

# A remark about weak fillings

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**Abstract** Let  $L$  be a closed manifold of dimension  $n \geq 2$  which admits a totally real embedding into  $\mathbb{C}^n$ . Let  $ST^*L$  be the space of rays of the cotangent bundle  $T^*L$  of  $L$ , and let  $DT^*L$  be the unit disk bundle of  $T^*L$  defined by any Riemannian metric on  $L$ . We observe that  $ST^*L$  endowed with its standard contact structure admits weak symplectic fillings  $W$  which are diffeomorphic to  $DT^*L$  and for which any closed Lagrangian submanifold  $N \subset W$  has the property that the map  $H_1(N, \mathbb{R}) \rightarrow H_1(W, \mathbb{R})$  has a nontrivial kernel. This relies on a variation on a theorem by Laudenbach and Sikorav.

## 1. Introduction

A field of hyperplanes  $\xi$  on a manifold  $M$  is a contact structure if one can locally find a 1-form  $\alpha$  such that  $\xi = \text{Ker}(\alpha)$  and such that  $d\alpha$  restricted to  $\xi$  is everywhere nondegenerate. This implies that  $M$  is odd-dimensional; we write  $\dim M = 2n - 1$ . We always assume that the manifolds and contact structures we consider are compatibly oriented. This means that  $\xi$  can be globally defined by a 1-form  $\alpha$  such that  $\alpha \wedge (d\alpha)^{n-1}$  defines the positive orientation of  $M$  and  $(d\alpha)^{n-1}$  defines the positive orientation of  $\xi$ . The restriction of  $d\alpha$  to  $\xi$  is a symplectic form whose conformal class depends only on  $\xi$ ; we denote this conformal class by  $\text{CS}_\xi$ .

There exist many notions of symplectic (or holomorphic) fillings of a contact manifold  $(M, \xi)$ . They are motivated by the fact that many examples of contact manifolds appear as boundaries of complex or symplectic manifolds with suitable convexity properties, although not all contact manifolds appear in this way. We start with an informal definition. A *filling* of  $(M, \xi)$  is a compact symplectic manifold with boundary  $(W, \omega)$  such that the oriented boundary of  $W$  is identified with  $M$  and such that the symplectic form  $\omega$  is compatible with  $\xi$  near the boundary of  $W$ . Depending on the additional properties one asks on  $(W, \omega)$  and on the exact compatibility condition required between  $\omega$  and  $\xi$ , one obtains many notions of fillings: weak fillings, strong fillings, exact fillings, Stein fillings, and so on. We refer the reader to [6] for the beginning of the theory of fillings of contact manifolds and to [4], [5], [7], [8], [12], [13], and [19] for more recent references. Here, we will simply use the following definition, which is classical

when  $M$  has dimension 3 and due to Massot, Niederkrüger, and Wendl [12] in higher dimensions.

DEFINITION 1.1

We say that  $(W, \omega)$  is a *weak filling* of  $(M, \xi)$  if the restriction  $\omega_\xi$  of  $\omega$  to the distribution  $\xi$  is symplectic and if each element in the ray  $\omega_\xi + \text{CS}_\xi$  is symplectic.

When  $M$  has dimension 3, this definition reduces to the fact that  $\omega_\xi$  is everywhere nondegenerate and defines the positive orientation of  $\xi$ . We refer the reader to [12] for some motivations for this definition and for more history on fillability problems in contact geometry. Here, the main observation that we will need is the following.

*If  $M$  bounds a smooth strictly pseudoconvex domain  $W$  in a Kähler manifold  $(X, \omega)$ , then  $(W, \omega)$  is a weak filling of  $(M, \xi)$ , where  $\xi$  is the contact structure given by the complex tangencies.*

We shall use this observation to produce exotic examples of weak fillings. The idea of considering open sets in Kähler manifolds which are strictly pseudoconvex but not *symplectically convex* is actually not new. It was used by Eliashberg and Gromov 25 years ago to motivate the various definitions of convexity in complex and symplectic geometry (see [6, Section 3.1.3]).

