SINGULAR HYPERSURFACES IN GENERAL RELATIVITY

BY A. H. TAUB

1. Introduction

The general theory of relativity is concerned with a four-dimensional Riemannian space, space-time, with a metric

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu. \tag{1.1}$$

The components of the metric tensor $g_{\mu\nu}$ represent the gravitational fields in the coordinate system x^μ. They are related to the matter present by means of the field equations

$$R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R = -kc^2T^{\mu\nu}, \tag{1.2}$$

where $R_{\mu\nu}$ is the Ricci tensor of the Riemannian space whose metric is given by equations (1.1), R is the scalar curvature of this space,

$$k = 8\pi G/c^2 \tag{1.3}$$

with G being Newton's constant of gravitation, c the special theory of relativity velocity of light, and $T^{\mu\nu}$ the stress energy tensor of the matter present. In writing equations (1.2) we have chosen the dimensions of ds to be those of time.

It is a consequence of equations (1.2) that

$$T^{\mu\nu}{}_{;\nu} = 0, \tag{1.4}$$

where the semicolon represents the covariant derivative with respect to the tensor $g_{\mu\nu}$ and the summation convention has been used. Equations (1.4) are the equations of motion of the matter present and restrict the specification of the tensor $T^{\mu\nu}$ in equations (1.2).

These equations have been discussed in great detail [1], [2] for the case where the space-time has plane symmetry and the matter present is a perfect fluid with stress energy tensor

$$T^{\mu\nu} = \sigma u^\mu u^\nu - g^{\mu\nu}p/c^2, \tag{1.5}$$

where u^μ is the four-dimensional velocity vector of the fluid and satisfies

$$g_{\mu\nu}\, u^\mu u^\nu = u^\mu u_\mu = 1, \tag{1.6}$$

p is the pressure, and

$$\sigma = \rho\left(1 + \frac{\varepsilon}{c^2} + \frac{p}{\rho c^2}\right), \tag{1.7}$$

where ρ is the density as measured by an observer at rest with respect to the fluid, and ε is the specific internal energy of the fluid measured similarly.

Received January 29, 1957.

It has been shown that solutions of equations (1.2) for $g_{\mu\nu}$, p, ρ, and u^μ may be found which reduce to solutions of problems in fluid flow in the special theory of relativity when $k = 0$. If we further take the limit of these solutions when $c \rightarrow \infty$, we obtain solutions to problems in classical hydrodynamics. Further, the approximation method which was used is such that any special relativistic solution of a plane-symmetric problem in hydrodynamics could be obtained by such a reduction process.

In particular those solutions of the special relativistic equations describing the motion of perfect fluids which are physically unacceptable in certain regions of Minkowski (flat) space-time have their counterpart among the solutions of equations (1.2). Such solutions are associated with the formation of shock waves in special relativity and in classical theory [3]. For this reason it was suggested that the theory of a perfect fluid in general relativity must allow for the existence of shock waves. That is, we must contemplate the existence of three-dimensional hypersurfaces in space-time across which there may be discontinuities in the stress energy tensor, the $g_{\mu\nu}$, and their derivatives.

If such hypersurfaces are to be considered, then we must consider equations (1.2) as holding on each side of such a hypersurface, and we must supplement these equations by conditions which relate the values of the $g_{\mu\nu}$, the derivatives of these quantities, and the stress energy tensor on both sides of such a hypersurface. The relations that must hold between the components of the stress energy tensor across a hypersurface of discontinuities must be the generalization of the Rankine-Hugoniot equations of classical and special relativistic hydrodynamics [3].

It is the purpose of this paper to derive and discuss a set of conditions of the type described above for a space-time in which the matter present is a perfect fluid. The method used is based on the existence of a variational principle from which the field equations and the equations of motion may be derived and may be applied to any space-time for which this is the case. The existence of such a variational principle for the case of a space-time in which there is a perfect fluid was shown in a previous publication [4].

If the field equations and the equations of motion, the conservation equations, can be derived from a variational principle, then we may generalize the variational principle by allowing the region of integration involved to include regions of space-time which contain hypersurfaces across which the matter distribution and the metric tensor and its derivatives are discontinuous. We may even vary these singular hypersurfaces. In the classical theory such a generalization is known to give the classical Rankine-Hugoniot equations [5].

It will be shown below that such a generalization of the variational principle leading to the field equations and the equations of motion for a space-time containing a perfect fluid leads to a general relativistic generalization of the Rankine-Hugoniot equations which reduces to the appropriate equations in the special relativistic and classical limits. We shall also obtain conditions that must be satisified by the $g_{\mu\nu}$ and their derivatives across hypersurfaces of discontinuities. Such conditions have been discussed by S. O'Brien and J. L. Synge [6] and by A. Lichnerowitz [7]. The results given below are obtained in a general coordinate system. When the coordinate system is chosen to be that used by O'Brien and Synge, the equations obtained reduce to those given by them.

A transformation of coordinates with a discontinuous second derivative is then shown to reduce these conditions to the requirement of Lichnerowicz, namely that $g_{\mu\nu}$ and the derivatives of the $g_{\mu\nu}$ be continuous in the new coordinate system.

2. The variational principle

It has been shown [4] that if

$$(2.1) \qquad I = \int_V (R - 2k\rho(c^2 + H + \tfrac{1}{2}\mu g_{\mu\nu} u^\mu u^\nu)) \sqrt{-g}\, d^4x = \int_V f\, d^4x,$$

where μ is a Lagrange multiplier and

$$(2.2) \qquad H = \varepsilon - TS,$$

with T the rest temperature and S the rest specific entropy, and where the region of integration is over a volume of space-time swept out by the world lines of an arbitrary number of fluid elements, then equations (1.2) and (1.4) are consequences of

$$\delta I = 0.$$

In this variation the field variables ρ, T, $g_{\mu\nu}$ and the particle paths are varied so that the conservation of mass

$$(2.3) \qquad (\rho u^\nu)_{;\nu} = 0$$

is always satisfied and the Lagrange multiplier μ is chosen so that equation (1.6) is satisfied.

We now wish to study the integral I and the variations produced in it by varying the same field variables in the case where there exists a three-dimensional hypersurface Σ which divides the region of integration V into two four-dimensional regions V_1 and V_2 in each of which the integrand exists and is integrable. We do not require that the integrand exist on the three-dimensional hypersurface Σ. We define I to be the sum of the result of integrating the integrand over the volumes V_1 and V_2. Thus

$$I = \int_{V_1} f\, d^4x + \int_{V_2} f\, d^4x \equiv \int_{V_1+V_2} f\, d^4x.$$

Because of this definition of I and because variations of the field quantities which vanish on the boundaries of the volume V need not vanish on the boundaries of the subvolumes V_1 and V_2, attention must be paid to those terms in f which involve divergences of expressions containing the field quantities and their variations, in the calculation of δI. Such terms will lead to expressions involving integrals over the hypersurface Σ. The integrands in these integrals will contain functions evaluated on both sides of the hypersurface Σ. Thus the admission of the dividing hypersurface Σ brings into the expression for δI terms of the type entering into the Rankine-Hugoniot equations and the looked for conditions involving the $g_{\mu\nu}$ and their derivatives.

