GENERALIZED INCIDENCE MATRICES OVER GROUP ALGEBRAS

BY D. R. HUGHES

1. Introduction

In previous papers [3, 4] the author has investigated certain matrix equations which must hold if a (v, k, λ) configuration is to possess collineations. These equations involved matrices with rational entries, and the Hasse-Minkowski theory of rational congruence was applied to give numerical conditions restricting the possible collineations of a (v, k, λ) configuration. The author has found that these rational matrix equations are in fact derivable from more "general" equations involving matrices over a group algebra, and that these latter equations yield at least one result which is not deductible by the rational congruence methods of the earlier papers; if π is a projective plane of order $n \equiv 2 \pmod 4$, $n \neq 2$, then π possesses no collineations of even order. However, the general problems presented by the group algebra equations appear to be difficult of solution.

2. Group algebra matrices

We shall rely heavily on [4] for background material, but a brief review of some basic topics will be given. Let v, k, λ be integers satisfying $v > k > \lambda > 0$ and $\lambda(v - 1) = k(k - 1)$, and let π be a collection of v *points* and v *lines*, together with an *incidence relation* satisfying: (i) each point (line) is on k lines (contains k points), and (ii) each pair of distinct points (lines) is on λ common lines (contains λ common points). Then π is a (v, k, λ) *configuration*, and we define the *order* n of π by $n = k - \lambda$; if $\lambda = 1$, then π is a *projective plane* of order n. A *collineation* of π is a one-to-one mapping of points onto points and lines onto lines which preserves incidence. A collineation group $\mathfrak{G}$ of π is called *standard* if every non-identity element of $\mathfrak{G}$ fixes the same set of points and lines; any collineation group of prime order is standard.

Suppose π is a (v, k, λ) configuration and $\mathfrak{G}$ is a collineation group of π, where $\mathfrak{G}$ has order m. From Theorem 2.3 of [4] we know that the number of transitive classes of points equals the number of transitive classes of lines (X and Y are in the same transitive class if and only if $X = Yb$ for some b in $\mathfrak{G}$). We number the transitive classes of points (lines) $1, 2, \cdots, w$, and let P_i (J_i) be an arbitrary but fixed point (line) in the i^{th} transitive class of points (lines). Let $\mathfrak{P}_i$ ($\mathfrak{J}_i$) be the subgroup of $\mathfrak{G}$ which fixes P_i (J_i), and let $\mathfrak{P}_i$ ($\mathfrak{J}_i$) have order r_i (s_i). Let D_{ij} be the set of all x in $\mathfrak{G}$ such that $P_i x$ is on J_j.

Let F be a field whose characteristic does not divide any of the numbers r_i or s_i; if $\mathfrak{K}$ is a group, we denote by $\mathfrak{A}(\mathfrak{K})$ the group algebra of $\mathfrak{K}$ over F.

Received November 19, 1956.

In particular, let $\mathfrak{A} = \mathfrak{A}(\mathfrak{G})$. Define the following elements in $\mathfrak{A}$:

$$\gamma = \sum x, \text{ all } x \text{ in } \mathfrak{G}; \quad \rho_i = \sum x, \text{ all } x \text{ in } \mathfrak{P}_i; \quad \sigma_i = \sum x, \text{ all } x \text{ in } \mathfrak{J}_i;$$
$$\delta_{ij} = \sum x, \text{ all } x \text{ in } D_{ij}.$$

Now let $\mathfrak{A}_w$ be the set of all $w \times w$ matrices over $\mathfrak{A}$, and define the following matrices, all in $\mathfrak{A}_w$:

$$C_1 = \operatorname{diag}(s_1^{-1}, s_2^{-1}, \cdots, s_w^{-1}), \qquad C_2 = \operatorname{diag}(r_1^{-1}, r_2^{-1}, \cdots, r_w^{-1}),$$
$$E_1 = \operatorname{diag}(\rho_1, \rho_2, \cdots, \rho_w), \qquad E_2 = \operatorname{diag}(\sigma_1, \sigma_2, \cdots, \sigma_w),$$
$$D = (\delta_{ij}), \qquad S = (\gamma_{ij}), \qquad \text{where each} \qquad \gamma_{ij} = \gamma.$$

If $\sum f_i x_i$ is in $\mathfrak{A}$, where each f_i is in F and each x_i in $\mathfrak{G}$, we define

$$\left(\sum f_i x_i\right)^* = \sum f_i x_i^{-1};$$

i.e., (*) is the operation of "conjugation." If $M = (\alpha_{ij})$ is in $\mathfrak{A}_w$, then the *conjugate transpose* of M is $M' = (\alpha_{ji}^*)$.