Consider now a manifold  $L$  and its cotangent bundle  $T^*L$ , endowed with its Liouville form  $\lambda = pdq$ . Let  $Y$  be the Liouville vector field of  $T^*L$  whose flow is given by  $(q, p) \mapsto (q, e^t p)$ . The space  $ST^*L$  of rays of  $T^*L$  carries a canonical contact structure  $\xi_{st}$  which can be defined as follows. Identify  $ST^*L$  with a hypersurface of  $T^*L$  transverse to  $Y$ , and take the kernel of the restriction of the Liouville form to  $ST^*L$ . The resulting hyperplane field does not depend on this identification. Finally, we will denote by  $DT^*L$  the unit disk bundle inside  $T^*L$  for an auxiliary Riemannian metric on  $L$ . We assume that  $\dim L = n \geq 2$ . We will prove the following result.

THEOREM 1.2

*Let  $L$  be closed. Assume that  $L$  admits a totally real embedding into  $\mathbb{C}^n$ . Then there exists an exact symplectic form  $\Omega$  on  $W = DT^*L$  such that  $(W, \Omega)$  is a weak filling of  $(ST^*L, \xi_{st})$  and such that, for any closed Lagrangian submanifold  $N \subset W$ , the induced map  $H_1(N, \mathbb{R}) \rightarrow H_1(W, \mathbb{R})$  has a nontrivial kernel.*

This applies for instance when  $L$  is a torus or any oriented closed 3-manifold. See [1] for a discussion of which manifolds admit totally real embeddings into  $\mathbb{C}^n$  and for more examples.

When  $L = T^2$ , weak fillings of  $(ST^*T^2, \xi_{st})$  which are different from  $DT^*T^2$  with its standard symplectic structure were already known (see, e.g., [9]). However, the examples in [9] were not cohomologically exact, as opposed to the

examples provided by Theorem 1.2. We also mention that Wendl [18] proved that every *exact* filling  $(W, \omega)$  of  $(ST^*T^2, \xi_{st})$  is symplectomorphic to a star-shaped open set of  $T^*T^2$ . See [18] for the definition of exact fillings. In particular, exact fillings of  $(ST^*T^2, \xi_{st})$  are diffeomorphic to  $DT^*T^2 \simeq T^2 \times D^2$  and contain an incompressible Lagrangian torus.

In cases where it is known that there is a unique exact filling of  $(ST^*L, \xi_{st})$ , the weak filling  $(W, \Omega)$  appearing in Theorem 1.2 can be symplectically embedded in  $T^*L$  with its standard symplectic structure as a deformation of the unit disk bundle. But the image of such an embedding will never contain the zero section by our result. This embedding result follows from the fact that one can glue a product to the boundary of  $W$  to obtain a manifold

$$W' = W \cup ST^*L \times [0, 1]$$

such that  $W'$  carries a symplectic structure which extends  $\Omega$  and which makes  $W'$  an exact filling of  $(ST^*L, \xi_{st})$ . The proof is essentially folkloric and relies on an argument of Eliashberg [3, Proposition 3.1] (see also [12, Section 2] for more details).

Let us now discuss the proof of Theorem 1.2. The weak fillings that we will consider will be small tubular neighborhoods of totally real submanifolds of the complex Euclidean space. We thus consider the space  $\mathbb{C}^n$  endowed with its standard symplectic form:

$$\omega_0 = \frac{i}{2} \sum_{j=1}^n dz_j \wedge d\bar{z}_j.$$

Denote by  $J_0$  the complex structure on  $\mathbb{C}^n$ . Fix a totally real embedding

$$j : L \rightarrow \mathbb{C}^n$$

of a closed manifold  $L$ . We assume that  $\dim L = n \geq 2$ . If  $j$  happens to be Lagrangian, we can always perturb it to a non-Lagrangian embedding, keeping it totally real. So from now on we assume that  $j$  is *not* Lagrangian. Let  $f : \mathbb{C}^n \rightarrow \mathbb{R}_+$  be the square of the distance to  $j(L)$ , that is, for  $p \in \mathbb{C}^n$  let

