A JACOBIAN CRITERION OF SIMPLE POINTS

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Previously Zariski [5] gave a Jacobian criterion of simplicity of points of an algebraic variety, and its generalization to the algebroid case[1] was treated by Samuel [4].[2] In the present paper, we shall give a proof of the criterion of simplicity. Although we shall treat the algebroid case, our proof is also valid for the algebraic case if formal power series rings are replaced by polynomial rings.

1. Derivations of a ring (cf. [3])

Let $\mathfrak{o}$ be a ring.[3] A *derivation* D of $\mathfrak{o}$ is an additive endomorphism of the total quotient ring L of $\mathfrak{o}$ which satisfies the following conditions: (1) $D(xy) = xDy + yDx$ for $x, y \in L$, (2) there exists an element d of $\mathfrak{o}$ which is not a zero-divisor such that $dDx \in \mathfrak{o}$ for $x \in \mathfrak{o}$. Here, if d can be chosen to be 1, we call D an *integral derivation* of $\mathfrak{o}$.

A derivation D of $\mathfrak{o}$ such that $D\mathfrak{o}' = 0$, $\mathfrak{o}'$ being a subring of $\mathfrak{o}$, is called a derivation over $\mathfrak{o}'$; if $D\mathfrak{o} = 0$, then we say that D is the zero derivation or the trivial derivation of $\mathfrak{o}$, and we denote it by 0.

The set of derivations of $\mathfrak{o}$ over a subring $\mathfrak{o}'$ is an L-module, which will be denoted by $\mathfrak{D}_{\mathfrak{o}/\mathfrak{o}'}$. Obviously $\mathfrak{D}_{\mathfrak{o}/\mathfrak{o}'}$ is generated by integral derivations. Linear dependence of derivations will always mean dependence in this module, hence over L, equivalently over $\mathfrak{o}$. The length of the module $\mathfrak{D}_{\mathfrak{o}/\mathfrak{o}'}$ (as an L-module) is called the *dimension* of $\mathfrak{D}_{\mathfrak{o}/\mathfrak{o}'}$ and is denoted by $\dim \mathfrak{D}_{\mathfrak{o}/\mathfrak{o}'}$ (the dimension of $\mathfrak{D}_{\mathfrak{o}/\mathfrak{o}'}$ may be infinite).

Let $\mathfrak{a}$ be an ideal of $\mathfrak{o}$ and let ϕ be the natural homomorphism from $\mathfrak{o}$ onto $\mathfrak{o}/\mathfrak{a}$. Let D be a derivation of $\mathfrak{o}$. Assume that there exists an element $d \in \mathfrak{o}$ which is not a zero-divisor modulo $\mathfrak{a}$ and is such that (i) dD is an integral derivation of $\mathfrak{o}$ and (ii) $dD\mathfrak{a} \subseteqq \mathfrak{a}$. Then we can define an operator D' in $\mathfrak{o}/\mathfrak{a}$ to be $D'(\phi(x)) = \phi(dDx)/\phi(d)$ $(x \in \mathfrak{o})$. D' can be uniquely extended to a derivation of $\mathfrak{o}/\mathfrak{a}$ (independently of the choice of d). The derivation obtained in this manner is called the derivation *induced* in $\mathfrak{o}/\mathfrak{a}$ by D.

2. Derivations of a local ring

LEMMA 1. *Let $\mathfrak{o}$ be a local ring with maximal ideal $\mathfrak{m}$, $\mathfrak{o}'$ a subring of $\mathfrak{o}$, and let $f_1, \cdots, f_r$ be a set of generators of $\mathfrak{m}$. Assume that a subset M of $\mathfrak{o}$ generates*

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[1] The notion of algebroid varieties over an algebraically closed field was introduced by Chevalley [1]. An algebroid variety can be defined similarly over an arbitrary field.

[2] Samuel's argument seems to be valid only if $[k:k^p]$ is finite. His treatment of the general case being too sketchy, we prefer to give a proof based upon other methods.

[3] A ring means a commutative ring with identity.

a subring $\mathfrak{o}''$ *over* $\mathfrak{o}'$ *such that* $\mathfrak{o}/\mathfrak{m} = \mathfrak{o}''/(\mathfrak{m} \cap \mathfrak{o}'')$. *If* D *and* D' *are derivations of* $\mathfrak{o}$ *over* $\mathfrak{o}'$ *such that* (1) $Dm = D'm$ *for all* $m \in M$ *and* (2) $Df_i = D'f_i$ *for every* i, *then we have* $D = D'$.