THEOREM 2.1. $DC_1D' = nE_1 + \lambda S$, $D'C_2D = nE_2 + \lambda S$, $DC_1S = SC_2D = kS$.

Proof. The proof of Theorem 2.1 of [4] essentially contains the desired result. We will demonstrate that the diagonal terms are "correct"; i.e., for a fixed i, show that $\sum_j \delta_{ij}\delta_{ij}^*/s_j = n\rho_i + \lambda\gamma$.

Let x be in $\mathfrak{G}$, x not in $\mathfrak{P}_i$. Consider the pair of (distinct) points P_i, $P_i x$; they are on λ common lines $J_j y$, and for each such common line, there are s_j choices of y, for $J_j y = J_j(\mathfrak{J}_j y)$. For each j and y, y^{-1} and xy^{-1} are in D_{ij}; hence $x = d_1 d_2^{-1}$, where d_1, d_2 are in D_{ij}, for λ choices of j (not all necessarily distinct) and s_j choices of d_1, d_2 for each j. Note that d_1 is not in $\mathfrak{P}_i d_2$. Thus in the sum $\sum_j \delta_{ij}\delta_{ij}^*/s_j$ every element of $\mathfrak{G}$ not in $\mathfrak{P}_i$ has coefficient λ. Now we note that $\mathfrak{P}_i D_{ij} \mathfrak{J}_j = D_{ij}$. If x is in $\mathfrak{P}_i$, then for each d_1 in D_{ij}, the element $d_2 = xd_1$ is also in D_{ij}, so $x = d_1 d_2^{-1}$ for a_{ij} choices of d_1, d_2 in D_{ij}, where a_{ij} is the number of elements in D_{ij}. Hence in the sum above, every element of $\mathfrak{P}_i$ has the coefficient $\sum_j a_{ij}/s_j$. But it is quite straightforward to verify that $\sum_j a_{ij}/s_j$ counts the number of lines through the point P_i, hence is equal to k. So $\sum_j \delta_{ij}\delta_{ij}^*/s_j = k\rho_i + \lambda(\gamma - \rho_i) = n\rho_i + \lambda\gamma$.

The off-diagonal elements of DC_1D' are computed similarly, considering the pair of points P_i and $P_j x$, where $i \neq j$ and there are no restrictions on x in $\mathfrak{G}$. The matrix $D'C_2D$ is computed by "dual" considerations, substituting lines for points in the above arguments. The final equations of Theorem 2.1 are easily verified.

Now suppose that $\mathfrak{B}$ is another algebra over the same field F, and $\mathfrak{H}$ is a (multiplicative) group of units in $\mathfrak{B}$. If ϕ is a homomorphism of $\mathfrak{G}$ onto $\mathfrak{H}$, then ϕ extends (linearly) to a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$; the mapping $\phi: M = (\alpha_{ij}) \to M\phi = (\alpha_{ij}\phi)$ is a homomorphism of $\mathfrak{A}_w$ into $\mathfrak{B}_w$. Hence we have the following as an immediate corollary of Theorem 2.1.

THEOREM 2.2. $D\phi\cdot C_1\cdot(D\phi)' = n\cdot E_1\phi + \lambda\cdot S\phi$, $(D\phi)'\cdot C_2\cdot D\phi = n\cdot E_2\phi + \lambda\cdot S\phi$, $D\phi\cdot C_1\cdot S\phi = S\phi\cdot C_2\cdot D\phi = k\cdot S\phi$.

Applications of Theorem 2.2 include the following:

(1) Let $\mathfrak{H}$ be a group, ϕ a homomorphism of $\mathfrak{G}$ onto $\mathfrak{H}$, and let $\mathfrak{B}$ be the group algebra $\mathfrak{A}(\mathfrak{H})$ of $\mathfrak{H}$ over F.

