In view of Theorem 1, $\bar{\pi}_1(K(Z_n, 1)) \approx Z_n$. Hence it follows from Axiom II′ and Proposition 19 that $\bar{\pi}_0(K(Z_n, 0))$ contains at most n elements. Hence $h_0(K(Z_n, 0))$, being an isomorphism into, must be onto, and so is therefore $h_0(K(X, 0))$ for every finite $X \in \mathcal{R}$.

Let Z_2' be the direct sum of a countably infinite number of copies of Z_2. For every map $f: Z_2 \to Z_2'$ there exists a commutative diagram

$$\begin{tikzcd} K(Z_2, 0) \arrow[r, "q"] \arrow[d, "K(f,0)"] & E \arrow[r, "p"] \arrow[d] & K(Z_2, 1) \arrow[d, "K(f,1)"] \\ K(Z_2', 0) \arrow[r, "q'"] & E' \arrow[r, "p'"] & K(Z_2', 1) \end{tikzcd}$$

such that both horizontal sequences are fibre sequences and E and E' are contractible. Hence commutativity holds in

$$\begin{tikzcd} \bar{\pi}_1(K(Z_2, 1)) \arrow[r, "\bar{\partial}_1(q,p)"] \arrow[d, "\bar{\pi}_1(K(f,1))"] & \bar{\pi}_0(K(Z_2, 0)) \arrow[d, "\bar{\pi}_0(K(f,0))"] & \pi_0(K(Z_2, 0)) \arrow[l, "h_0(K(Z_2,0))"'] \arrow[d, "\bar{\pi}_0(K(f,0))"] \\ \bar{\pi}_1(K(Z_2', 1)) \arrow[r, "\bar{\partial}_1(q',p')"] & \bar{\pi}_0(K(Z_2', 0)) & \pi_0(K(Z_2', 0)). \arrow[l, "h_0(K(Z_2',0))"'] \end{tikzcd}$$

In view of Axiom II′ and Proposition 19 the map $\bar{\partial}_1(q', p')$ is onto. Hence in view of Theorem 1, there exists for every element $\xi \in \bar{\pi}_0(K(Z_2', 0))$ a map $f: Z_2 \to Z_2'$ such that $\xi \subset$ *image* $(\bar{\partial}_1(q', p') \circ \bar{\pi}_1(K(f, 1)))$, and because $h_0(K(Z_2', 0))$ is onto, it follows that $\xi \subset$ *image* $h_0(K(Z_2', 0))$. Therefore $h_0(K(Z_2', 0))$ is onto, and so is $h_0(K(X, 0))$ for every $X \in \mathcal{R}$ which is countably infinite.

This completes the proof of Proposition 25.

PROPOSITION 26. *For every fibre sequence* $F \xrightarrow{q} E \xrightarrow{p} B$ *the diagram*

$$\begin{tikzcd} \pi_1(B) \arrow[r, "\partial_1(q,p)"] \arrow[d, "h_1(B)"] & \pi_0(F) \arrow[d, "h_0(F)"] \\ \bar{\pi}_1(B) \arrow[r, "\bar{\partial}_1(q,p)"] & \bar{\pi}_0(F) \end{tikzcd}$$

is either always commutative or always anticommutative.

Proof. Consider a fibre sequence

$$K(Z, 0) \xrightarrow{t} E \xrightarrow{s} K(Z, 1),$$

where E is contractible. It then follows from naturality considerations that it suffices to show that for a generator $\zeta \in \pi_1(K(Z, 1))$

$$\{(\bar{\partial}_1(t, s) \circ h_1(K(Z, 1))\}\zeta = \{h_0(K(Z, 0)) \circ \partial_1(t, s)\}\zeta^n,$$

where $n = \pm 1$.