Proof. By the definition of derivations, we may assume that D and D' are integral derivations of $\mathfrak{o}$. If $f \in \mathfrak{o}$ is the limit of a convergent sequence $\{g_i\}$, then Df and $D'f$ are the limits of the sequences $\{Dg_i\}$ and $\{D'g_i\}$ respectively (because $D\mathfrak{m}^{i+1} \subseteqq \mathfrak{m}^i$, $D'\mathfrak{m}^{i+1} \subseteqq \mathfrak{m}^i$). By our assumption, every element f of $\mathfrak{o}$ can be expressed as a power series in $f_1, \cdots, f_r$ with coefficients in $\mathfrak{o}''$. Therefore $Df = D'f$.

Remark. Every derivation of $\mathfrak{o}$ has a unique extension to a derivation of the completion of $\mathfrak{o}$. (Possibility of extension can be easily verified, and the uniqueness follows from Lemma 1.)

Next we consider a special case where $\mathfrak{o}$ is a formal power series ring over a field. Let k be a field, let $X_1, \cdots, X_n$ be indeterminates, and let A be the formal power series ring in $X_1, \cdots, X_n$ over k. Then (i) there exists, for each $i = 1, \cdots, n$, a derivation D_i over k such that $D_iX_i = 1$ and $D_iX_j = 0$ if $i \neq j$; the derivation D_i is called the partial derivation of A and is denoted by $\partial/\partial X_i$. (ii) If D is a derivation of k over a subfield k', then there exists one and only one derivation of A over $k'\{X_1, \cdots, X_n\}$ which coincides with D on k; if D' is this derivation and if f is an element of the field of quotients of A, we shall denote by f^D the element $D'f$. Let $f_1, \cdots, f_r$ be elements of the field of quotients of A. (1) The matrix $(\partial f_i/\partial X_j)$ (with r rows and n columns) is called the Jacobian matrix of $f_1, \cdots, f_r$ and is denoted by $J(f_1, \cdots, f_r)$. (2) Let k^* be a subfield of k such that $[k:k^*]$ is finite and let $\{D_1, \cdots, D_s\}$ be a k-basis of $\mathfrak{D}_{k/k^*}$. Then the matrix $(\partial f_i/\partial X_j, f_i^{D_t})$ (with r rows and $n + s$ columns) is called a mixed Jacobian matrix of $f_1, \cdots, f_r$ with respect to k^* and is denoted by $J^*(f_1, \cdots, f_r; D_1, \cdots, D_s)$ or by $J^*(f_1, \cdots, f_r; k^*)$.[4] (Observe that $J(f_1, \cdots, f_r) = J^*(f_1, \cdots, f_r; k)$.)

We recall the definition of p-independence. Let K be a field of characteristic $p \neq 0$, and let K^* be a subfield of K which contains K^p. We say that elements $z_1, \cdots, z_m$ of K are p-independent over K^* if $[K^*(z_1, \cdots, z_m):K^*] = p^m$. Now we shall prove the following

LEMMA 2. *Let* k^* *be a subfield of* k *such that* $[k:k^*]$ *is finite, and let* $\mathfrak{a}$ *be an ideal of* A. *Then every derivation of* $A/\mathfrak{a}$ *over* k^* *is induced by a derivation of* A *over* k^*.

Proof. (i) If k is of characteristic $p \neq 0$, every derivation of A or of $A/\mathfrak{a}$ is a derivation over k^p, and therefore we may assume that k^* contains k^p. Let $z_1, \cdots, z_m$ be p-independent elements of k over k^* such that $k = k^*(z_1, \cdots, z_m)$. Let D_i be the derivation of k over k^* such that $D_iz_i = 1$ and $D_iz_j = 0$ if $i \neq j$. Let D' be an integral derivation of $A/\mathfrak{a}$ over k^*. Set

[4] $J^*(f_1, \cdots, f_r; k^*)$ is substantially unique, i.e., change of the base of $\mathfrak{D}_{k/k^*}$ corresponds to a linear transformation of columns.

$u_i' = D'z_i$ and $v_j' = D'x_j$, where x_j is the $\mathfrak{a}$-residue of X_j. Let u_i and v_j be representatives of u_i' and v_j' in A. Set $D = \sum u_i D_i + \sum v_j \partial/\partial X_j$. Then D induces D' by Lemma 1. Thus every integral derivation of $A/\mathfrak{a}$ over k^* is induced by a derivation of A. Therefore every derivation of $A/\mathfrak{a}$ over k^* is induced by a derivation of A (over k^*).