Such terms will also arise when only the hypersurface Σ is varied. If the hypersurface Σ is described by the equation

$$x^\mu = x^\mu(\alpha, \beta, \gamma) \tag{2.4}$$

and a nearby hypersurface is defined by the equation

$$x^\mu = x^\mu(\alpha, \beta, \gamma) + e\, \Xi^\mu(\alpha, \beta, \gamma), \tag{2.5}$$

then

$$\delta_\Sigma I = \int_\Sigma [f]\, \Xi^\mu \lambda_\mu \, d\Sigma, \tag{2.6}$$

where λ_μ is the normal to the hypersurface Σ, that is

$$\lambda_\mu \, d\Sigma = \varepsilon_{\mu\nu\sigma\tau} \frac{\partial x^\nu}{\partial \alpha} \frac{\partial x^\sigma}{\partial \beta} \frac{\partial x^\tau}{\partial \gamma} \, d\alpha \, d\beta \, d\gamma \tag{2.7}$$

and

$$[f] = \lim_{\varepsilon \to 0} \{f(x^\mu - \varepsilon \Xi^\mu) - f(x^\mu + \varepsilon \Xi^\mu)\} = f_1 - f_2 . \tag{2.8}$$

3. The particle path

The fluid motion is described by the four-dimensional velocity vector field u^μ which satisfies equation (1.6). The ordinary differential equations

$$\frac{dx^\mu}{ds} = u^\mu(x), \tag{3.1}$$

where s is the proper time, have as solutions

$$x^\mu = x^\mu(u, v, w, s), \tag{3.2}$$

where the parameters u, v, and w may be defined by the conditions

$$\begin{aligned} u &= x^1(u, v, w, 0), \\ v &= x^2(u, v, w, 0), \\ w &= x^3(u, v, w, 0). \end{aligned} \tag{3.3}$$

For fixed values of u, v, w, equations (3.2) define a curve in space-time. This curve will be called the world line of the fluid particle u, v, w or a particle path. Equations (3.2) describe a three-parameter family of curves in space-time or equivalently a transformation of coordinates from the system x^μ to the coordinate system $x^{*\mu}$ where

$$x^{*1} = u, \qquad x^{*2} = v, \qquad x^{*3} = w, \qquad x^{*4} = s. \tag{3.4}$$

We shall use the $x^{*\mu}$ coordinate system in much of the subsequent discussion, and therefore we derive various relations between quantities in the two coordinate systems.

Varied particle paths will be given by the equations

$$x^\mu = x^\mu(u, v, w, s) + e\xi^\mu(u, v, w, s), \tag{3.5}$$

and these equations define a transformation of coordinates from the $x^{*\mu}$ system to the x^μ one, different from that given by equations (3.2).

If the equation of a varied hypersurface Σ in the x^* coordinate system is

$$x^{*\mu} = x^{*\mu}(\alpha, \beta, \gamma) + e\Xi^{*\mu}(\alpha, \beta, \gamma), \tag{3.6}$$

and if equations (3.6) are substituted into equations (3.5), we then obtain equations (2.5) for the varied hypersurface Σ in the x coordinate system. Thus it follows from this fact that

$$\left(\frac{\partial x^\mu}{\partial e}\right)_{e=0} = \Xi^\mu = \left(\frac{\partial x^\mu}{\partial x^{*\rho}}\right)_{e=0} \Xi^{*\rho} + \xi^\mu, \tag{3.7}$$

where the partial derivative occurring on the left of this equation is taken for fixed α, β, and γ, and the partial derivatives occurring on the right-hand side of this equation are evaluated from equations (3.5) with e constant and equal to zero.

It follows from equation (3.5) that for fixed x^μ

$$\left(\frac{\partial x^{*\rho}}{\partial e}\right)_{e=0} = -\left(\frac{\partial x^{*\rho}}{\partial x^\mu}\right)_{e=0} \xi^\mu = -\xi^{*\mu}. \tag{3.8}$$

If in the x^* cordinate system we consider a family of metric tensors

$$g^*{}_{\sigma\tau}(e) = g^*{}_{\sigma\tau} + e\eta^*{}_{\sigma\tau}, \tag{3.9}$$

then this defines a family of tensors in the x coordinate system through the equations (3.5) and the transformation law

$$g_{\mu\nu}(x, e) = g^*{}_{\sigma\tau} \frac{\partial x^{*\sigma}}{\partial x^\mu} \frac{\partial x^{*\tau}}{\partial x^\nu}. \tag{3.10}$$

We define

$$\delta g_{\mu\nu} = \left(\frac{\partial g_{\mu\nu}}{\partial e}\right)_{e=0},$$

where x^μ is fixed in this differentiation.

It then follows from equations (3.9) and (3.10) that

$$\delta g_{\mu\nu} = -\xi_{\mu;\nu} - \xi_{\nu;\mu} + \eta_{\mu\nu}, \tag{3.11}$$

where

$$\eta_{\mu\nu} = \eta^*{}_{\sigma\tau} \left(\frac{\partial x^{*\sigma}}{\partial x^\mu}\right)_{e=0} \left(\frac{\partial x^{*\tau}}{\partial x^\nu}\right)_{e=0}. \tag{3.12}$$

We shall find it convenient to compute the variation of I given by equation (2.1) by evaluating the integrand in the x^* coordinate system and varying the particle paths in accordance with equation (3.5) and the $g^*_{\mu\nu}$ in accordance with equation (3.8). We note that the variation of the particle paths given by equation (3.5) induces a variation of the velocity field u^μ given by the equation

$$\left(\frac{\partial u^\mu}{\partial e}\right)_{e=0} = \frac{\partial \xi^\mu}{\partial s} = \frac{\partial \xi^\mu}{\partial x^\rho} u^\rho. \tag{3.13}$$

In this equation the values of x^* are fixed in the differentiation occurring on the left-hand side of the equation. At fixed values of the x^μ coordinates

$$\delta u^\mu = \left(\frac{\partial u^\mu}{\partial x^{*\rho}} \frac{\partial x^{*\rho}}{\partial e} + \frac{\partial \xi^\mu}{\partial s}\right)_{e=0} = -u^\mu{}_{;\rho}\, \xi^\rho + u^\rho \xi^\mu{}_{;\rho}\,, \tag{3.14}$$

where we have used the notation

$$u^\mu = \frac{\partial x^\mu}{\partial s}, \tag{3.15}$$

and the functions entering into the right-hand side of this equation are those in equation (3.2).