$$f(p) = \inf_{q \in j(L)} d(p, q)^2,$$

where  $d$  is the Euclidean distance in  $\mathbb{C}^n$ . The function  $f$  is smooth and strictly plurisubharmonic near  $j(L)$  (see [2, Section 2.7]). Let

$$V_\varepsilon := \{p \in \mathbb{C}^n, f(p) \leq \varepsilon\}.$$

In the following we will say that a submanifold  $N$  of an ambient manifold  $W$  is  $H_1$ -embedded if the map  $H_1(N, \mathbb{R}) \rightarrow H_1(W, \mathbb{R})$  induced by the inclusion is injective. We have the following result.

**THEOREM 1.3**

*If  $\varepsilon$  is small enough, then the symplectic manifold  $(V_\varepsilon, \omega_0)$  does not contain any  $H_1$ -embedded closed Lagrangian submanifolds.*

This result is an immediate application or rather variation on a theorem by Laudenbach and Sikorav [11]. We will prove it in Section 2. To deduce Theorem 1.2 from Theorem 1.3, we simply make the following observations. Let  $\xi_\varepsilon$  denote the field of complex tangencies on the boundary  $M_\varepsilon$  of  $V_\varepsilon$ :

$$\xi_\varepsilon(p) = T_p M_\varepsilon \cap J_0 T_p M_\varepsilon.$$

Then, for small enough  $\varepsilon$ ,  $(M_\varepsilon, \xi_\varepsilon)$  is a contact manifold,  $(V_\varepsilon, \omega_0)$  is a weak filling of  $(M_\varepsilon, \xi_\varepsilon)$ , and  $M_\varepsilon$  and  $V_\varepsilon$  are, respectively, diffeomorphic to  $ST^*L$  and  $DT^*L$ . The only thing left to conclude the proof of Theorem 1.2 is to observe that  $(M_\varepsilon, \xi_\varepsilon)$  and  $(ST^*L, \xi_{st})$  are actually contactomorphic. This is certainly well known to experts, but we include a proof in Section 3, due to the lack of a convenient reference.

## 2. Displaceability and homologically essential Lagrangians

In the following, for each subset  $X \subset \mathbb{C}^n$  and each positive number  $\varepsilon$ , we will denote by  $V_\varepsilon(X)$  the set of points of  $\mathbb{C}^n$  at distance at most  $\sqrt{\varepsilon}$  from  $X$ .

We start by recalling the theorem by Laudenbach and Sikorav [11] alluded to earlier. Let  $P$  be a closed manifold. Then we have the following result.

### THEOREM 2.1 ([11])

*If a sequence of smooth Lagrangian embeddings  $\varphi_\ell : P \rightarrow \mathbb{C}^n$   $C^0$ -converges to a smooth embedding  $\varphi_\infty : P \rightarrow \mathbb{C}^n$ , then  $\varphi_\infty(P)$  is still Lagrangian.*

This can be seen as a generalization of the famous result by Eliashberg and Gromov about  $C^0$ -limits of symplectic diffeomorphisms; it also holds in more general symplectic manifolds under an additional hypothesis (see [11]). Now let  $\varphi_\ell$  and  $\varphi_\infty$  be embeddings as in the statement of the theorem. The  $C^0$ -convergence of  $\varphi_\ell$  to  $\varphi_\infty$  implies that, for  $\varepsilon$  small enough and for  $\ell$  large enough, the inclusion

$$\varphi_\ell(P) \hookrightarrow V_\varepsilon(\varphi_\infty(P))$$

induces an isomorphism on fundamental groups. In particular, it induces an injection on the first homology groups. We will see by closely inspecting Laudenbach and Sikorav's proof that this property, namely, the presence of  $H_1$ -embedded Lagrangians in arbitrarily small neighborhoods of  $\varphi_\infty(P)$  is sufficient to imply that  $\varphi_\infty(P)$  is itself Lagrangian. It is not even necessary to assume that these Lagrangians are diffeomorphic to  $P$ . In other words, we will prove the following.