(ii) If k is of characteristic zero, we may assume that $k = k^*$ because k is separable over k^*. Then using only the partial derivations $\partial/\partial X_i$, we can prove the assertion in the same way as above.

3. The criterion

LEMMA 3. *Let K be a field of characteristic $p \neq 0$, K' a finite algebraic extension of K, and let K^* be a subfield of K such that $[K:K^*]$ is finite. Then there exists a subfield K^{**} of K^* such that $[K:K^{**}]$ is finite and such that* $\dim \mathfrak{D}_{K/K^{**}} = \dim \mathfrak{D}_{K'/K^{**}}$.

Proof. We shall prove the assertion by induction on $[K':K]$. If $[K':K] = 1$, then the assertion is obvious. Assume that $K' \neq K$. If a is an element of K' which does not belong to K, then, by our induction hypothesis, there exists a field K^{**} ($K^{**} \subseteqq K^*$, $[K^*:K^{**}] < \infty$) such that

$$\dim \mathfrak{D}_{K'/K^{**}} = \dim \mathfrak{D}_{K(a)/K^{**}}.$$

Now if a is separable over K, then $\dim \mathfrak{D}_{K/K^{**}} = \dim \mathfrak{D}_{K(a)/K^{**}}$, and hence $\dim \mathfrak{D}_{K'/K^{**}} = \dim \mathfrak{D}_{K/K^{**}}$. Hence if there exists an element a of K' which does not belong to K and which is separable over K, then our assertion is proved. Now we treat the case where K' is purely inseparable over K. Let $a \in K'$ be such that $a \notin K$, $a^p \in K$. Then $a^p \notin K^p$. If $a^p \in K^p(K^*)$, then let K_1 be a subfield of K^* such that $[K:K_1] < \infty$ and such that $a^p \notin K^p(K_1)$. Then considering K_1 instead of K^*, we may assume that $a^p \notin K^p(K^*)$. There exists a field K^{**} ($K^{**} \subseteqq K^*$, $[K:K^{**}] < \infty$) such that $\dim \mathfrak{D}_{K'/K^{**}} = \dim \mathfrak{D}_{K(a)/K^{**}}$. Since $a^p \in K$ and $a \notin K$, a derivation D of K has an extension D'' to $K(a)$ if and only if $D(a^p) = 0$, and when that is so then $D''a$ can be assigned arbitrarily in $K(a)$. Hence $\dim \mathfrak{D}_{K(a)/K^{**}} = 1 + \dim \mathfrak{D}_{K/K^{**}(a^p)}$. Since $a^p \notin K^p(K^{**})$, we have $\dim \mathfrak{D}_{K/K^{**}} = 1 + \dim \mathfrak{D}_{K/K^{**}(a^p)}$. Hence $\dim \mathfrak{D}_{K(a)/K^{**}} = \dim \mathfrak{D}_{K/K^{**}}$, and $\dim \mathfrak{D}_{K'/K^{**}} = \dim \mathfrak{D}_{K/K^{**}}$, as was asserted.

Remark. If one K^{**} is given as above, then every subfield of K^{**} with finite index satisfies the same condition, as is easily seen from the proof above.

THEOREM. *Let A be the formal power series ring in indeterminates $X_1, \cdots, X_n$ over a field k, let $\mathfrak{p} \subseteqq \mathfrak{q}$ be prime ideals of A and set $R = A_\mathfrak{q}$. Furthermore, let $\{f_1, \cdots, f_r\}$ be a set of generators of $\mathfrak{p}$. Then the following holds:*

(1) *If $A/\mathfrak{q}$ is separably generated*[5] *over k, $R/\mathfrak{p}R$ is a regular local ring if and only if* rank $(J(f_1, \cdots, f_r)$ modulo $\mathfrak{q})$ = rank $\mathfrak{p}$.

[5] $A/\mathfrak{q}$ is separably generated over k if and only if there exists a system of parameters $y_1, \cdots, y_d$ such that $A/\mathfrak{q}$ is separable over the formal power series ring $k\{y_1, \cdots, y_d\}$. See [2].

(2) *If k is of characteristic $p \neq 0$, then $R/\mathfrak{p}R$ is a regular local ring if and only if there exists a subfield k^* of k such that $[k:k^*]$ is finite and such that* rank $(J^*(f_1, \cdots, f_r; k^*)$ modulo $\mathfrak{q}) =$ rank $\mathfrak{p}$.