We shall need the result of evaluating

$$\frac{\partial}{\partial e} (g^*_{\mu\nu}\, u^{*\mu} u^{*\nu})_{e=0} = \left(\frac{\partial}{\partial e} \left(g_{\mu\nu}(x(x^*)) u^\mu(x^*) u^\nu(x^*)\right)\right)_{e=0} = \left(\frac{\partial f}{\partial e}\right)_{e=0}$$

at fixed values of $x^{*\mu}$. It follows from equation (3.13) that

$$\begin{aligned} \left(\frac{\partial f}{\partial e}\right)_{e=0} &= \left(\frac{\partial g_{\mu\nu}}{\partial x^\rho} \frac{\partial x^\rho}{\partial e} + 2 g_{\mu\nu}\, u^\mu \frac{\partial u^\nu}{\partial e}\right)_{e=0} \\ &= \left(\frac{\partial g_{\mu\nu}}{\partial x^\rho}\, \xi^\rho + 2 g_{\mu\nu}\, u^\mu u^\rho \frac{\partial \xi^\nu}{\partial x^\rho}\right)_{e=0} \\ &= 2 g_{\mu\nu}\, u^\mu \xi^\nu{}_{;\rho}\, u^\rho = 2 g^*_{\mu\nu}\, u^{*\mu} \xi^{*\mu}{}_{;\rho}\, u^{*\rho}. \end{aligned} \tag{3.16}$$

The varied proper density in the x^μ coordinate system will be written as

$$\rho(x, e) = \rho(x) + e\delta\rho(x).$$

These proper densities may be considered as functions of the variables $x^{*\mu}$ defined by the equations

$$\rho^*(x^*, e) = \rho(x(x^*)) + e\delta\rho(x(x^*)).$$

Then

$$\delta\rho^* = \left(\frac{\partial \rho^*}{\partial e}\right)_{e=0} = \left(\frac{\partial \rho}{\partial x^\mu}\, \xi^\mu + \delta\rho\right)_{e=0}, \tag{3.17}$$

where the differentiation on the left takes place for constant values of the $x^{*\mu}$.

It has been shown [4] that the equation of conservation of mass, equation (2.3), is equivalent to the condition that

$$(3.18)\qquad M = \rho\sqrt{-g}\,\varepsilon_{\lambda\mu\nu\tau}\frac{\partial x^{\lambda}}{\partial u}\frac{\partial x^{\mu}}{\partial v}\frac{\partial x^{\nu}}{\partial w}\frac{\partial x^{\tau}}{\partial s}$$

be independent of s. We restrict the variations of ρ, $g_{\mu\nu}$, and the particle paths so that

$$\left(\frac{\partial M}{\partial e}\right)_{e=0} = 0,$$

where the differentiation is taken at a fixed value of $x^{*\mu}$. Since

$$\frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial e} = \tfrac{1}{2}g^{\mu\nu}\frac{\partial g_{\mu\nu}}{\partial e},$$

it is a consequence of equations (3.18), (3.17), (3.10), and (3.5) that

$$\frac{1}{M}\left(\frac{\partial M}{\partial e}\right)_{e=0} = \frac{1}{\rho}\left(\delta\rho + \frac{\partial\rho}{\partial x^{\mu}}\xi^{\mu}\right) + \tfrac{1}{2}g^{\mu\nu}\eta_{\mu\nu} + \xi^{\mu}{}_{;\mu}$$
$$= \frac{\delta\rho^*}{\rho^*} + \tfrac{1}{2}g^{*\mu\nu}\eta^*{}_{\mu\nu} + \xi^{*\mu}{}_{;\mu} = 0.$$

4. The variation of the scalar curvatur

It is convenient to write equation (2.1) as

$$(4.1)\qquad I = I_1 + I_2,$$

where

$$(4.2)\qquad I_1 = \int_{V_1+V_2} R\sqrt{-g}\,d^4x$$

and

$$(4.3)\qquad I_2 = -2k\int_{V_1+V_2}\rho(c^2 + H + \tfrac{1}{2}\mu g_{\mu\nu}u^{\mu}u^{\nu})\,d^4x.$$

In this section we shall discuss the variation of I_1 due to variations of the $g_{\mu\nu}$ in an arbitrary coordinate system.

From the definition of the scalar curvature, we have

$$(4.4)\qquad \begin{aligned}\sqrt{-g}\,R &= \sqrt{-g}\,g^{\mu\nu}R_{\mu\nu}\\ &= \sqrt{-g}\,g^{\mu\nu}\left(-\left\{{\sigma \atop \mu\nu}\right\}_{,\sigma} + \left\{{\sigma \atop \mu\sigma}\right\}_{,\nu} - \left\{{\sigma \atop \rho\sigma}\right\}\left\{{\rho \atop \mu\nu}\right\} + \left\{{\sigma \atop \rho\nu}\right\}\left\{{\rho \atop \mu\sigma}\right\}\right),\end{aligned}$$

where $\left\{{\sigma \atop \mu\nu}\right\}$ are the Christoffel symbols of the second kind computed from the $g_{\mu\nu}$ and the comma denotes the ordinary derivative. It is well known that

$$(4.5)\qquad \delta R_{\mu\nu} = -\left(\delta\left\{{\sigma \atop \mu\nu}\right\}\right)_{;\sigma} + \left(\delta\left\{{\sigma \atop \mu\sigma}\right\}\right)_{;\nu},$$

where the semicolon denotes the covariant derivative with respect to the metric tensor $g_{\mu\nu}$.

Since

$$\begin{Bmatrix} \sigma \\ \mu\nu \end{Bmatrix} = \tfrac{1}{2} g^{\sigma\rho}(g_{\mu\rho,\nu} + g_{\nu\rho,\mu} - g_{\mu\nu,\rho}),$$

it follows that

$$\delta \begin{Bmatrix} \sigma \\ \mu\nu \end{Bmatrix} = \tfrac{1}{2} g^{\sigma\rho}(\delta g_{\mu\rho;\nu} + \delta g_{\nu\rho;\mu} - \delta g_{\mu\nu;\rho}).$$

Hence

$$\sqrt{-g}\, g^{\mu\nu} \delta R_{\mu\nu} = \sqrt{-g}\, ((g^{\alpha\rho} g^{\beta\sigma} - g^{\rho\sigma} g^{\alpha\beta}) \delta g_{\alpha\rho;\beta})_{;\sigma}.$$

Therefore

$$\delta(\sqrt{-g}\, R) = \sqrt{-g}\, (-R^{\mu\nu} + \tfrac{1}{2} g^{\mu\nu} R) \delta g_{\mu\nu} + \sqrt{-g}\, ((g^{\alpha\rho} g^{\beta\sigma} - g^{\rho\sigma} g^{\alpha\beta}) \delta g_{\alpha\rho;\beta})_{;\sigma}.$$