(2) Let F be the field of rationals, $\mathfrak{B} = F$, and $\mathfrak{H}$ the identity subgroup of F. Then Theorem 2.2 of [4] is a corollary of Theorem 2.2 above.

(3) Suppose the order of G is a prime p and F is taken as a complex number field which contains the p^{th} roots of unity. Let $\mathfrak{H}$ be the group of p^{th} roots of unity and ϕ an isomorphism of $\mathfrak{G}$ onto $\mathfrak{H}$. This example is treated in more detail in the next section.

3. Collineation groups of prime order

Let us first assume that $\mathfrak{G}$ is standard; then (see Section 3 of [4]) $C_1 = C_2$ contains $t = (v - N)/m$ elements $+1$ and N elements $1/m$ on its main diagonal, where N is the number of points of π fixed by every element of $\mathfrak{G}$. Furthermore, $E_1 = E_2$ contains t elements $+1$ and N elements γ on its main diagonal. Let us write all the matrices defined in the last section so that the first t rows and columns correspond to the "non-fixed" points and lines, and the last N rows and columns correspond to the fixed points and lines. Let A_1 be the submatrix of order t in the upper left corner of D; it is evident that all of the elements of D which lie outside of A_1 are either 0 or γ.

Now we make the further assumption that $\mathfrak{G}$ has prime order p. Let ϕ be an isomorphism of $\mathfrak{G}$ onto the p^{th} roots of unity, where we are assuming that F is some complex number field containing these roots. Define

$$A = A_1\phi\,.$$

All of the elements of $D\phi$, excepting the elements in A, are zero, since for $p \neq 1$, the sum of the p^{th} roots of unity is zero. Thus, "pulling out" the nonzero part of the matrix equations, Theorem 2.2 implies:

THEOREM 3.1. *$AA' = nI$, where I is the identity matrix of order $t = (v - N)/p$.*

In Theorem 3.1, A is a matrix of order t all of whose entries are sums (with $+1$ or 0 as coefficients) of p^{th} roots of unity, and A' is the "ordinary" conjugate transpose of A. The author knows of no method of analyzing the equation of Theorem 3.1 in general. But at least one fragmentary result is possible, a result which indeed has a good deal of interest in its own right.

Now we assume that $\lambda = 1$, $p = 2$, $n \equiv 2 \pmod 4$. Since n is even but not a square, a theorem due to Baer [1] asserts that a collineation of order two of a projective plane of order n must fix all of the points on a line K, all of the lines through a point Q, where Q is on K, and nothing else; thus $t = n^2/2$.

Besides K, let the lines through Q be $L_1, L_2, \cdots, L_n$, and besides Q, let the points on K be $R_1, R_2, \cdots, R_n$. Each line L_i (point R_i) is incident with $n/2$ of the transitive classes of points (lines), besides Q (K). For convenience, let us redesignate the points on the lines L_i as follows: the $n/2$ "base points" that are on L_i will be P_{ij}, $j = 1, 2, \cdots, n/2$. The row of the matrix A which corresponds to the point P_{ij} will be called V_{ij}. By properly choosing the points P_{1j} on L_1 and the lines through the various R_j,[1] each row V_{1j} can be assumed to contain n ones and $t - n$ zeros. Suppose $i \neq 1$; since the row V_{ij} has inner product n with itself and zero with every row V_{1x}, V_{ij} must contain $n/2$ elements $+1$, $n/2$ elements -1, and $t - n$ elements zero. Suppose x is fixed, $x = 1, 2, \cdots, n$, and let $i \neq j$. Then there is no column which contains a nonzero element in both rows V_{xi} and V_{xj}, since the points P_{xi} and P_{xj} are on the line L_x; on the other hand, in a given column, some one of the positions in the set of rows V_{xi} is not zero, since any line (except K) through a point R_j meets L_x in a point different from Q.