Proof. (i) We begin with a special case where $\mathfrak{p} = \mathfrak{q}$. In this case, $R/\mathfrak{p}R$ is a field ($=$ the residue class field of the local ring R), and hence we have to show that rank $(J(f_1, \cdots, f_r)$ modulo $\mathfrak{q}) =$ rank $\mathfrak{q}$ in case (1), and that rank $(J^*(f_1, \cdots, f_r; k^*)$ modulo $\mathfrak{q}) =$ rank $\mathfrak{q}$ for a suitable k^* such that $[k:k^*] < \infty$, in case (2). Let $y_1, \cdots, y_s$ be a system of parameters of $A/\mathfrak{q}$; here, if $A/\mathfrak{q}$ is separably generated over k, then we choose the y_i's so that $A/\mathfrak{q}$ is separable over $k\{y_1, \cdots, y_s\}$.[5]

(1) *Separable case.* $\dim \mathfrak{D}_{(A/\mathfrak{q})/k}$ is obviously equal to s. Hence the set of vectors $(Dx_1, \cdots, Dx_n)$, with $D \in \mathfrak{D}_{(A/\mathfrak{q})/k}$ and $x_i =$ (the $\mathfrak{q}$-residue of X_i), is a vector space of dimension s over $R/\mathfrak{q}R$. Since $J(f_1, \cdots, f_r)$ modulo $\mathfrak{q}$ is the matrix of coefficients of linear equations of $(Dx_1, \cdots, Dx_n)$ by virtue of Lemma 2, we see that rank $(J(f_1, \cdots, f_r$ modulo$)\ \mathfrak{q}) = n - s =$ rank $\mathfrak{q}$.

(2) Assume that k is of characteristic $p \neq 0$. We shall make use of the following variation of Lemma 3:

LEMMA 3*. *Let $K' = A/\mathfrak{q}$, let K be a field between $k\{y_1, \cdots, y_s\}$ and K', and let k^* be a subfield of k such that $[k:k^*] < \infty$ and such that $k^p \subseteqq k^*$. Then there exists a subfield k^{**} between k^p and k^* such that $[k:k^{**}] < \infty$ and such that* $\dim \mathfrak{D}_{K'/K^{**}} = \dim \mathfrak{D}_{K/K^{**}}$, *where K^{**} is the field of quotients of* $k^{**}\{y_1^p, \cdots, y_s^p\}$.

The proof is similar to that of Lemma 3.

Apply Lemma 3* in the case where K is the field of quotients of $k\{y_1, \cdots, y_s\}$. Let $z_1, \cdots, z_m$ be p-independent elements over k^{**} such that $k = k^{**}(z_1, \cdots, z_m)$. Consider the vector space

$$\{(Dx_1, \cdots, Dx_n, Dz_1, \cdots, Dz_m); D \in \mathfrak{D}_{K'/K^{**}}\}.$$

Since $\dim \mathfrak{D}_{K/K^{**}} = s + m$, $\dim \mathfrak{D}_{K'/K^{**}} = s + m$, and therefore the rank of $J^*(f_1, \cdots, f_r; k^{**}) = (n + m) - (s + m) = n - s =$ rank $\mathfrak{q}$.

(ii) We shall treat now the case where $\mathfrak{p} \neq \mathfrak{q}$. Let $g_1, \cdots, g_t$ be elements of $\mathfrak{q}$ such that together with the f_i they generate $\mathfrak{q}$. Then (i) shows that $J(f_1, \cdots, f_r, g_1, \cdots, g_t)$ or $J^*(f_1, \cdots, f_r, g_1, \cdots, g_t; k^{**})$ modulo $\mathfrak{q}$ is of rank equal to rank $\mathfrak{q}$. We shall treat now the second case; the first case can be regarded as a special case where $k = k^{**}$. Assume first that $R/\mathfrak{p}R$ is regular. Then there exists a regular system of parameters $u_1, \cdots, u_a, u_{a+1}, \cdots, u_b$ of R such that $u_1, \cdots, u_a$ generate $\mathfrak{p}R$. Then

$$\text{rank } (J^*(u_1, \cdots, u_b; k^{**}) \text{ modulo } \mathfrak{q}) = \text{rank } \mathfrak{q} = b,$$

hence

$$\text{rank } (J^*(u_1, \cdots, u_a; k^{**}) \text{ modulo } \mathfrak{q}) = a = \text{rank } \mathfrak{p}$$

and

$$\text{rank } (J^*(f_1, \cdots, f_r; k^{**}) \text{ modulo } \mathfrak{q}) = \text{rank } \mathfrak{p}.$$