Hence under variations of the $g_{\mu\nu}$ alone, we have

$$\begin{aligned} I = &\int_{V_1+V_2} \sqrt{-g}\, (-R^{\mu\nu} + \tfrac{1}{2} g^{\mu\nu} R) \delta g_{\mu\nu}\, d^4x \\ &+ \int_{V_1+V_2} \sqrt{-g}\, ((g^{\alpha\rho} g^{\beta\sigma} - g^{\rho\sigma} g^{\alpha\beta}) \delta g_{\alpha\rho;\beta})_{;\sigma}\, d^4x. \end{aligned} \tag{4.6}$$

The second integral being a divergence may be written in terms of an integral over the boundaries of the subvolumes V_1 and V_2. We shall assume that $\delta g_{\mu\nu}$ vanishes on all boundaries except the hypersurface Σ. In that case we may write

$$\begin{aligned} \delta I_1 = &\int_{V_1+V_2} \sqrt{-g}\, (-R^{\mu\nu} + \tfrac{1}{2} g^{\mu\nu} R) \delta g_{\mu\nu}\, d^4x \\ &+ \int_{\Sigma} [\sqrt{-g}\, (g^{\alpha\rho} g^{\beta\sigma} - g^{\rho\sigma} g^{\alpha\beta}) \delta g_{\alpha\rho;\beta}]\, \lambda_\sigma\, d\Sigma, \end{aligned} \tag{4.7}$$

where the notation defined by equations (2.7) and (2.8) has been used.

If in addition to varying the $g_{\mu\nu}$ we vary the hypersurface of discontinuity, we obtain

$$\begin{aligned} \delta I_1 = &\int_{V_1+V_2} \sqrt{-g}\, (-R^{\mu\nu} + \tfrac{1}{2} g^{\mu\nu} R) \delta g_{\mu\nu}\, d^4x \\ &+ \int_{\Sigma} [\sqrt{-g}\, (g^{\alpha\rho} g^{\beta\sigma} - g^{\rho\sigma} g^{\alpha\beta}) \delta g_{\alpha\rho;\beta} + R\Xi^\sigma] \lambda_\sigma\, d\Sigma. \end{aligned} \tag{4.8}$$

Equation (4.8) holds in an arbitrary coordinate system, and in particular it may be applied in the $x^{*\mu}$ one in which case we have

$$\delta g^*_{\mu\nu} = \eta^*_{\mu\nu} \tag{4.9}$$

as follows from equation (3.9), and $\Xi^{*\mu}$ defines the equation of the varied hypersurface in accordance with equation (3.6). It may be verified that if in equation (4.8) applied in the x^μ coordinate system we substitute for $\delta g_{\mu\nu}$ from equation (3.11) and for Ξ^σ from equation (3.7), we obtain equation (4.8) applied in the $x^{*\mu}$ coordinate system, that is we obtain

$$\delta I_1 = \int_{V_1+V_2} \sqrt{-g^*}\,(-R^{*\mu\nu} + \tfrac{1}{2}g^{*\mu\nu}R^*)\eta^*_{\mu\nu}\, du\, dv\, dw\, ds \tag{4.10}$$
$$+ \int_\Sigma [\sqrt{-g^*}\,(g^{*\alpha\rho}g^{*\beta\sigma} - g^{*\rho\sigma}g^{*\alpha\beta})\eta^*_{\alpha\rho;\beta} + R^*\Xi^{*\sigma}]\lambda^*_\sigma\, d\Sigma^*.$$

5. The variation of I_2

We now discuss the variation of I_2 as was done previously [4] but now include a variation of the hypersurface Σ. We obtain

$$-\frac{\delta I_2}{2k} = \int_{V_1+V_2} \rho^*\left(\frac{p^*}{\rho^{*2}}\,\delta\rho^* - S^*\delta T^* + \tfrac{1}{2}\,\mu^* u^{*\mu}u^{*\nu}\eta^*_{\mu\nu}\right. \tag{5.1}$$
$$\left. + \mu^* g^*_{\mu\nu}u^{*\mu}u^{*\rho}\xi^{*\nu}{}_{;\rho}\right)\sqrt{-g^*}\, du\, dv\, dw\, ds$$
$$+ \int_\Sigma \left[\rho^*\sqrt{-g^*}\left(c^2 + H^* + \frac{\mu^*}{2}\,g^*_{\mu\nu}u^{*\mu}u^{*\nu}\right)\right]\Xi^{*\mu}\lambda^*_\mu\, d\Sigma^*.$$

The hypersurface integral on the right of this expression arises from the variation of the surface Σ, the volume integral is derived by evaluating I_2 in the $x^{*\mu}$ coordinate system and varying the functions $\rho^*(x^*)$, $T^*(x^*)$, and $x^\mu(x^*)$ in accordance with equation (3.9); equation (3.6) has been used in deriving equation (5.1) as well as the equation

$$dH = \frac{p}{\rho^2}\,d\rho - SdT,$$

which follows from (2.2) and the definitions of temperature and entropy.

If we now substitute from equation (3.19) into (5.1), we may write the latter equation as

$$-\frac{\delta I_2}{k} = \int_{V_1+V_2} (\theta^{*\mu\nu}(\eta^*_{\mu\nu} + 2\xi^*_{\mu;\nu}) - 2\rho^* S^* u^{*\rho}(\delta\alpha^*)_{,\rho})\sqrt{-g}^*du\, dv\, dw\, ds \tag{5.2}$$
$$+ \int_\Sigma [\rho^*(c^2 + H^* + \tfrac{1}{2}\mu^* g^*_{\mu\nu}u^{*\mu}u^\nu)\sqrt{-g^*}]\,\Xi^\mu\,\lambda^*_\mu\, d\Sigma^*,$$

where

$$\theta^{*\mu\nu} = \rho^*\mu^* u^{*\mu}u^{*\nu} - p^* g^{*\mu\nu}, \tag{5.3}$$

and we have written

$$\delta T^* = (\delta\alpha^*)_{,\rho}\, u^{*\rho}. \tag{5.4}$$

After integration by parts we may write equation (5.2) as

$$
\begin{aligned}
-\frac{\delta I_2}{k} = \int_{V_1+V_2} & (\theta^{*\mu\nu}\eta^*_{\mu\nu} - 2\theta^{*\nu}{}_{\mu;\nu}\xi^{*\mu} + 2(\rho^* S^* u^{*\mu})_{;\mu}\delta\alpha^*) \\
& \cdot\sqrt{-g^*}\, du\, dv\, dw\, ds \\
+ \int_\Sigma & [(\rho^*(c^2 + H^* + \tfrac{1}{2}\mu^* g^*_{\mu\nu} u^{*\mu} u^{*\nu})\Xi\, \mu^* + 2\theta^{*\mu}{}_\nu \xi^{*\nu} - 2\rho^* S^* u^{*\mu}\delta\alpha^*) \\
& \cdot\sqrt{-g^*}]\, \lambda^*_\mu\, d\Sigma^*.
\end{aligned}
\tag{5.5}
$$