### PROPOSITION 2.2

*Let  $P \subset \mathbb{C}^n$  be a closed submanifold of real dimension  $n$ . Assume that there exists a sequence  $(\varepsilon_k)$  of positive numbers converging to 0 such that, for all  $k$ ,  $V_{\varepsilon_k}(P)$  contains a closed  $H_1$ -embedded Lagrangian submanifold. Then  $P$  is Lagrangian.*

It is clear that this proposition implies Theorem 1.3 since the embedding  $j$  from the previous section was assumed to be totally real but not Lagrangian.

We now turn to the proof of Proposition 2.2. We follow the proof of Laudenbach and Sikorav and check that it still applies under our hypothesis. We will use the following result of Gromov [10] (see also [17, Proposition 1.2]).

*If a closed Lagrangian submanifold  $N \subset \mathbb{C}^n$  is contained in  $\mathbb{C}^{n-1} \times B(\alpha)$ , then there exists a nonconstant holomorphic disk with boundary on  $N$  of area smaller than  $\pi\alpha^2$ . In particular, there is a loop  $\gamma$  in  $N$  such that  $0 < \langle [\lambda_0], \gamma \rangle < \pi\alpha^2$  where  $[\lambda_0]$  is the Liouville class of  $N$ . Here  $B(\alpha)$  stands for the ball of radius  $\alpha$  in the complex plane.*

We argue by contradiction and assume that  $P$  is not Lagrangian. Exactly as in [11], we can assume that the normal bundle of  $P$  in  $\mathbb{C}^n$  has a nowhere-vanishing section (up to replacing  $P$  by  $P \times S^1$  in  $\mathbb{C}^{n+1}$ ). We can also assume that the sequence  $(\varepsilon_k)$  is decreasing. Let  $L_k$  be a closed  $H_1$ -embedded Lagrangian submanifold of  $V_{\varepsilon_k}(P)$ . By [11, Theorem 1] (see also [16]), there exists a Hamiltonian function  $H$  with flow  $\phi_H^t$  such that  $\phi_H^t(P) \cap P$  is empty for all  $t \in (0, \delta)$  for some positive number  $\delta$ . We fix a sequence  $t_\ell$  of positive real numbers converging to 0 and for each  $\ell$  we choose  $\varepsilon_{k_\ell} > 0$  such that

$$(2.i) \quad \phi_H^{t_\ell}(V_{\varepsilon_{k_\ell}}(P)) \cap V_{\varepsilon_{k_\ell}}(P) = \emptyset.$$

In particular, the isotopy  $(\phi_H^t)_{0 \leq t \leq t_\ell}$  displaces the Lagrangian submanifold  $L_{k_\ell}$ . By reparameterizing it, one can find a Hamiltonian isotopy  $(\varphi^t)_{0 \leq t \leq t_\ell}$  which still satisfies (2.i) and which has the additional property that it is constant equal to the identity (resp.,  $\varphi^{t_\ell}$ ) for  $t$  close to 0 (resp., close to  $t_\ell$ ). From this fact, one can construct a particular Lagrangian embedding of  $L_{k_\ell} \times \mathbb{S}^1$  in  $\mathbb{C}^{n+1}$ ; namely, we have the following result.

**PROPOSITION 2.3**

*Identify the circle  $\mathbb{S}^1$  with  $\mathbb{R}/2\mathbb{Z}$ . There exists a Lagrangian embedding  $\Phi_\ell : L_{k_\ell} \times \mathbb{S}^1 \rightarrow \mathbb{C}^n \times \mathbb{C}$  with the following properties.*

- (1) *The Liouville class of  $\Phi_\ell$  is of the form  $([\lambda_\ell], 0) \in H^1(L_{k_\ell} \times \mathbb{S}^1, \mathbb{R})$ .*
- (2) *If  $\underline{\pi}$  denotes the natural projection  $\mathbb{C}^n \times \mathbb{C} \rightarrow \mathbb{C}^n$ , then the map  $\underline{\pi} \circ \Phi_\ell : L_{k_\ell} \times \mathbb{S}^1 \rightarrow \mathbb{C}^n$  is given by  $\underline{\pi} \circ \Phi_\ell(x, t) = \varphi^{t-t_\ell}(x)$  for  $t \in [0, 1]$  and  $\underline{\pi} \circ \Phi_\ell(x, t) = \varphi^{(2-t) \cdot t_\ell}(x)$  for  $t \in [1, 2]$ .*
- (3) *The image of  $\Phi_\ell$  is contained in  $\mathbb{C}^n \times B(\alpha_\ell)$  where  $\alpha_\ell$  goes to 0 as  $\ell$  goes to infinity.*