Conversely, assume that rank $\left(J^*(f_1, \cdots, f_r; k^{**}) \text{ modulo } \mathfrak{q}\right) = \text{rank } \mathfrak{p}$. We may assume that rank $\left(J^*(f_1, \cdots, f_a; k^{**}) \text{ modulo } \mathfrak{q}\right) = \text{rank } \mathfrak{p} = a$. Assume that $c_1, \cdots, c_a$ are elements of R such that $\sum c_i f_i \in \mathfrak{q}^2R$. Then for every integral derivation D of A, $D(\sum c_i f_i) \in \mathfrak{q}R$. But $D(\sum c_i f_i) = \sum f_i Dc_i + \sum c_i Df_i \equiv \sum c_i Df_i \pmod{\mathfrak{q}R}$. Since

$$\text{rank } \left(J^*(f_1, \cdots, f_a; k^{**}) \text{ modulo } \mathfrak{q}\right) = a,$$

we have $c_i \in \mathfrak{q}R$ for every i. Therefore there exists a regular system of parameters of R which contains $f_1, \cdots, f_a$ as a subset. Since $a = \text{rank } \mathfrak{p}$, we have $\mathfrak{p}R = \sum_1^a f_i R$ and $R/\mathfrak{p}R$ is a regular local ring. Thus the proof is completed.

Remark 1. We have also proved that if $f_1, \cdots, f_a$ are elements of $\mathfrak{p}R$ and if rank $\left(J^*(f_1, \cdots, f_a; k^{**}) \text{ modulo } \mathfrak{q}R\right) = a = \text{rank } \mathfrak{p}$, then $\mathfrak{p}R$ is generated by the f_i, and $R/\mathfrak{p}R$ is a regular local ring. Consequently,

$$\text{rank } \left(J^*(f_1, \cdots, f_r; k^{**}) \text{ modulo } \mathfrak{q}\right)$$

is not greater than rank $\mathfrak{p}$ for any elements $f_1, \cdots, f_r$ of $\mathfrak{p}$. The same is true of Jacobian matrices.

Remark 2. The choice of k^* in the above theorem is indefinite in some sense. (i) Let k' be the subfield of k generated by the coefficients of the f_i over k^p. If $[k':k^p]$ is finite (an assumption which is always satisfied if we apply our proof to the algebraic case), then one k^* can be chosen as follows: Let $z_1, \cdots, z_m$ be p-independent elements of k' over k^p such that $k' = k^p(z_1, \cdots, z_m)$. Let k^* be a maximal subfield of k among those containing k^p and over which $z_1, \cdots, z_m$ are p-independent. Then $k = k^*(z_1, \cdots, z_m)$, and this k^* can be used as the k^* in the theorem, because if k^{**} is a subfield of k^* such that $[k:k^{**}]$ is finite, then $J^*(f_1, \cdots, f_r; k^{**})$ is obtained by adjoining zero columns to $J^*(f_1, \cdots, f_r; k^*)$. (ii) In the general case, if we consider mixed Jacobian matrices with infinite columns, that is, if we consider a base of $\mathfrak{D}_{k/k^p}$ and define mixed Jacobian matrix similarly, then we have the same result as in our theorem.

Remark 3. As was stated before, our proof can be applied to the algebraic case. Furthermore, our proof can be applied to another case, which may be called the analytic case. Namely, let k be a field with an Archimedean or a non-Archimedean valuation v, and let $X_1, \cdots, X_n$ be indeterminates. Let A be the set of convergent power series (under the valuation v) in $X_1, \cdots, X_n$ with coefficients in k. Then for this regular local ring A, our proof can be applied, and we have the same result as in our theorem.

References

1. C. Chevalley, *Intersections of algebraic and algebroid varieties*, Trans. Amer. Math. Soc., vol. 57 (1945), pp. 1–85.
2. M. Nagata, *Note on complete local integrity domains*, Mem. Coll. Sci. Univ. Kyoto Ser. A, Math., vol. 28 (1953–54), pp. 271–278.

3. ———, *A general theory of algebraic geometry over Dedekind domains, II*, to appear in Amer. J. Math.
4. P. SAMUEL, *Simple subvarieties of analytic varieties*, Proc. Nat. Acad. Sci. U.S.A., vol. 41 (1955), pp. 647–650.
5. O. ZARISKI, *The concept of a simple point of an abstract algebraic variety*, Trans. Amer. Math. Soc., vol. 62 (1947), pp. 1–52.

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