We then have by combining equations (5.5) and (4.10)

$$
\begin{aligned}
-\delta I = \int_{V_1+V_2} & \sqrt{-g}\, ((R^{*\mu\nu} - \tfrac{1}{2}g^{*\mu\nu}R^* + k\theta^{*\mu\nu})\eta^*_{\mu\nu} \\
& - 2k\theta^{*\nu}{}_{\mu;\nu}\xi^{*\mu} + 2k\rho^* S^* u^{*\mu}\delta\alpha^*)\, du\, dv\, dw\, ds \\
+ \int_\Sigma & [(2k\,\{\rho^*(c^2 + H^* + \tfrac{1}{2}\mu^* g^*_{\mu\nu}u^{*\mu}u^{*\nu}) - R^*\}\, \Xi^{*\mu} \\
& -(g^{*\alpha\rho}g^{*\beta\mu} - g^{*\rho\mu}g^{*\alpha\beta})\eta^*_{\alpha\rho;\beta} - 2\theta^{*\mu}{}_\nu\, \xi^{*\nu} \\
& + 2\rho^* S^* u^{*\mu}\delta\alpha^*)\sqrt{-g^*}]\lambda^*_\mu\, d\Sigma^*.
\end{aligned}
\tag{5.6}
$$

6. The equations of motion and boundary conditions

The requirement that

$$\delta I = 0 \tag{6.1}$$

for arbitrary $\eta^*_{\mu\nu}$, $\xi^{*\mu}$, and $\delta\alpha^*$ which vanish on the hypersurface Σ leads to the results obtained previously:

$$R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R + k\theta^{\mu\nu} = 0, \tag{6.2}$$

$$\theta^{\mu\nu}{}_{;\nu} = 0, \tag{6.3}$$

and

$$\mu = c^2 + \varepsilon + p/\rho. \tag{6.4}$$

Since our equations are tensor equations, we may write them in an arbitrary coordinate system. This will be done henceforth. Equation (6.4) must be satisfied in order that equation (1.6) hold. It then follows from equations (6.4) and (5.3) that

$$\theta^{\mu\nu} = c^2 T^{\mu\nu},$$

where $T^{\mu\nu}$ is given by equation (1.5).

We further require that (6.1) hold for more general $\eta^*_{\mu\nu}$ and $\xi^{*\mu}$, namely those which may take on arbitrary values in the regions V_1 and V_2 and which may take on arbitrary values on the hypersurface Σ. Then we must have in addition to equations (6.2) through (6.4) the conditions

$$[\theta^{\mu}\nu\sqrt{-g}\,\lambda_{\mu}] = 0, \tag{6.5}$$

$$[\sqrt{-g}\,\lambda_{\mu}(g^{\alpha\rho}g^{\beta\mu} - g^{\rho\mu}g^{\alpha\beta})] = 0, \tag{6.6}$$

and

$$\left[\sqrt{-g}\,\lambda_{\mu}(g^{\alpha\rho}g^{\beta\mu} - g^{\rho\mu}g^{\alpha\beta})\left(\delta_{\alpha}^{\sigma}\begin{Bmatrix}\lambda\\ \rho\beta\end{Bmatrix} + \delta_{\rho}^{\sigma}\begin{Bmatrix}\lambda\\ \alpha\beta\end{Bmatrix}\right)\right] = 0. \tag{6.7}$$

Equations (6.6) and (6.7) arise from setting the coefficients of the derivatives of $\eta_{\alpha\rho}$ and the coefficients of $\eta_{\alpha\rho}$ in the second term of the integrand of the hypersurface integral in equation (5.6) separately equal to zero. This must be done if the hypersurface integral is to vanish under the conditions stated above.

Equations (6.5), (6.6), and (6.7) are the generalized Rankine-Hugoniot equations and the looked for conditions involving the jump in the $g_{\mu\nu}$ and their derivations. They will be discussed in Section 8.

In the remainder of this section we discuss the remaining terms in the hypersurface integral in equation (5.6). That is, we consider the equation

$$\left[\left(2k\left\{\rho c^2 + H + \frac{\mu}{2}g_{\mu\nu}\,u^{\mu}u^{\nu}\right) - R\right\}\Xi^{\mu} + 2k\,Su^{\mu}\delta\alpha\right)\sqrt{-g}\,\lambda_{\mu}\right] = 0 \tag{6.8}$$

which must be satisfied if δI is to equal zero. This equation may be regarded as a relation between the variations of T and the variations of the surface. It is the general relativistic analogue of a relation previously found in the classical theory [5].

It follows from equations (1.6) and (6.2) to (6.4) that this equation may be written as

$$\left[\sqrt{-g}\,\rho\left(c^2 + \varepsilon - TS + \frac{2p}{\rho}\right)\Xi^{\mu}\lambda_{\mu}\right] = -[\rho\sqrt{-g}\,u^{\mu}\lambda_{\mu}\,S\delta\alpha]. \tag{6.9}$$

Our result is then: the variational principle

$$\delta I = 0$$

under arbitrary variations of the $g_{\mu\nu}$, the particle paths, the rest density, the rest temperature, and the hypersurface Σ, subject to conditions (3.19), (1.6), and (6.9), implies equations (6.2) and (6.3) in the regions where the dependent variables have the required number of derivatives and equations (6.5), (6.6), and (6.7) hold across hypersurfaces of discontinuity.

7. The generalization of the conservation of mass equation

The equation describing the conservation of mass in regions where ρ and u^{μ} have derivatives is equation (2.3) which may be written as

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\sigma}}(\sqrt{-g}\,\rho u^{\sigma}) = 0. \tag{7.1}$$

If this equation is integrated over a region, it may be written as

$$\int \sqrt{-g}\,\rho u^{\sigma}\lambda_{\sigma}\,d\tau = 0, \tag{7.2}$$

where $d\tau$ is the invariant three-dimensional volume element of the boundary of the region of integration and λ_σ is the normal vector to the boundary. If we suppose that the region of integration is a shell of thickness ε enclosing the hypersurface Σ, we obtain from equation (7.2) in the limit as ε goes to zero the condition

$$[\sqrt{-g}\,\rho u^{\sigma}\lambda_{\sigma}] = 0, \tag{7.3}$$

where now λ_σ is the normal to the hypersurface Σ.

This equation will be taken as the generalization of the equation of conservation of mass and must be considered on a par with equation (6.5). Properly speaking the five equations (7.3) and (6.5) are the generalizations of the Rankine-Hugoniot equations. They reduce to the equations found earlier [3] in case the space-time is flat.