Now we will construct a new matrix B as follows: the i^{th} row V_i of B will be formed by superimposing all of the rows V_{ij}, $j = 1, 2, \cdots, n/2$, onto one another. This never superimposes a nonzero element onto a nonzero element, but on the other hand, no position in B contains a zero. So B is an $n \times t$ matrix consisting entirely of $\pm$ 1's, whose first row contains nothing but $+1$'s. The inner product of V_i and V_j is the sum of all the inner products V_{ix} by V_{jy}, $x, y = 1, 2, \cdots, n/2$. So $BB' = tI$, where I is the identity matrix of order n.

THEOREM 3.2. *If π is a projective plane of order $n \equiv 2 \pmod 4$, and if π possesses a collineation of order two, then $n = 2$.*

Proof. Assume that $n \neq 2$. Then B has at least three rows, and by rearranging the columns if necessary, the first three rows appear as follows:

<table>
<tr><td colspan="4">$+1 \quad (t)$</td></tr>
<tr><td colspan="2">$+1 \quad (t/2)$</td><td colspan="2">$-1 \quad (t/2)$</td></tr>
<tr><td>$+1 \quad (a)$</td><td>$-1 \quad (t/2 - a)$</td><td>$+1 \quad (b)$</td><td>$-1 \quad (t/2 - b)$</td></tr>
</table>

where the numbers in parentheses denote the length of the block. Taking the inner product of the first and third rows, we have:

$$a - (t/2 - a) + b - (t/2 - b) = 0,$$

whence $b = t/2 - a$. Then taking the inner product of the second and third rows,

$$a - (t/2 - a) - (t/2 - a) + a = 0,$$

whence $t = 4a \equiv 0 \pmod 4$. But since $n \equiv 2 \pmod 4$, it is easy to see that $t = n^2/2 \equiv 2 \pmod 4$. This is a contradiction, so we must have $n = 2$.

[1] I.e., each line $J_i \neq K$ which contains a point R_i will be chosen to contain some point P_{1j}, and never a point $P_{1j}\,x$, for $x \neq 1$.

Corollary. *If π is a projective plane of order $n \equiv 2 \pmod 4$, $n \neq 2$, then π possesses no collineations of even order.*

It will be observed that the proof of Theorem 3.2 is exactly the same as the proof that a Hadamard matrix of order $\neq 1, 2$, must have order divisible by 4.

4. Some applications

In [6] Ostrom has proved that if π is a projective plane of order n, where n is odd and not a square, possessing a doubly transitive collineation group, then π is Desarguesian (see [7], say, for definition). Since a doubly transitive permutation group has even order, we can use Theorem 3.2 to extend this result.

Theorem 4.1. *If π is a projective plane of order $n \equiv 2 \pmod 4$ possessing a collineation group $\mathfrak{G}$ which is doubly transitive on the points of π, then $n = 2$ (whence π is certainly Desarguesian).*

Another result related to [6] is the following (in fact, the corresponding theorem in [6] requires *double* transitivity):

Theorem 4.2. *If π is a projective plane of order $n \equiv 2 \pmod 4$ and if π possesses a collineation group $\mathfrak{G}$ which fixes a line K and is transitive on the points off of K, then $n = 2$.*

Proof. No nonidentity element of $\mathfrak{G}$ fixes all of the points off of K. Hence $\mathfrak{G}$, as a collineation group of π, is isomorphic to the permutation group which results from restricting $\mathfrak{G}$ to the points off of K. There are n^2 of these points, and since the degree of a transitive permutation group divides its order, $\mathfrak{G}$ has even order, so the theorem is proved.

Further applications of Theorem 3.2 are found in the theory of partially transitive projective planes; see [2] for definition and discussion. If π is a partially transitive plane of type (1a) or (2), and if π has order $n \equiv 2 \pmod 4$, then it is easy to see from the table in Section 3 of [2] that π has a collineation of order two; so $n = 2$. Furthermore, if π is of type (1b), with abelian $\mathfrak{G}$, then from the theorems on multipliers, the mapping $(x) \to (x^n)$, $[Dx] \to [Dax^n]$, for some fixed a in $\mathfrak{G}$, is a collineation of order two; so if π has order $n \equiv 2 \pmod 4$, then $n = 2$.