Suppose $n \equiv 0 \pmod{i}$ where $i > 1$. Let $f: Z \to Z_i$ be a homomorphism onto. Then there exists a commutative diagram

$$\begin{array}{ccccc} K(Z,0) & \xrightarrow{t} & E & \xrightarrow{s} & K(Z,1) \\ \downarrow{\scriptstyle K(f,0)} & & \downarrow & & \downarrow{\scriptstyle K(f,1)} \\ K(Z_i,0) & \xrightarrow{t_i} & E_i & \xrightarrow{s_i} & K(Z_i,1) \end{array}$$

such that both horizontal sequences are fibre sequences and E and E_i are contractible. As $\bar{\partial}_1(t_i, s_i)$ is an isomorphism (see the proof of Proposition 25), it follows from the naturality of ∂_1, $\bar{\partial}_1$, and h_0 that

$$\{\bar{\pi}_0(K(f, 0)) \circ \bar{\partial}_1(t, s) \circ h_1(K(Z, 1))\}\zeta$$
$$= \{\bar{\partial}_1(t_i, s_i) \circ \bar{\pi}_1(K(f, 1)) \circ h_1(K(Z, 1))\}\zeta \neq 1,$$

while

$$\{\bar{\pi}_0(K(f, 0)) \circ h_0(K(Z, 0)) \circ \partial_1(t, s)\}\zeta^n$$
$$= \{h_0(K(Z_i, 0)) \circ \pi_0(K(f, 0)) \circ \partial_1(t, s)\}\zeta^n$$
$$= \{h_0(K(Z_i, 0)) \circ \partial_1(t_i, s_i) \circ \pi_1(K(f, 1))\}\zeta^n = 1.$$

This is a contradiction. Hence $n \not\equiv 0 \pmod{i}$ for all $i > 1$, ie., $n = \pm 1$.

This completes the proof of Theorem 3.

Appendix

7. The influence of Axioms I and II (or II′) on $\pi_1(K(Z, 1))$

In the above axiomatization the group Z and hence the group $\pi_1(K(Z, 1))$ played an important role, somewhat similar to the role of the coefficient group in homology theory. Hence one may ask the question:

Is it possible to weaken or change Axiom III in such a manner that $\pi_1(K(Z, 1))$ need not necessarily be infinite cyclic or trivial, but may be any (abelian) group?

The answer to this question is negative. In fact we will show

THEOREM 4. *Let $\{\pi_i, \partial_i\}$ be a theory of homotopy groups on the category $\mathcal{S}_c$, which satisfies only Axioms I and II (or I and II′). Then $\pi_1(K(Z, 1))$ is torsion-free and abelian.*

An example of such a theory of homotopy groups on $\mathcal{S}_c$ which satisfies Axioms I and II but not Axiom III can, for instance, be obtained by taking the direct sum (or product) of a number of copies of the usual homotopy groups.

Proof of Theorem 4. By Theorem 3, Axioms I and II′ imply Axiom II. As in the proof of Proposition 7 no use was made of Axioms III and IV, Theorem 4 is an immediate consequence of the following group theoretical lemma.

LEMMA 4. *Let $M:\mathcal{G}_c \to \mathcal{G}$ be a functor which preserves short exact sequences. Then $M(Z)$ is torsion-free and abelian.*

Proof. As in the proof of Proposition 11 no use was made of Proposition 8, it follows that $M(Z)$ is abelian.

For A abelian, let $\dot{i}_A^n : A \to A$ denote the map given by $\dot{i}_A^n \alpha = \alpha^n$ for all $\alpha \in A$. We now show that $M(\dot{i}_A^n) = \dot{i}_{M(A)}^n$.

Clearly $M(\dot{i}_A^1) = \dot{i}_{M(A)}^1$. Suppose it has already been shown that

$$M(\dot{i}_A^{n-1}) = \dot{i}_{M(A)}^{n-1}.$$

The map $\dot{i}_A^n$ is the composition

$$A \xrightarrow{d} A + A \xrightarrow{\dot{i}_A^1 + \dot{i}_A^{n-1}} A + A \xrightarrow{q} A,$$

where $d:A \to A + A$ is the diagonal map and q is as in the proof of Proposition 11. It is readily verified that $M(d)$ is again the diagonal map. As application of the functor M to the above sequence yields a similar sequence, it follows that $M(\dot{i}_A^n) = \dot{i}_{M(A)}^n$.

Let $n > 1$ and consider the exact sequence

$$1 \to Z \xrightarrow{\dot{i}_Z^n} Z \to Z_n \to 1.$$

Application of the functor M yields the exact sequence

$$1 \to M(Z) \xrightarrow{\dot{i}_{M(Z)}^n} M(Z) \to M(Z_n) \to 1.$$

Hence $M(Z)$ is torsion-free.

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