8. The conditions (6.6) and (6.7)

In order to simplify the discussion of these equations, we assume that we have chosen the coordinate systems in the regions V_1 and V_2 of space-time in such a way that the hypersurface Σ is given by the equation

$$x^1 = 0. \tag{8.1}$$

That is, we have chosen the coordinate system so that equations (2.4) are

$$x^2 = \alpha \qquad x^3 = \beta \qquad x^4 = \gamma.$$

Then it follows from equation (2.7) that

$$\lambda_\mu = \varepsilon_{\mu\nu\sigma\tau}\frac{\partial x^\nu}{\partial\alpha}\frac{\partial x^\sigma}{\partial\beta}\frac{\partial x^\tau}{\partial\gamma} = \delta^1_\mu. \tag{8 2}$$

If the hypersurface Σ is to be free of singularities, and if we choose our coordinate system so that equation (8.1) holds, then the line element on Σ is given by

$$d\sigma^2 = g_{ij}\,dx^i\,dx^j \qquad (i, j = 2, 3, 4), \tag{8.3}$$

where $g_{ij}(0, x^2, x^3, x^4)$ are continuous functions of the variables x^2, x^3, x^4 and have at least second derivatives with respect to these variables. We may suppose that the choice of coordinate systems in the regions V_1 and V_2 is such that the same line element for Σ is obtained by taking equation (8.1) in V_1 and the same equation in V_2 to define Σ. That is, we may suppose that as a result of our choice of coordinate systems we have

$$[g_{ij}] = 0. \tag{8.4}$$

Since equations (8.3) are to hold identically in x^2, x^3, x^4 at an arbitrary point of Σ, we must also have

$$\left[\frac{\partial g_{ij}}{\partial x^k}\right] = 0. \tag{8.5}$$

The terms

$$\sqrt{-g}\, \lambda_\mu\, dx^2\, dx^3\, dx^4 = \sqrt{-g}\, \delta^1_\mu\, dx^2\, dx^3\, dx^4$$

in the hypersurface integrals considered in sections 4 and 5 which gave rise to the appearance of $\sqrt{-g}$ in equations (6.4), (6.5), and (6.7) represent the invariant volume measure on the hypersurface Σ and have to do only with the geometry of Σ. Hence if Σ is to be without singularities, we must have

$$[\sqrt{-g}\, \lambda_\mu] = [\sqrt{-g}\, \delta^1_\mu] = 0. \tag{8.6}$$

It follows from the identity

$$\sqrt{-g}\,(g^{\alpha\rho}g^{\beta\mu} - g^{\rho\mu}g^{\alpha\beta}) = \frac{1}{\sqrt{-g}}\, \varepsilon^{\sigma\tau\alpha\beta}\varepsilon^{\gamma\delta\rho\mu} g_{\sigma\gamma}\, g_{\tau\delta}$$

and equation (8.6) that equation (6.6) becomes

$$[\varepsilon^{\sigma\tau\alpha\beta}\varepsilon^{\gamma\delta\rho 1} g_{\sigma\gamma}\, g_{\tau\delta}] = 0.$$

From the properties of the ε's it follows that

$$\varepsilon^{\sigma\tau\alpha\beta}\varepsilon^{jkl1}[g_{\sigma j}\, g_{\tau k}] = 0.$$

These equations are equivalent to the equations

$$[g_{\alpha l}\, g_{mn} - g_{\beta l}\, g_{\alpha m}] = 0.$$

If in these equations we set $\alpha = \beta = 1$, they become identically satisfied. It is therefore sufficient to treat the case $\alpha = 1$, $\beta = n$. Then we have

$$[g_{1l}\, g_{mn} - g_{nl}\, g_{1m}] = 0.$$

We may now assume that at an arbitrary point of Σ the coordinate system is chosen so that $g_{mn} = 0$ unless $m = n$. Then choosing $m = n$ and $l \neq n$ in the above equation, we obtain the condition

$$[g_{mn}\, g_{1l}] = 0.$$

That is, g_{1l} is continuous across Σ. Since g is also continuous across Σ (cf. (8.6)), we see that equations (6.6) and the requirements on our coordinate system and on the hypersurface Σ imply that

$$[g_{\mu\nu}] = 0, \tag{8.7}$$

that is, the $g_{\mu\nu}$ are continuous across Σ.

Since equation (8.7) is an identity in x^2, x^3, x^4, we must have

$$\left[\frac{\partial g_{\mu\nu}}{\partial x^i}\right] = 0 \tag{8.8}$$

in the coordinate system in which equation (8.2) holds.

In view of the symmetry of the Christoffel symbols and equations (6.7), we may write equations (6.7) as

$$\lambda_\mu(g^{\alpha\rho}g^{\beta\mu} - g^{\rho\mu}g^{\alpha\beta})\left[\left\{\begin{matrix}\lambda\\ \alpha\beta\end{matrix}\right\}\right] = 0.$$

We may write these equations as

$$\lambda_\mu(\delta^\alpha_\rho\, g^{\beta\mu} - \delta^\mu_\rho\, g^{\alpha\beta})\left[g_{\nu\lambda}\left\{\begin{matrix}\lambda\\ \alpha\beta\end{matrix}\right\}\right] = 0,$$

where

$$\left[g_{\nu\lambda}\left\{\begin{matrix}\lambda\\ \alpha\beta\end{matrix}\right\}\right] = \frac{1}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right].$$

Thus in the coordinate system in which (8.2) holds, equations (6.7) may be written as

$$g^{\beta 1}\left[\frac{\partial g_{\rho\nu}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\rho} - \frac{\partial g_{\rho\beta}}{\partial x^\nu}\right] - \delta^1_\rho\, g^{\alpha\beta}\left[2\,\frac{\partial g_{\alpha\nu}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right] = 0. \tag{8.9}$$

If in this equation we set $\nu = \rho = 1$, we obtain

$$g^{\beta 1}\left[\frac{\partial g_{11}}{\partial x^\beta}\right] - g^{\alpha\beta}\left[2\,\frac{\partial g_{\alpha 1}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^1}\right] = 0.$$

In view of equation (8.8) this may be written as

$$g^{ij}\left[\frac{\partial g_{ij}}{\partial x^1}\right] = 0. \tag{8.10}$$

Next, set $\rho = 1$, $\nu = n$ in equation (8.9) and obtain

$$g^{m1}\left[\frac{\partial g_{mn}}{\partial x^1}\right] = 0 \tag{8.11}$$

as a consequence of (8.8).