Finally, specific consideration of the case $n = 10$, as the smallest order for which the existence of the plane is undecided, is of interest. From the theorems of [4] and Theorem 3.2 above, the only possible collineations of a plane π of order 10, whose order is prime, are:

(1) Order 3, fixed points Q_i, $i = 1, 2$, on a line K_0, and a point Q_0 not on K_0, together with the lines K_0 and $K_i = Q_0 Q_i$, $i = 1, 2$.

(2) Same as (1), excepting that there are 8 points Q_i on K_0, etc.

(3) Order 11, fixed point Q_0, fixed line K_0, Q_0 not on K_0.

(4) Order 5, fixed point Q_0, fixed line K_0, Q_0 on K_0.

(5) Order 5, where the fixed points are all the points on a line K_0, and the fixed lines are all the lines through a point Q_0, Q_0 on K_0.

However, let us consider (5) in more detail. The techniques of [7] allow one to prove rather directly that if the collineation exists, then π must possess a planar ternary ring (i.e., with K_0 as L_∞, Q_0 as (∞)) whose additive loop contains a subgroup of order 5. Since the additive loop (strictly, its Cayley table) is one of a set of nine mutually orthogonal latin squares, Mann's result [5] assures us that this cannot occur. So we are left with only the first four cases to consider. It does not seem unreasonable to hope that some combination of theory and computing will allow these cases to be rejected, thus proving the following:

Conjecture. If there exists a projective plane of order 10, then it possesses no nonidentity collineations.

Since every finite projective plane known at the present time possesses collineations of order two, Theorem 3.2 offers new evidence that no plane exists for $n \equiv 2 \pmod 4$, $n \neq 2$. Unfortunately, we appear to possess insufficient machinery with which to attack this problem yet.

5. Remarks

(1) With respect to double transitivity, Marshall Hall and the author have proved the following more general extension of [6].

THEOREM 5.1. *If π is a projective plane of order n, where n is even and not a square, and if π possesses a collineation group $\mathfrak{G}$ doubly transitive on points, then π is Desarguesian.*

Proof. Let L be any line of π and P any point on L; define $\mathfrak{G}(L, P)$ to be the collineation group of π which fixes every point on L and every line through P. In a recent paper[2] on finite Fano planes, A. M. Gleason has proved that if $\mathfrak{G}(L, P) \neq 1$ for every line L of π and every point P on L, then π is Desarguesian.

Since $\mathfrak{G}$ is doubly transitive, there is a collineation x in $\mathfrak{G}$ of order two. As remarked above, x must fix every point on some line L and every line through some point P on L. Since $\mathfrak{G}$ is certainly transitive on lines, this implies that for every line L of π, there is some point P on L for which

$$\mathfrak{G}(L, P) \neq 1.$$

But because of double transitivity on points, P can be mapped onto any point Q on L by a collineation fixing L; so every $\mathfrak{G}(L, P) \neq 1$, and π is Desarguesian.

Note that this proof, as well as the proof in [6] is actually valid for those square values of n for which there is no plane of order $n^{1/2}$, e.g., $n = 36$.

(2) In a project currently underway at The Ohio State University, E. T.

[2] *Finite Fano planes*, Amer. J. Math., vol. 78 (1956), pp. 797–807.

Parker has shown the collineation of order 11 of a plane of order 10 cannot occur. So only three types are left to investigate; these, however, appear to be a good deal more difficult to handle.

Bibliography

1. Reinhold Baer, *Projectivities with fixed points on every line of the plane*, Bull. Amer. Math. Soc., vol. 52 (1942), pp. 273–286.
2. D. R. Hughes, *Partial difference sets*, Amer. J. Math., vol. 78 (1956), pp. 650–674.
3. ———, *Regular collineation groups*, Proc. Amer. Math. Soc., vol. 8 (1957), pp. 165–168.
4. ———, *Collineations and generalized incidence matrices*, Trans. Amer. Math. Soc., to appear.
5. H. B. Mann, *On orthogonal Latin squares*, Bull. Amer. Math. Soc., vol. 50 (1944), pp. 249–257.
6. T. G. Ostrom, *Double transitivity in finite projective planes*, Canadian J. Math., vol. 8 (1956), pp. 563–567.
7. Günter Pickert, *Projektive Ebenen*, Berlin, 1955.

The Ohio State University
Columbus, Ohio