This proposition is very classical; we refer the reader to [10, Section 2.3.B'\_3] or [15] for its proof. Exactly as in [11], we will now use Proposition 2.3 to finish the proof of Proposition 2.2. According to Gromov's result mentioned above, there exists a nonconstant holomorphic disk  $g^\ell = (g_1^\ell, g_2^\ell) : D \rightarrow \mathbb{C}^n \times \mathbb{C}$  with boundary contained in  $\Phi_\ell(L_{k_\ell} \times \mathbb{S}^1)$  of area at most  $\pi\alpha_\ell^2$ . Let  $p : L_{k_\ell} \times \mathbb{S}^1 \rightarrow L_{k_\ell}$  be the natural projection. The map

$$h_\ell := p \circ \Phi_\ell^{-1} \circ g^\ell : \partial D \rightarrow L_{k_\ell}$$

represents a nontrivial homology class by Proposition 2.3(1). Recall that the sequence  $(\varepsilon_k)$  is decreasing; hence, we can consider the map

$$i \circ h_\ell : \partial D \rightarrow V_{\varepsilon_1}(P),$$

where  $i : L_{k_\ell} \rightarrow V_{\varepsilon_1}(P)$  is the inclusion. It still represents a nontrivial homology class since  $L_{k_\ell}$  is  $H_1$ -embedded.

#### LEMMA 2.4

*The maps  $i \circ h_\ell$  and  $g_1^\ell$  considered as maps from  $\partial D$  to  $V_{\varepsilon_1}(P)$  are homotopic.*

*Proof*

Define  $v_\ell : \partial D \rightarrow L_{k_\ell} \times \mathbb{S}^1$  by  $v_\ell = \Phi_\ell^{-1} \circ g^\ell$ . We have  $i \circ h_\ell = i \circ p \circ v_\ell$  and  $g_1^\ell = \underline{\pi} \circ \Phi_\ell \circ v_\ell$ . Proposition 2.3(2) implies that the two maps

$$i \circ p, \underline{\pi} \circ \Phi_\ell : L_{k_\ell} \times \mathbb{S}^1 \rightarrow V_{\varepsilon_1}(P)$$

are homotopic; hence,  $i \circ h_\ell$  and  $g_1^\ell$  are homotopic.  $\square$

Hence,  $g_1^\ell(\partial D)$  represents a nontrivial homology class in  $H_1(V_{\varepsilon_1}(P), \mathbb{R})$ . However,  $g_1^\ell(\partial D)$  bounds the disk  $g_1^\ell(D)$  whose area is bounded above by the area of  $g^\ell(D)$ , which goes to 0 as  $\ell$  goes to infinity. This gives a contradiction with the following lemma from [11, p. 165] applied to  $A = V_{\varepsilon_1}(P)$ .

#### LEMMA 2.5

*Let  $A$  be a compact domain in  $\mathbb{C}^n$ . Then there exists  $\delta(A) > 0$  such that, for any smooth map  $u : D \rightarrow \mathbb{C}^n$  with boundary contained in  $A$  and such that  $[u(\partial D)] \neq 0$  in  $H_1(A, \mathbb{R})$ , one has*

$$\text{area}(D) \geq \delta(A).$$

This concludes the proof of Proposition 2.2 and thus of Theorem 1.3. As the reader will have noticed, we have only repeated the proof of Laudénbach and Sikorav [11].