Finally set $\rho = r$, $\nu = n$ in equation (8.9) and obtain

$$g^{11}\left[\frac{\partial g_{rn}}{\partial x^1}\right] = 0. \tag{8.12}$$

Equations (8.10), (8.11), and (8.12) are the conditions found by O'Brien and Synge [6]. If the hypersurface Σ is such that λ_σ is not a null vector (if Σ is nowhere tangent to the light cone), $g^{11} \neq 0$, and these three equations become

$$\left[\frac{\partial g_{rn}}{\partial x^1}\right] = 0. \tag{8.13}$$

That is, the derivatives of g_{rn} in the above coordinate system are continuous across Σ. The four quantities, $\partial g_{1\mu}/\partial x^1$, may be discontinuous across Σ.

However, it has been pointed out by A. Lichnerowicz [7] that if these four quantities are discontinuous across Σ in the coordinate system chosen, we may make a nonanalytic transformation of coordinates which preserves the coordinates of each point of Σ and the region V_1 and which changes the coordinates of points in the region V_2 in such a way that the discontinuities in

$\partial g_{1\mu}/\partial x^1$ are removed. If the original coordinate system is such that the hypersurface is defined by equation (8.1) and the region V_1 is characterized by $x^1 < 0$ and the region V_2 by $x^1 > 0$, the transformation which removes the discontinuity in $\partial g_{1\mu}/\partial x^1$ is

$$\begin{aligned} \bar{x}^\mu &= x^\mu + \tfrac{1}{2}(x^1)^2\phi^\mu(x^i) \qquad & x^1 > 0 \\ \bar{x}^\mu &= x^\mu & x^1 \leqq 0. \end{aligned} \qquad (\mu = 1, 2, 3, 4; i = 2, 3, 4) \tag{8.14}$$

It follows from these equations that for $x^1 > 0$

$$\frac{\partial \bar{x}^\mu}{\partial x^i} = \delta_i^\mu + \frac{(x^1)^2}{2}\frac{\partial \phi^\mu}{\partial x^i}$$

$$\frac{\partial \bar{x}^\mu}{\partial x^1} = \delta_1^\mu + x^1\phi^\mu$$

$$\frac{\partial^2 \bar{x}^\mu}{\partial x^i \partial x^j} = \frac{(x^1)^2}{2}\frac{\partial^2 \phi^\mu}{\partial x^i \partial x^j}$$

$$\frac{\partial^2 \bar{x}^\mu}{\partial x^1 \partial x^i} = \frac{\partial^2 \bar{x}^\mu}{\partial x^i \partial x^1} = x^1\frac{\partial \phi^\mu}{\partial x^i}$$

$$\frac{\partial^2 \bar{x}^\mu}{(\partial x^1)^2} = \phi^\mu,$$

whereas for $x^1 \leqq 0$ we have

$$\frac{\partial \bar{x}^\mu}{\partial x^\nu} = \delta_\nu^\mu,$$

and all second derivatives vanish. Hence the transformation and its first derivatives are continuous across Σ.

We shall use the subscript $+$ to denote the limit of a function of x^1 as x^1 tends to zero through positive values. Thus

$$f_+ = \lim_{\substack{x^1>0 \\ x^1\to 0}} f(x^1).$$

With this notation it follows from the above equations that

$$\bar{g}_+^{\mu\nu} = \left(g^{\sigma\tau}\frac{\partial \bar{x}^\mu}{\partial x^\sigma}\frac{\partial \bar{x}^\nu}{\partial x^\tau}\right)_+ = g_+^{\mu\nu}$$

and

$$\begin{aligned} \left(\frac{\partial \bar{g}^{\mu\nu}}{\partial \bar{x}^1}\right)_+ &= \left\{\left(\frac{\partial g^{\sigma\tau}}{\partial x^\rho}\frac{\partial \bar{x}^\mu}{\partial x^\sigma}\frac{\partial \bar{x}^\nu}{\partial x^\tau} + g^{\sigma\tau}\frac{\partial^2 \bar{x}^\mu}{\partial x^\sigma \partial x^\rho}\frac{\partial \bar{x}^\nu}{\partial x^\tau} + g^{\sigma\tau}\frac{\partial \bar{x}^\mu}{\partial x^\sigma}\frac{\partial^2 \bar{x}^\mu}{\partial x^\rho \partial x^\tau}\right)\frac{\partial x^\rho}{\partial \bar{x}^1}\right\}_+ \\ &= \left(\frac{\partial g^{\mu\nu}}{\partial x^1}\right)_+ + g_+^{1\nu}\phi^\mu + g_+^{1\mu}\phi^\nu. \end{aligned}$$

Hence

$$\left(\frac{\partial \bar{g}_{1\rho}}{\partial \bar{x}^1}\right)_+ = -\left(\frac{\partial \bar{g}^{\mu\nu}}{\partial \bar{x}^1}\right)_+ \bar{g}_{+\mu 1}\,\bar{g}_{+\nu\rho} = -\left(\frac{\partial g_1}{\partial x^1}\right)_+ - \delta^1_\rho \phi_1 - \phi_\rho .$$

We may thus determine ϕ^μ so that the left-hand side of this equation is any prescribed function of the x^i, in particular, so that the left-hand side is

$$\lim_{\substack{x^1<0 \\ x^1\to 0}} \left(\frac{\partial \bar{g}_1}{\partial \bar{x}^1}\right) = \left(\frac{\partial \bar{g}_1}{\partial \bar{x}^1}\right)_- .$$

Thus we may remove the discontinuity in the derivatives of the $g_{\mu\nu}$ by suitably modifying our coordinate system.

However, in a general coordinate system, the conditions that the $g_{\mu\nu}$ must satisfy across Σ are equations (6.6) and (6.7). In order to remove the discontinuities which may appear in the derivatives of the $g_{\mu\nu}$, a transformation of coordinates which has discontinuous second derivatives is required. Thus the introduction of singular hypersurfaces in a space-time in which the variational principle discussed above is to hold implies that we must formulate the general theory of relativity in terms of a Riemannian space in which the transformations between admissible coordinate systems are at most required to be continuous and have continuous first derivatives across the singular hypersurfaces. The formulation given by Lichnerowicz has more restricted admissible coordinate systems. He requires the transformations between them to have continuous second derivatives.

It is known that the interior and exterior Schwarzschild solution for a spherically symmetric distribution of an incompressible fluid have a discontinuity in the derivatives of the $g_{\mu\nu}$ with respect to the spatial coordinate r in the usually used coordinate system. In this coordinate system the boundary is given by $r = r_0$. This discontinuity is in accordance with equations (6.6) and (6.7). A similar situation arises in the plane-symmetric case [2].