### 3. Identification of the contact structure

Recall from the introduction that  $M_\varepsilon$  is the boundary of the  $\varepsilon$ -tubular neighborhood of a totally real submanifold  $j(L)$  of  $\mathbb{C}^n$ , endowed with its canonical contact structure  $\xi_\varepsilon$ . We prove here the following result.

#### PROPOSITION 3.1

*The manifold  $(M_\varepsilon, \xi_\varepsilon)$  is contactomorphic to  $(ST^*L, \xi_{st})$ .*

In what follows, we will identify  $L$  and  $j(L)$  and will not mention the map  $j$  anymore. Hence, we will think of  $TL$  as a subbundle of  $L \times \mathbb{C}^n$ .

First, we will have to deal with the fact that the normal bundle  $NL$  of  $L$  in  $\mathbb{C}^n$  need not coincide with the image under  $J_0$  of its tangent bundle; indeed,  $L$

is not assumed to be Lagrangian. For this, we choose once and for all a smooth map

$$\phi : TL \times [0, 1] \rightarrow L \times \mathbb{C}^n$$

with the following properties:

- the map  $\phi$  is of the form  $\phi(x, v, t) = (x, A(x, v, t))$ ,
- for each  $t$ ,  $\phi_t := \phi(\cdot, \cdot, t)$  is an isomorphism onto a rank  $n$  subbundle of  $L \times \mathbb{C}^n$  which is transverse to  $TL$ ,
- $\phi_0(TL) = NL$ ,
- $\phi_1 = J_0$  and hence  $\phi_1(TL) = J_0TL$ ,
- the metric on  $TL$  induced by the embedding  $\phi_t$  and by the Euclidean metric on  $\mathbb{C}^n$  does not depend on  $t$ ; it is denoted by  $|\cdot|$ .

It is easy to construct a map  $\phi$  satisfying the first four properties, and one can then always achieve the last one by composing with a bundle automorphism  $TL \rightarrow TL$  depending on  $t$ . In the following we will denote by  $D_\delta(TL)$  the open disk bundle of radius  $\delta$  in  $TL$  for the metric  $|\cdot|$  induced by  $\phi$ . Consider now the maps

$$\theta_t : TL \rightarrow \mathbb{C}^n \quad (t \in [0, 1])$$

defined by  $\theta_t(x, v) = j(x) + A(x, v, t)$ . There exists  $\delta_1 > 0$  such that, for each  $t$  in  $[0, 1]$ , the map  $\theta_t$  is injective and a local diffeomorphism on  $D_{\delta_1}(TL)$ . The open set

$$\theta_t(D_{\delta_1}(TL)) \subset \mathbb{C}^n$$

is a tubular neighborhood of  $L$  modeled on a varying subbundle of  $L \times \mathbb{C}^n$ , which is always transverse to  $TL$ . Let  $J^t$  be the almost-complex structure on  $D_{\delta_1}(TL)$  which is the pullback of  $J_0$  by  $\theta_t$ . Now recall the following formula from [2, Section 2.2]. If  $\varphi$  is a function defined on an open set  $V$  of  $\mathbb{C}^n$ , if  $x$  is a point of  $V$ , and if  $u \in \mathbb{C}^n$ , then we have

$$-dd^{\mathbb{C}}\varphi_x(u, J_0u) = \text{Hess}_{\varphi,x}(u) + \text{Hess}_{\varphi,x}(J_0u),$$

where  $\text{Hess}_{\varphi,x}$  is the Hessian of  $\varphi$  at  $x$  and  $d^{\mathbb{C}}\varphi = d\varphi \circ J_0$ . Let  $h : D_{\delta_1}(TL) \rightarrow \mathbb{R}_+$  be the function  $h(x, v) = |v|^2$ . By applying the above formula to the functions  $h \circ \theta_t^{-1}$  for  $x \in L$  one finds

$$-dd_{J^t}^{\mathbb{C}}h > 0 \quad (t \in [0, 1])$$

along the zero section  $L \subset D_{\delta_1}(TL)$ . Here we have used the notation

$$d_{J^t}^{\mathbb{C}}h = dh \circ J^t.$$

This implies that there exists a positive number  $\delta_2 < \delta_1$  such that

$$-dd_{J^t}^{\mathbb{C}}h > 0$$

on all of  $D_{\delta_2}(TL)$  for all  $t$  in  $[0, 1]$ . Fix any number  $\varepsilon \in (0, \delta_2^2)$ , and let

$$S_\varepsilon = \{h = \varepsilon\} \subset D_{\delta_2}(TL).$$

This is the sphere bundle of radius  $\sqrt{\varepsilon}$ . Let  $\xi(t) = TS_\varepsilon \cap J^t TS_\varepsilon$  be the field of complex tangencies for  $J^t$  in  $S_\varepsilon$ . All the  $\xi(t)$ 's are contact structures since  $h$  is  $J^t$ -convex on  $D_{\delta_2}(TL)$ . Note that  $\xi(0) = \xi_\varepsilon$  is the contact structure that we want to identify with the canonical contact structure on  $ST^*L$ . By Gray's theorem, all the  $\xi(t)$ 's are isomorphic; hence, it is enough to prove that  $(S_\varepsilon, \xi(1))$  is contactomorphic to  $(ST^*L, \xi_{st})$ .

Recall now that the Riemannian metric  $|\cdot|$  on  $L$  induces a decomposition of the tangent bundle of  $TL$  into horizontal and vertical subbundles. At each point  $(x, v) \in TL$  one has a decomposition

$$T_{(x,v)}TL = H(x, v) \oplus V(x, v),$$

where  $V(x, v)$  is the tangent space to the fiber of the projection  $\pi : TTL \rightarrow L$  at  $(x, v)$ , and  $H(x, v)$  is the horizontal subspace defined by the Levi-Civita connection of  $|\cdot|$  (see [14, Section 1.3]). Both  $V(x, v)$  and  $H(x, v)$  are canonically identified with  $T_xL$ . Hence, one has an identification

$$T_{(x,v)}TL \simeq T_xL \times T_xL,$$

where the first factor is horizontal and the second vertical. Let  $J^*$  be the almost-complex structure on  $TL$  defined by  $(u, v) \mapsto (-v, u)$  under the previous identification (see [14, Section 1.3.2] for more details). Let  $\xi^*$  be the field of complex tangencies for  $J^*$  on  $S_\varepsilon$ . Then  $(S_\varepsilon, \xi^*)$  is contactomorphic to  $(ST^*L, \xi_{st})$  (see [14]).

We now want to find a path of almost-complex structures from  $J^*$  to  $J^1$  to relate the contact structures  $\xi^*$  and  $\xi(1)$ . But along the zero section both  $J^1$  and  $J^*$  interchange the vertical and horizontal subbundles, and actually the hypothesis  $\phi_1 = J_0$  made earlier implies that  $J^* = J^1$  along the zero section. Since  $J^*$  is tamed by a symplectic form on  $TL$ , this implies that  $J^1$  and  $J^*$  are tamed by a common symplectic form, say,  $\Omega$ , on  $D_{\delta_3}(TL)$  for  $\delta_3 \in (0, \delta_2)$  small enough. Now we can find a path  $J(t)$  of almost-complex structures on  $D_{\delta_3}(TL)$  such that  $J(0) = J^*$  and  $J(1) = J^1$  and such that  $J(t)$  does not depend on  $t$  along the zero section. Since

$$-dd_{J(t)}^{\mathbb{C}}h > 0$$

along the zero section for all  $t$ , we can once again shrink our neighborhood and conclude that

$$-dd_{J(t)}^{\mathbb{C}}h > 0$$

for all  $t$  on  $D_{\delta_4}(TL)$  for some  $\delta_4 \in (0, \delta_3)$ . Appealing to Gray's theorem again and shrinking  $\varepsilon$  if necessary, we conclude that  $(S_\varepsilon, \xi(1))$  and  $(S_\varepsilon, \xi^*)$  are contactomorphic. This concludes the proof of Proposition 3.1.

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## References

- [1] M. Audin, *Fibrés normaux d'immersions en dimension double, points doubles d'immersions lagrangiennes et plongements totalement réels*, Comment. Math. Helv. **63** (1988), 593–623. MR 0966952. DOI 10.1007/BF02566781.
- [2] K. Cieliebak and Y. Eliashberg, *From Stein to Weinstein and back: Symplectic geometry of affine complex manifolds*, Amer. Math. Soc. Colloq. Publ. **59**, Amer. Math. Soc., Providence, 2012. MR 3012475. DOI 10.1090/coll/059.
- [3] Y. Eliashberg, *On symplectic manifolds with some contact properties*, J. Differential Geom. **33** (1991), 233–238. MR 1085141.
- [4] ———, *Unique holomorphically fillable contact structure on the 3-torus*, Int. Math. Res. Not. IMRN **1996**, no. 2, 77–82. MR 1383953. DOI 10.1155/S1073792896000074.
- [5] ———, *A few remarks about symplectic fillings*, Geom. Topol. **8** (2004), 277–293. MR 2023279. DOI 10.2140/gt.2004.8.277.
- [6] Y. Eliashberg and M. Gromov, “Convex symplectic manifolds” in *Several Complex Variables and Complex Geometry, Part 2 (Santa Cruz, CA, 1989)*, Proc. Sympos. Pure Math. **52**, Amer. Math. Soc., Providence, 1991, 135–162. MR 1128541. DOI 10.1090/pspum/052.2/1128541.
- [7] J. B. Etnyre and K. Honda, *Tight contact structures with no symplectic fillings*, Invent. Math. **148** (2002), 609–626. MR 1908061. DOI 10.1007/s002220100204.
- [8] D. T. Gay, *Four-dimensional symplectic cobordisms containing three-handles*, Geom. Topol. **10** (2006), 1749–1759. MR 2284049. DOI 10.2140/gt.2006.10.1749.
- [9] E. Giroux, *Une structure de contact, même tendue, est plus ou moins tordue*, Ann. Sci. Éc. Norm. Supér. (4) **27** (1994), 697–705. MR 1307678.
- [10] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), 307–347. MR 0809718. DOI 10.1007/BF01388806.
- [11] F. Laudenbach and J.-C. Sikorav, *Hamiltonian disjunction and limits of Lagrangian submanifolds*, Int. Math. Res. Not. IMRN **1994**, no. 4, 161–168. MR 1266111. DOI 10.1155/S1073792894000176.
- [12] P. Massot, K. Niederkrüger, and C. Wendl, *Weak and strong fillability of higher dimensional contact manifolds*, Invent. Math. **192** (2013), 287–373. MR 3044125. DOI 10.1007/s00222-012-0412-5.
- [13] K. Niederkrüger and C. Wendl, *Weak symplectic fillings and holomorphic curves*, Ann. Sci. Éc. Norm. Supér. (4) **44** (2011), 801–853. MR 2931519.
- [14] G. P. Paternain, *Geodesic Flows*, Progr. Math. **180**, Birkhäuser Boston, Boston, 1999. MR 1712465. DOI 10.1007/978-1-4612-1600-1.
- [15] L. Polterovich, *Symplectic displacement energy for Lagrangian submanifolds*, Ergodic Theory Dynam. Systems **13** (1993), 357–367. MR 1235478. DOI 10.1017/S0143385700007410.
- [16] ———, “An obstacle to non-Lagrangian intersections,” in *The Floer Memorial Volume*, Progr. Math. **133**, Birkhäuser, Basel, 1995, 575–586. MR 1362842.

- [17] J.-C. Sikorav, *Quelques propriétés des plongements lagrangiens*, Mém. Soc. Math. France (N.S.) **46** (1991), 151–167. [MR 1125841](#).
- [18] C. Wendl, *Strongly fillable contact manifolds and J-holomorphic foliations*, Duke Math. J. **151** (2010), 337–384. [MR 2605865](#).  
[DOI 10.1215/00127094-2010-001](#).
- [19] ———, *A hierarchy of local symplectic filling obstructions for contact 3-manifolds*, Duke Math. J. **162** (2013), 2197–2283. [MR 3102479](#).  
[DOI 10.1215/00127094-2348333](#).

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