9. The Rankine-Hugoniot equations

These equations are equations (6.5) and (7.3). In view of the fact that we have the $g_{\mu\nu}$ continuous across Σ, they may be written as

(9.1) $$[\lambda_\mu T^\mu{}_\nu] = \lambda_\mu(T_+{}^\mu{}_\nu - T_-{}^\mu{}_\nu) = 0$$

and

(9.2) $$[\rho u^\mu \lambda_\mu] = \lambda_\mu(\rho_+ u_+{}^\mu - \rho_- u_-{}^\mu) = 0.$$

The last of these equations may be written as

(9.3) $$\rho_+ \lambda_\mu u_+{}^\mu = \rho_- \lambda_\mu u_-{}^\mu = m.$$

The quantity m represents the rate at which matter is approaching the hypersurface Σ, and this is equal to the rate at which it is leaving this hypersurface.

There are two types of discontinuities: (1) those for which

$$m = 0, \tag{9.4}$$

and (2) those for which

$$m \neq 0. \tag{9.5}$$

In case $m = 0$ the discontinuity is either a slip-stream, a density discontinuity, or both when $T^{\mu}{}_{\nu}$ is the stress energy tensor for a perfect fluid. For in that case equations (9.1) may be written as

$$m\left(\frac{\sigma}{\rho}\right)_+ u_{+\mu} - \frac{p_+}{c^2}\lambda_\mu = m\left(\frac{\sigma}{\rho}\right)_- u_{-\mu} - \frac{p_-}{c^2}\lambda_\mu, \tag{9.6}$$

which reduces to

$$p_+ = p_- \tag{9.7}$$

when $m = 0$.

In a case where $m = 0$, $\rho_+ = 0$, and $\rho_- \neq 0$ (a density discontinuity), we obtain from equation (9.3)

$$\lambda_\mu u_-{}^{\mu} = 0. \tag{9.8}$$

Thus, the hypersurface Σ is made up of the world lines of particles of the fluid. Equations (9.7) with $p_+ = 0$ and (9.8) are the conditions usually assumed at the boundary between a distribution of matter and empty space-time.

We may have $m = 0$, $\rho_+ \neq 0$, and $\rho_- \neq 0$. In this case it follows that

$$\lambda_\mu(u_+{}^{\mu} - u_-{}^{\mu}) = 0, \tag{9.9}$$

that is, the component of flow normal to the hypersurface is continuous. In this case a discontinuity in tangential velocity occurs (a slip-stream is said to exist), or there is a discontinuity in density, or both possibilities may occur.

In case $m \neq 0$ the hypersurface Σ will be said to be a shock front when the vector normal to Σ is a space-like vector. In that case equations (9.1) and (9.2) evaluated at an arbitrary point of Σ in a coordinate system, chosen so that the $g_{\mu\nu}$ have the values $g_{\mu\nu} = 0$, $\mu \neq \nu$, $g_{11} = g_{22} = g_{33} = g_{44} = -1$, become the Rankine-Hugoniot equations in special relativity. These equations have been shown [3] to reduce to the classical Rankine-Hugoniot equations in the limit $c \to \infty$.

It should be noted that in such a case the normal component of the velocity vector u^{μ} is discontinuous across the hypersurface Σ. This means that the transformation of coordinates given by equations (3.2) is such that it is continuous but has discontinuous first derivatives across Σ. Thus the mathematical formulation of general relativity must be broadened even further than indicated above. It must allow for transformations of admissible coordinate systems which are continuous across Σ but may have discontinuous first derivatives across this hypersurface.

10. Generalizations

It should be pointed out that equations (6.5), (6.6), and (6.7) will arise from any variational principle of the form

$$\delta I = 0,$$

where the variations may include the variations of the hypersurface Σ and

$$I = I_1 + \int_{V_1+V_2} f(g, \phi^{(\alpha)}, x)\, d^4x,$$

where I_1 is given by equation (4.2) and the integrand f depends on the $g_{\mu\nu}$ and not the derivatives of this tensor. It may also depend on other field variables $\phi^{(\alpha)}(x)$. In such a case the tensor $\theta^{\mu\nu}$ entering in equation (6.5) is given by

$$\theta^{\mu\nu} = \frac{\partial f}{\partial g_{\mu\nu}}.$$

Additional equations holding across Σ which involve conditions on the discontinuities of the variables $\phi^{(\alpha)}$ and their derivatives will also arise in such a case. They did not in the case of the perfect fluid because the Euler equations of the variational principle which led to the hydrodynamical equations of motion were a consequence of the field equations for the $g_{\mu\nu}$.

The hypersurfaces Σ may be considered as a mathematical abstraction corresponding to a small region of space-time in which abrupt changes in various field variables take place. This is the interpretation placed on such hypersurfaces in classical theory. The relations (6.5), (6.6), and (6.7) are then to be interpreted as the relations that must hold between the field variables before and after such transitions.

This remark enables us to see under what conditions we may apply these equations even when λ_μ is a time-like vector. Suppose that in a region of space-time bounded by two nearby space-like hypersurfaces Σ_+ and Σ_- the stress energy tensor is of one form but that outside this region it is of another form, say that of a perfect fluid given by equation (1.5) in the region bounded by Σ_- and that of a perfect fluid and a radiation field in the region bounded by Σ_+. In the limit as $\Sigma_+ \to \Sigma_- \to \Sigma$ the theory given above should hold. Thus if for a short interval of time there are some processes taking place which would affect the gravitational field present, we may take account of such processes to the extent that the limiting configuration is an approximation to the one described. That is, we consider the singular hypersurface

Σ as providing an abrupt transition from a space-time with a perfect fluid to a space-time with a perfect fluid and radiation present. A more detailed theory would consider the transition in the form of a stress energy tensor as taking place over a zone of space-time limited by the space-like hypersurfaces Σ_- and Σ_+.

In classical theory the approximation of a hydrodynamics shock wave by a mathematical discontinuity has been shown to be a fruitful simplifying approximation. The generalization of these waves to singular hypersurfaces in space-time is a natural one, and a range of phenomena may be studied in terms of them.

REFERENCES

1. A. H. TAUB, *Isentropic hydrodynamics in plane symmetric space-times*, Physical Rev., vol. 103 (1956), pp. 454–467.
2. ———, *Approximate solutions of the Einstein equations for isentropic motions of plane-symmetric distributions of perfect fluids*, to appear in Physical Rev., August 1, 1957.
3. ———, *Relativistic Rankine-Hugoniot equations*, Physical Rev., vol. 74 (1948), pp. 328–334.
4. ———, *General relativistic variational principle for perfect fluids*, Physical Rev., vol. 94 (1954), pp. 1468–1470.
5. ———, *On Hamilton's principle for perfect compressible fluids*, Proceedings of First Symposium of Applied Mathematics, American Mathematical Society, New York, 1949, pp. 148–157.
6. STEPHEN O'BRIEN AND JOHN L. SYNGE, *Jump conditions at discontinuities in general relativity*, Communications of the Dublin Institute for Advanced Studies, Ser. A, no. 9 (1953).
7. A. LICHNEROWICZ, *Théories relativistes de la gravitation et de l'électromagnetisme*, Masson et cie, Paris, 1955.

UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS