On an invariance property of the space of smooth vectors

Karl-Hermann Neeb, Hadi Salmasian, and Christoph Zellner

Abstract Let $(\pi, \mathcal{H})$ be a continuous unitary representation of the (infinite-dimensional) Lie group G, and let $\gamma\colon \mathbb{R} \to \mathrm{Aut}(G)$ be a group homomorphism which defines a continuous action of $\mathbb{R}$ on G by Lie group automorphisms. Let $\pi^{\#}(g,t) = \pi(g)U_t$ be a continuous unitary representation of the semidirect product group $G \rtimes_\gamma \mathbb{R}$ on $\mathcal{H}$. The first main theorem of the present note provides criteria for the invariance of the space $\mathcal{H}^\infty$ of smooth vectors of π under the operators $U_f = \int_{\mathbb{R}} f(t)U_t\,dt$ for $f \in L^1(\mathbb{R})$ and $f \in \mathcal{S}(\mathbb{R})$, respectively. When $\mathfrak{g}$ is complete and the actions of $\mathbb{R}$ on G and $\mathfrak{g}$ are continuous, we use the above theorem to show that, for suitably defined spectral subspaces $\mathfrak{g}_{\mathbb{C}}(E)$, $E \subseteq \mathbb{R}$, in the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and $\mathcal{H}^\infty(F)$, $F \subseteq \mathbb{R}$, for U_t in $\mathcal{H}^\infty$, we have

$$\mathrm{d}\pi\big(\mathfrak{g}_{\mathbb{C}}(E)\big)\mathcal{H}^\infty(F) \subseteq \mathcal{H}^\infty(E+F).$$

1. Introduction

For a complex Lie algebra $\mathfrak{g}$ with a root decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$ and the corresponding $\mathfrak{h}$-weight spaces V_β in a $\mathfrak{g}$-module, one has the elementary relation

$$\mathfrak{g}_\alpha.V_\beta \subseteq V_{\beta+\alpha},$$

which is of central importance in understanding the structure of the action of $\mathfrak{g}$ on V (see [Hu], [B1]). The main point of the present note is to provide a generalization of this relation to unitary representations of infinite-dimensional Lie groups. The results of this note are used in our forthcoming articles [NS2] and [MN].

To make our results as flexible as possible, we consider the following setting. Let G be a locally convex Lie group with Lie algebra $\mathfrak{g}$ and a smooth exponential map $\exp_G : \mathfrak{g} \to G$ denoted by $e^x := \exp_G(x)$ (see [N2]). We denote the group of smooth automorphisms of G by $\mathrm{Aut}(G)$. We further consider a one-parameter group $\gamma\colon \mathbb{R} \to \mathrm{Aut}(G)$, $t \mapsto \gamma_t$ defining a continuous action of $\mathbb{R}$ on G. Then

Kyoto Journal of Mathematics, Vol. 55, No. 3 (2015), 501–515
DOI 10.1215/21562261-3089019, © 2015 by Kyoto University
Received January 6, 2014. Revised April 24, 2014. Accepted April 25, 2014.
2010 Mathematics Subject Classification: 22E65, 22E45, 17B65.
Salmasian's work supported by a Natural Sciences and Engineering Research Council Discovery Grant.
Neeb and Zellner's work supported by Deutsche Forschungsgemeinschaft grant NE 413/7-2 in the framework of Schwerpunktprogramme "Representation Theory."

the semidirect product $G \rtimes_\gamma \mathbb{R}$ is a topological group whose continuous unitary representations $(\pi^\#, \mathcal{H})$ can be written as $\pi^\#(g,t) = \pi(g)U_t$, where $(\pi, \mathcal{H})$ is a unitary representation of G and $(U_t)_{t\in\mathbb{R}}$ a continuous unitary one-parameter group satisfying

$$U_t \pi(g) U_t^* = \pi\big(\gamma_t(g)\big) \quad \text{for } t \in \mathbb{R}, g \in G.$$

For $f \in L^1(\mathbb{R})$, we then obtain a bounded operator $U_f = \int_{\mathbb{R}} f(t)U_t\, dt \in B(\mathcal{H})$. We call $v \in \mathcal{H}$ *smooth* if the orbit map $\pi^v\colon G \to \mathcal{H}, g \mapsto \pi(g)v$ is smooth and write $\mathcal{H}^\infty$ for the subspace of smooth vectors. Then

$$\mathrm{d}\pi(x)v := \frac{d}{dt}\Big|_{t=0} \pi(e^{tx})v, \quad \text{for } v \in \mathcal{H}^\infty, x \in \mathfrak{g},$$

defines by complex linear extension a representation $\mathrm{d}\pi\colon \mathfrak{g}_\mathbb{C} \to \mathrm{End}(\mathcal{H}^\infty)$.

Let $\mathrm{Aut}(\mathfrak{g})$ denote the group of continuous automorphisms of $\mathfrak{g}$. Our first main result asserts that if $(\mathbf{L}(\gamma_t))_{t\in\mathbb{R}}$ is equicontinuous in $\mathrm{Aut}(\mathfrak{g})$, then $\mathcal{H}^\infty$ is invariant under the operators U_f, $f \in L^1(\mathbb{R})$. Under the weaker assumption that $(\mathbf{L}(\gamma_t))_{t\in\mathbb{R}}$ is polynomially bounded, we still have the invariance under U_f, $f \in \mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ denotes the space of Schwartz functions. The main point of this result is that it permits us to localize the U-spectrum within the space of smooth vectors because the U-spectrum of a vector of the form $U_f v$ is contained in $\mathrm{supp}(\widehat{f})$ (see Theorem 2.3).

To turn this into an effective tool to analyze positive energy representations, that is, representations where $\mathrm{Spec}(U)$ is bounded from below, we need to know how the U-spectrum of an element v changes when we apply elements of $\mathfrak{g}_\mathbb{C}$. This is clarified by Theorem 3.1, where we show that, when $\mathfrak{g}$ is complete and the action of $\mathbb{R}$ on $\mathfrak{g}$ is continuous, for suitably defined spectral subspaces $\mathfrak{g}_\mathbb{C}(E)$, $E \subseteq \mathbb{R}$, and $\mathcal{H}^\infty(F) := \mathcal{H}(F) \cap \mathcal{H}^\infty$, $F \subseteq \mathbb{R}$, corresponding to U in $\mathcal{H}$, we have

$$\mathrm{d}\pi\big(\mathfrak{g}_\mathbb{C}(E)\big)\mathcal{H}^\infty(F) \subseteq \mathcal{H}^\infty(E+F).$$

Note that all this applies, in particular, to the special case where $\gamma_t(g) = e^{tx}ge^{-tx}$ for $x \in \mathfrak{g}$, provided that the one-parameter group $\mathrm{Ad}(e^{tx})$ is equicontinuous (resp., polynomially bounded). In this context the results of the present paper are used in the forthcoming articles [NS2] and [MN].

2. The invariance theorem

We prepare the proof of Theorem 2.3 with the following lemma. For $U \subset G$ open and $h : U \to \mathbb{C}$ a smooth map, we define the derivative of h along a left-invariant vector field by

$$L_x h : U \to \mathbb{C}, \qquad L_x h(g) := \lim_{s\to 0} \frac{1}{s}\big(h(ge^{sx}) - h(g)\big)$$

for $x \in \mathfrak{g}, g \in U$. We refer to [N2] for the basic facts and definitions concerning calculus in locally convex spaces and the corresponding manifold and Lie group concepts (see also [Ha] and [N1]).

LEMMA 2.1

Let $K \in \mathbb{N}$, let $W \subset G$ be open, and let $\Phi : W \to \mathfrak{g}$ be a chart. Let $F : \mathbb{R} \times W \to \mathbb{C}$ satisfy the following properties.

(a) *The map $F_t \colon W \to \mathbb{C}, F_t(g) := F(t,g)$ is in $C^\infty(W,\mathbb{C})$ for every fixed $t \in \mathbb{R}$.*

(b) *For every $g_\circ \in W$ and every $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ satisfying $k \le K$, there exist an open $g_\circ$-neighborhood $U_{g_\circ,k} \subset W$ and an open $\mathbf{0}$-neighborhood $V_{g_\circ,k} \subseteq \mathfrak{g}$ such that*

$$\sup\big\{\big|L_{x_1}\cdots L_{x_k}F_t(g)\big| : g \in U_{g_\circ,k}, x_1,\dots,x_k \in V_{g_\circ,k}, t \in \mathbb{R}\big\} < \infty. \tag{1}$$

Then, for every $g_\circ \in W$ and every $k \in \mathbb{N}_0$ with $k \le K$, there exist an open $g_\circ$-neighborhood $\widetilde{U}_{g_\circ,k} \subset W$ and an open $\mathbf{0}$-neighborhood $\widetilde{V}_{g_\circ,k} \subseteq \mathfrak{g}$ such that

$$\sup\big\{\big|\mathrm{d}^k\widetilde{F}_t(u)(x_1,\dots,x_k)\big| : u \in \Phi(\widetilde{U}_{g_\circ,k}), x_1,\dots,x_k \in \widetilde{V}_{g_\circ,k}, t \in \mathbb{R}\big\} < \infty,$$

where $\widetilde{F}_t := F_t \circ \Phi^{-1}$.

Proof

Let $g_\circ \in W$, and set $\ell_{g_\circ}(g) := g_\circ g$ for every $g \in G$. The operators L_x satisfy the relation $L_x(F_t \circ \ell_{g_\circ}) = L_x(F_t) \circ \ell_{g_\circ}$. Thus (after replacing F_t by $F_t \circ \ell_{g_\circ}$, W by $g_\circ^{-1}(W)$, and Φ by $\Phi \circ \ell_{g_\circ}$) we may assume without loss of generality that $g_\circ = \mathbf{1}$. Moreover, we may assume that $\Phi(\mathbf{1}) = \mathbf{0}$. Let $V := \Phi(W) \subset \mathfrak{g}$. Replacing F_t by $F_t \circ \Phi^{-1} = \widetilde{F}_t$, we can assume that F_t is defined on the open $\mathbf{0}$-neighborhood $V \subset \mathfrak{g}$. We will consider V as a local Lie group with the multiplication induced from G.

Step 1. Choose $U \subset V$ open such that $U = U^{-1}$, $\mathbf{0} \in U$, and $UU \subset U$. Our goal is to prove (by induction on k) that, for every $k \in \mathbb{N}$ with $k \le K$ and every $u \in U$, there exist a u-neighborhood $U_{u,k} \subset U$ and a $\mathbf{0}$-neighborhood $V_{u,k} \subset \mathfrak{g}$ such that

$$\sup\big\{\big|\mathrm{d}^k F_t(u')(x_1,\dots,x_k)\big| : u' \in U_{u,k}, x_1,\dots,x_k \in V_{u,k}, t \in \mathbb{R}\big\} < \infty.$$

Then the special case $u = \mathbf{0}$ yields the assertion of the lemma for g_0.

Step 2. Fix $u \in U$, and fix $k \in \mathbb{N}$ with $k \le K$. By [NS1, Lemma 2.2.1], we have

$$\frac{\partial^k}{\partial t_1 \cdots \partial t_k} F_t(g e^{t_1x_1+\cdots+t_kx_k})\Big|_{t_1=\cdots=t_k=0} = \frac{1}{k!}\sum_{\sigma\in S_k} L_{x_{\sigma(1)}}\cdots L_{x_{\sigma(k)}}F_t(g) \tag{2}$$

for every $g \in U$. From (2) and (1) it follows that there exist open sets $u \in U_{u,k}^{(1)} \subset U$ and $\mathbf{0} \in V_{u,k}^{(1)} \subset \mathfrak{g}$ such that

$$\sup\Big\{\big|h_t(g,x_1,\dots,x_k)\big| : g \in U_{u,k}^{(1)}, x_1,\dots,x_k \in V_{u,k}^{(1)}, t \in \mathbb{R}\Big\} < \infty, \tag{3}$$

where

$$h_t(g,x_1,\dots,x_k) := \frac{\partial^k}{\partial t_1 \cdots \partial t_k} F_t(g e^{t_1x_1+\cdots+t_kx_k})\Big|_{t_1=\cdots=t_k=0}.$$

Next we use [NS1, Lemma 2.1.3] on the left-hand side of (2) to write

$$\begin{aligned}(4)\qquad &\frac{\partial^k}{\partial t_1\cdots\partial t_k}F_t(ge^{t_1x_1+\cdots+t_kx_k})\Big|_{t_1=\cdots=t_k=0}\\ &\quad=\sum_{\{A_1,\dots,A_m\}\in\mathcal{P}_k}\mathrm{d}^mF_t(g)\big(v_{|A_1|}(g,x_{A_1}),\dots,v_{|A_m|}(g,x_{A_m})\big),\end{aligned}$$

where $\mathcal{P}_k$ is the set of partitions of $\{1,\dots,k\}$, and for every set $A=\{a_1,\dots,a_p\}\subseteq\{1,\dots,k\}$, we define $x_A:=(x_{a_1},\dots,x_{a_p})$ and

$$v_p:U\times\mathfrak{g}^p\to\mathfrak{g},\quad v_p(g,x_A):=\frac{\partial^p}{\partial t_{a_1}\cdots\partial t_{a_p}}(ge^{t_{a_1}x_{a_1}+\cdots+t_{a_p}x_{a_p}})\Big|_{t_{a_1}=\cdots=t_{a_p}=0}.$$

The term on the right-hand side of (4) corresponding to the partition $\{\{1\},\dots,\{k\}\}$ is

$$(5)\qquad \mathrm{d}^kF_t(g)\big(v_1(g,x_1),\dots,v_1(g,x_k)\big),$$

and the remaining terms are partial derivatives of order strictly less than k. Let us denote the sum of these remaining terms by $A_t(g,x_1,\dots,x_k)$. Since the maps $v_p(\cdot,\cdot)$ are smooth, we can assume (by the induction hypothesis) that there exist open sets $u\in U^{(2)}_{u,k}\subset U^{(1)}_{u,k}$ and $\mathbf{0}\in V^{(2)}_{u,k}\subset V^{(1)}_{u,k}$ and a constant $M>0$ (depending only on F) such that

$$\big|A_t(g,x_1,\dots,x_k)\big|<M\quad\text{for every } g\in U^{(2)}_{u,k},x_1,\dots,x_k\in V^{(2)}_{u,k},\text{ and } t\in\mathbb{R}.$$

Thus, given the upper bound (3) and the expression (5) for the first term in the summation, to complete the proof of the claim in Step 1, it suffices to prove the following statement: there exist open sets $u\subset U'\subset U^{(2)}_{u,k}$ and $\mathbf{0}\in V'\subset\mathfrak{g}$ such that, for every $g\in U'$ and every $y\in V'$, the equation

$$(6)\qquad v_1(g,x)=y$$

has a solution $x\in V^{(2)}_{u,k}$.

Next we prove the latter statement. First note that $v_1(g,x)=\mathrm{d}\ell_g(\mathbf{0})(x)$, where $\ell_g(h)=gh$, and the chain rule implies that the solution to (6) is given by $x=\mathrm{d}\ell_{g^{-1}}(g)(y)$. From the smoothness (in fact only continuity) of the map

$$\varphi:U\times\mathfrak{g}\to\mathfrak{g},\varphi(g,y):=\mathrm{d}\ell_{g^{-1}}(g)(y)$$

and the relation $\varphi(u,\mathbf{0})=\mathbf{0}$ it follows that there exist U' and V' such that $\varphi(U'\times V')\subset V^{(2)}_{u,k}$. □

DEFINITION 2.2

Let E be a locally convex space, and let $\alpha:\mathbb{R}\to\mathrm{GL}(E)$, $t\mapsto\alpha_t$ be a group homomorphism. Then α is called

(a) *equicontinuous* if the subset $\{\alpha_t:t\in\mathbb{R}\}\subset\mathrm{End}(E)$ is equicontinuous (cf. Definition A.1);

(b) *polynomially bounded* if for every continuous seminorm p on E there exists an $N\in\mathbb{N}_0$ such that $\{(1+|t|^N)^{-1}\alpha_t:t\in\mathbb{R}\}$ is an equicontinuous subset

of $\mathrm{Hom}(E,(E,p))$, where (E,p) denotes E endowed with the topology defined by the single seminorm p.

THEOREM 2.3 (ZELLNER'S INVARIANCE THEOREM)
Let $\gamma:\mathbb{R}\to\mathrm{Aut}(G)$ be a one-parameter group, and let $\alpha:\mathbb{R}\to\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$ be defined by $\alpha_t:=\mathbf{L}(\gamma_t)_{\mathbb{C}}\in\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$ for $t\in\mathbb{R}$. Assume that γ defines a continuous action of $\mathbb{R}$ on G. Let $\pi^{\#}:G\rtimes_{\gamma}\mathbb{R}\to U(\mathcal{H})$, $(g,t)\mapsto\pi(g)U_t$ be a continuous unitary representation, and let $\mathcal{H}^{\infty}$ be the space of smooth vectors with respect to π. For $f\in L^1(\mathbb{R})$, let $U_f=\int_{\mathbb{R}}f(t)U_t\,dt\in B(\mathcal{H})$. Assume that at least one of the following conditions hold:

(a) *α is equicontinuous and $f\in L^1(\mathbb{R})$, or*
(b) *α is polynomially bounded and $f\in\mathcal{S}(\mathbb{R})$.*

Then $U_f\mathcal{H}^{\infty}\subseteq\mathcal{H}^{\infty}$ and

$$\mathrm{d}\pi(y_1)\cdots\mathrm{d}\pi(y_n)U_fv=\int_{\mathbb{R}}f(t)U_t\mathrm{d}\pi\big(\alpha_{-t}(y_1)\big)\cdots\mathrm{d}\pi\big(\alpha_{-t}(y_n)\big)v\,dt \tag{7}$$

for $y_1,\dots,y_n\in\mathfrak{g}_{\mathbb{C}}$ and $v\in\mathcal{H}^{\infty}$.

Proof
Let $v\in\mathcal{H}^{\infty}$, let $w\in\mathcal{H}$, and consider

$$F:\mathbb{R}\times G\to\mathbb{C},\quad (t,g)\mapsto\big\langle\pi(g)U_tv,w\big\rangle.$$

We set $F_t(g):=F(t,g)$. Since $\pi(g)U_t=\pi^{\#}(g,t)=U_t\pi(\gamma_{-t}g)$ and $v\in\mathcal{H}^{\infty}$, we conclude that $U_t\mathcal{H}^{\infty}\subseteq\mathcal{H}^{\infty}$ and $F_t\in C^{\infty}(G)$. Note that

$$\begin{aligned}L_{x_1}\cdots L_{x_k}F_t(g)&=\big\langle\pi(g)\mathrm{d}\pi(x_1)\cdots\mathrm{d}\pi(x_k)U_tv,w\big\rangle\\&=\big\langle\pi(g)U_t\mathrm{d}\pi\big(\alpha_{-t}(x_1)\big)\cdots\mathrm{d}\pi\big(\alpha_{-t}(x_k)\big)v,w\big\rangle\end{aligned}$$

for $x_1,\dots,x_k\in\mathfrak{g}$. Since $v\in\mathcal{H}^{\infty}$, the k-linear map

$$\mathfrak{g}^k\to\mathcal{H},\quad (x_1,\dots,x_k)\mapsto\mathrm{d}\pi(x_1)\cdots\mathrm{d}\pi(x_k)v$$

is continuous. From Proposition A.3 we thus obtain for every $k\in\mathbb{N}$ a continuous seminorm p_k on $\mathfrak{g}$ such that

$$\big\|\mathrm{d}\pi(x_1)\cdots\mathrm{d}\pi(x_k)v\big\|\le p_k(x_1)\cdots p_k(x_k)\quad\text{for all } x_1,\dots,x_k\in\mathfrak{g}.$$

We conclude that

$$\big|L_{x_1}\cdots L_{x_k}F_t(g)\big|\le p_k\big(\alpha_{-t}(x_1)\big)\cdots p_k\big(\alpha_{-t}(x_k)\big)\cdot\|w\|. \tag{8}$$

(a) Now assume first that α is equicontinuous and $f\in L^1(\mathbb{R})$. By Proposition A.2 we find for every $k\in\mathbb{N}$ a continuous seminorm q_k on $\mathfrak{g}$ such that $p_k(\alpha_{-t}(x))\le q_k(x)$ holds for all $t\in\mathbb{R}$, $x\in\mathfrak{g}$. Let $U_k:=\{x\in\mathfrak{g}:q_k(x)<1\}$. Then we obtain from (8) that

$$\sup\big\{|L_{x_1}\cdots L_{x_k}F_t(g)|:g\in G,x_1,\dots,x_k\in U_k,t\in\mathbb{R}\big\}\le\|w\|<\infty. \tag{9}$$

Let $g_0\in G$, and choose a chart $\Phi:W\to\mathfrak{g}$ with $W\subset G$ an open neighborhood of g_0. Now (9) implies that $F|_{\mathbb{R}\times W}$ satisfies the assumptions of Lemma 2.1. Thus, for

every $u_0 \in \Phi(W)$ and $k \in \mathbb{N}$, there exist an open u_0-neighborhood $U_{u_0,k} \subset \Phi(W)$ and an open 0-neighborhood $V_{u_0,k} \subset \mathfrak{g}$ such that

$$\sup\big\{\big|\mathrm{d}^k \widetilde{F}_t(u)(x_1,\ldots,x_k)\big| : u \in U_{u_0,k}, x_1,\ldots,x_k \in V_{u_0,k}, t \in \mathbb{R}\big\} < \infty,$$

where $\widetilde{F}_t := F_t \circ \Phi^{-1}$. Since $f \in L^1(\mathbb{R},\mathbb{C})$, Lemma A.4 yields that the map

$$\Phi(W) \to \mathbb{C}, \quad u \mapsto \int_{\mathbb{R}} f(t)\widetilde{F}_t(u)\,dt = \int f(t)\big\langle \pi\big(\Phi^{-1}(u)\big)U_t v, w\big\rangle\,dt$$

is smooth. We conclude that

$$G \to \mathbb{C}, \quad g \mapsto \big\langle \pi(g)\pi(f)v, w\big\rangle = \int_{\mathbb{R}} f(t)\big\langle \pi(g)U_t v, w\big\rangle\,dt$$

is smooth for every $w \in \mathcal{H}$. With $w = \pi(f)v$ we now obtain from [N3, Theorem 7.2] that $\pi(f)v \in \mathcal{H}^\infty$. Finally, (7) follows from the corresponding relation for the functions F_t. This proves (a).

(b) Now assume that α is polynomially bounded, and assume that $f \in \mathcal{S}(\mathbb{R})$. Then there exists for every $k \in \mathbb{N}$ a continuous seminorm q'_k on $\mathfrak{g}$ and $N_k \in \mathbb{N}_0$ such that

$$p_k\big(\alpha_t(x)\big) \le \big(1 + |t|^{N_k}\big)q'_k(x) \quad \text{for all } x \in \mathfrak{g}, t \in \mathbb{R}.$$

From (8) we thus obtain that

$$\big|L_{x_1}\cdots L_{x_k}F_t(g)\big| \le \big(1 + |t|^{N_k}\big)^k q'_k(x_1)\cdots q'_k(x_k)\cdot\|w\|. \tag{10}$$

Let $g_0 \in G$, and choose a chart $\Phi : W \to \mathfrak{g}$ with $W \subset G$ an open neighborhood of g_0. Now fix $K \in \mathbb{N}$, set $M_K := \max\{N_1,\ldots,N_K\}$, and set

$$U'_K := \big\{x \in \mathfrak{g} : q'_1(x) < 1, \ldots, q'_K(x) < 1\big\}.$$

Moreover, define $H^{(K)}(t,g) := (1 + |t|^{M_K})^{-K}F(t,g)$, and define $H_t^{(K)}(g) := H^{(K)}(t,g)$. From (10) we obtain that

$$\sup\big\{\big|L_{x_1}\cdots L_{x_k}H_t^{(K)}(g)\big| : g \in G, x_1,\ldots,x_k \in U'_K, t \in \mathbb{R}\big\} \le \|w\| < \infty$$

for all $k \le K$. Thus Lemma 2.1, applied to $H^{(K)}|_{\mathbb{R}\times W}$, implies that for every $u_0 \in \Phi(W)$ there exist an open u_0-neighborhood $U_{u_0,K} \subset \Phi(W)$ and an open 0-neighborhood $V_{u_0,K} \subset \mathfrak{g}$ such that

$$\sup\big\{\big|\mathrm{d}^K \widetilde{H}_t^{(K)}(u)(x_1,\ldots,x_K)\big| : u \in U_{u_0,K}, x_1,\ldots,x_K \in V_{u_0,K}, t \in \mathbb{R}\big\} < \infty, \tag{11}$$

where $\widetilde{H}_t^{(K)} := H_t^{(K)} \circ \Phi^{-1}$. Consider

$$\widehat{F} : \mathbb{R} \times \Phi(W) \to \mathbb{C}, \quad (t,u) \mapsto f(t)F\big(t, \Phi^{-1}(u)\big),$$

and set $\widehat{F}_t(u) := \widehat{F}(t,u)$. Then, for every $K \in \mathbb{N}$,

$$\mathrm{d}^K \widehat{F}_t(u)(x_1,\ldots,x_K) = \big(1 + |t|^{M_K}\big)^K f(t)\cdot \mathrm{d}^K \widetilde{H}_t^{(K)}(u)(x_1,\ldots,x_K).$$

Since $f \in \mathcal{S}(\mathbb{R},\mathbb{C})$ we have $(1 + |t|^{M_K})^K f(t) \in L^1(\mathbb{R},\mathbb{C})$ for all $K \in \mathbb{N}$. Thus (11) and Lemma A.4 show that the map

$$\Phi(W) \to \mathbb{C}, \quad u \mapsto \int_{\mathbb{R}} \widehat{F}(t,u)\,dt = \int_{\mathbb{R}} f(t)\big\langle \pi\big(\Phi^{-1}(u)\big)U_t v, w\big\rangle\,dt$$

is smooth. We conclude that

$$G \to \mathbb{C}, \quad g \mapsto \langle \pi(g)\pi(f)v, w\rangle = \int_{\mathbb{R}} f(t)\langle \pi(g)U_t v, w\rangle \, dt$$

is smooth for all $w \in \mathcal{H}$. As above we obtain with [N3, Theorem 7.2] that $\pi(f)v \in \mathcal{H}^\infty$ and that (7) holds. □

REMARK 2.4

In the situation of Theorem 2.3, assume that α has the infinitesimal generator $A : D(A) \to \mathfrak{g}$. Then growth bounds of α can often be determined in terms of the generator A. In particular, if $\mathfrak{g}$ is finite-dimensional, then α is polynomially bounded if and only if the spectrum of A is purely imaginary. However, for an infinite-dimensional Hilbert space $\mathcal{H}$ there is a one-parameter group $\alpha : \mathbb{R} \to B(\mathcal{H})$ with $\|\alpha_t\| = e^{|t|}$ whose generator has purely imaginary spectrum (cf. [vN, Example 1.2.4]).

REMARK 2.5

In the situation of Theorem 2.3, let B denote the self-adjoint generator of U_t. Assume that $f \in \mathcal{S}(\mathbb{R})$, and define $\widehat{f}(s) := \int_{\mathbb{R}} f(t)e^{ist}\, dt$. Then $\int_{\mathbb{R}} f(t)U_t v \, dt = \widehat{f}(B)v$, where $\widehat{f}(B)$ is defined by the functional calculus of B. Since the map $\mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R}), f \mapsto \widehat{f}$ is a bijection, we see that $h(B)v \in \mathcal{H}^\infty$ for all $v \in \mathcal{H}^\infty$, $h \in \mathcal{S}(\mathbb{R})$.

DEFINITION 2.6

An element $x \in \mathfrak{g}$ is called *elliptic* if the subgroup $\mathrm{Ad}(e^{\mathbb{R}x}) \subset \mathrm{End}(\mathfrak{g})$ is equicontinuous.

COROLLARY 2.7

Let $\pi : G \to U(\mathcal{H})$ be a continuous unitary representation, let $x \in \mathfrak{g}$, and set $\alpha_t := \mathrm{Ad}(e^{tx})$. Assume either that x is elliptic and $f \in L^1(\mathbb{R})$ or that $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathfrak{g}_\mathbb{C})$, as defined in Theorem 2.3, is polynomially bounded and $f \in \mathcal{S}(\mathbb{R})$. Then $U_f\mathcal{H}^\infty \subseteq \mathcal{H}^\infty$.

Proof

Define $\gamma : \mathbb{R} \times G \to G, (t, g) \mapsto e^{tx}ge^{-tx}$. Then $\pi^\#(g, t) := \pi(ge^{tx})$ is a continuous unitary representation. As $\alpha_t = \mathbf{L}(\gamma_t) = \mathrm{Ad}(e^{tx})$, the assertion follows from Theorem 2.3. □

3. The spectral translation formula

Let γ and α be as in Theorem 2.3. We assume, in addition, that $\mathfrak{g}$ is complete and that γ defines continuous actions of $\mathbb{R}$ on G and $\mathfrak{g}$. If α is equicontinuous, then we define the spectrum $\mathrm{Spec}_\alpha(x)$ of an element $x \in \mathfrak{g}_\mathbb{C}$ and the Arveson spectral subspace $\mathfrak{g}_\mathbb{C}(E)$ for $E \subset \mathbb{R}$ as in Definition A.5(b). A continuous unitary one-parameter group $(U_t)_{t\in\mathbb{R}}$ on $\mathcal{H}$ is clearly equicontinuous. Therefore we

can consider $\mathrm{Spec}_U(v)$ for $v \in \mathcal{H}$ and the Arveson spectral subspaces $\mathcal{H}(E)$ and $\mathcal{H}^\infty(E) := \mathcal{H}^\infty \cap \mathcal{H}(E)$. If α is only polynomially bounded, then we likewise define the spectrum $\mathrm{Spec}_\alpha(x;\mathcal{S})$ of an element $x \in \mathfrak{g}_\mathbb{C}$ and the Arveson spectral subspace $\mathfrak{g}_\mathbb{C}(E;\mathcal{S})$ for $E \subset \mathbb{R}$ (see Definition A.5(a)). By Lemma A.9 we have that $\mathrm{Spec}_\alpha(x) = \mathrm{Spec}_\alpha(x;\mathcal{S})$ and $\mathfrak{g}_\mathbb{C}(E) = \mathfrak{g}_\mathbb{C}(E;\mathcal{S})$ if α is equicontinuous.

THEOREM 3.1 (SPECTRAL TRANSLATION FORMULA)

Assume that $\mathfrak{g}$ is a complete locally convex Lie algebra, that $\gamma : \mathbb{R} \to \mathrm{Aut}(G)$ defines a continuous action of $\mathbb{R}$ on G, and that $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathfrak{g}_\mathbb{C})$, as defined in Theorem 2.3, defines a continuous action of $\mathbb{R}$ on $\mathfrak{g}_\mathbb{C}$. Let $\pi^\#(g,t) = \pi(g)U_t$ be a continuous unitary representation of $G \rtimes_\gamma \mathbb{R}$ on $\mathcal{H}$, and let $\mathcal{H}^\infty$ be the space of smooth vectors with respect to π.

(a) *Assume that α is equicontinuous. Then, for any subsets $E, F \subseteq \mathbb{R}$, we have that*

$$\mathrm{d}\pi\big(\mathfrak{g}_\mathbb{C}(E)\big)\mathcal{H}^\infty(F) \subseteq \mathcal{H}^\infty(E+F).$$

(b) *Assume that α is polynomially bounded. Then, for any subsets $E, F \subseteq \mathbb{R}$, we have that*

$$\mathrm{d}\pi\big(\mathfrak{g}_\mathbb{C}(E;\mathcal{S})\big)\mathcal{H}^\infty(F) \subseteq \mathcal{H}^\infty(E+F).$$

Proof

(a) Assume that α is equicontinuous. From (7) we recall for $v \in \mathcal{H}^\infty$ and $f \in L^1(\mathbb{R})$ the relation

$$\mathrm{d}\pi(y)U_f v = \int_\mathbb{R} f(t)U_t \mathrm{d}\pi\big(\alpha_{-t}(y)\big)v\,dt. \tag{12}$$

Fix $v \in \mathcal{H}^\infty$, and consider the bilinear map

$$\beta\colon \mathfrak{g}_\mathbb{C} \times L^1(\mathbb{R}) \to \mathcal{H}, \quad (y,f) \mapsto \mathrm{d}\pi(y)U_f v.$$

Since the map $\mathfrak{g}_\mathbb{C} \to \mathcal{H}, y \mapsto \mathrm{d}\pi(y)v$ is continuous and $\alpha_\mathbb{R} \subset \mathrm{End}(\mathfrak{g}_\mathbb{C})$ is equicontinuous, there is a continuous seminorm q on $\mathfrak{g}_\mathbb{C}$ with $\|\mathrm{d}\pi(\alpha_t(y))v\| \le q(y)$ for all $y \in \mathfrak{g}_\mathbb{C}$, $t \in \mathbb{R}$. Now let $x \in \mathfrak{g}_\mathbb{C}$, and let $f \in L^1(\mathbb{R})$. Then by (12) we obtain $\|\beta(x,f)\| \le \|f\|_{L^1(\mathbb{R})}q(x)$, so that β is continuous by Proposition A.3. Since the integrated representation U of $L^1(\mathbb{R})$ on $\mathcal{H}$ is continuous, the annihilator ideal

$$L^1(\mathbb{R})_v := \big\{f \in L^1(\mathbb{R})\colon U_f v = 0\big\}$$

is closed and therefore translation invariant. Note that it is a two-sided ideal because $L^1(\mathbb{R})$ is commutative. It follows that the left regular representation of $\mathbb{R}$ on $L^1(\mathbb{R})$ defined by $\lambda_t f(t') := f(t'-t)$, for $t, t' \in \mathbb{R}$ and $f \in L^1(\mathbb{R})$, factors to a continuous and equicontinuous representation of $\mathbb{R}$ on the Banach space $\mathcal{A} := L^1(\mathbb{R})/L^1(\mathbb{R})_v$. Write $\overline{f}$ for the image of $f \in L^1(\mathbb{R})$ in $\mathcal{A}$. Then the corresponding integrated representation of $L^1(\mathbb{R})$ on $\mathcal{A}$ is $\lambda_f \overline{h} := \overline{f * h}$, where $f, h \in L^1(\mathbb{R})$. For every $\overline{h} \in \mathcal{A}$, we consider $\mathrm{Spec}_\lambda(\overline{h})$ (see Definition A.5(b)).

Set $F := \mathrm{Spec}_U(v)$. For $f \in L^1(\mathbb{R})$ and $h \in L^1(\mathbb{R})_v$, the commutativity of $L^1(\mathbb{R})$ implies that $f * h = h * f \in L^1(\mathbb{R})_v$, so that $\lambda_h \overline{f} = \overline{h * f} = 0$. It follows that $\mathrm{Spec}_\lambda(\overline{f}) \subseteq F$ for every $\overline{f} \in \mathcal{A}$.

Now fix $E \subseteq \mathbb{R}$. As β is continuous, [N4, Proposition A.14] implies that, for every $y \in \mathfrak{g}_{\mathbb{C}}(E)$, we have

$$\mathrm{Spec}_U\big(\mathrm{d}\pi(y)U_f v\big) \subseteq \overline{E+F} \quad \text{for } f \in L^1(\mathbb{R}).$$

Next we observe that (12) implies that, for any δ-sequence δ_n in $L^1(\mathbb{R})$, we have

$$\mathrm{d}\pi(y)U_{\delta_n} v \to \mathrm{d}\pi(y)v.$$

Since $\mathcal{H}(\overline{E+F})$ is a closed subspace of $\mathcal{H}$, we obtain $\mathrm{Spec}_U(\mathrm{d}\pi(y)v) \subseteq \overline{E+F}$ for every $y \in \mathfrak{g}_{\mathbb{C}}(E)$.

(b) Now assume that α is polynomially bounded. Let $v \in \mathcal{H}^\infty$. Then the bilinear map

$$\beta\colon \mathfrak{g}_{\mathbb{C}} \times \mathcal{S}(\mathbb{R}) \to \mathcal{H}, \quad (y, f) \mapsto \mathrm{d}\pi(y)U_f v$$

is continuous, which follows from an argument similar to that used in (a). From the continuity of the inclusion $\mathcal{S}(\mathbb{R}) \hookrightarrow L^1(\mathbb{R})$ and the closedness of $L^1(\mathbb{R})_v$, it follows that the annihilator ideal $\mathcal{S}(\mathbb{R})_v := \{f \in \mathcal{S}(\mathbb{R}) : U_f v = 0\}$ is closed in $\mathcal{S}(\mathbb{R})$. Now let λ be the left regular representation of $\mathbb{R}$ on $\mathcal{S}(\mathbb{R})$ defined by $\lambda_t f(t') := f(t'-t)$ for $t, t' \in \mathbb{R}$ and $f \in \mathcal{S}(\mathbb{R})$. From the relation $U_t U_f v = U_{\lambda_t f} v$ for $t \in \mathbb{R}$ and $f \in \mathcal{S}(\mathbb{R})$, it follows that $\mathcal{S}(\mathbb{R})_v$ is translation invariant. The argument given in (a) for the case of $L^1(\mathbb{R})$ can be adapted to show that $\mathrm{Spec}_\lambda(\overline{f}, \mathcal{S}) \subseteq \mathrm{Spec}_U(v; \mathcal{S}) = \mathrm{Spec}_U(v)$ for every $\overline{f} := f + \mathcal{S}(\mathbb{R})_v$, where $f \in \mathcal{S}(\mathbb{R})$ (cf. Lemma A.9). Since β is continuous, we can now apply Proposition A.10 and complete the proof as in (a). □

PROPOSITION 3.2

Let γ and α be as in Theorem 2.3, let $\pi^\#(g,t) = \pi(g)U_t$ be a continuous unitary representation of $G \rtimes_\gamma \mathbb{R}$ on $\mathcal{H}$, and let $\mathcal{H}^\infty$ be the space of smooth vectors with respect to π. Let B denote the self-adjoint generator of U_t, and let P_B be its spectral measure. We have $P_B(E)\mathcal{H} = \mathcal{H}(E)$ for every closed subset $E \subset \mathbb{R}$. If α is polynomially bounded and π is smooth, then $\mathcal{H}^\infty \cap P_B(E)\mathcal{H}$ is dense in $P_B(E)\mathcal{H}$ for every open subset $E \subset \mathbb{R}$.

Proof

It is easy to verify that $P_B(E)\mathcal{H} = \mathcal{H}(E)$ holds for every closed subset $E \subset \mathbb{R}$. Now let $E \subset \mathbb{R}$ be open. Choose compact subsets $K_n \subset E$, $n \in \mathbb{N}$, with $K_n \subset K_{n+1}$ and $E = \bigcup_n K_n$. Let $v \in P_B(E)\mathcal{H}$, and let $\varepsilon > 0$. By the smooth Urysohn lemma we may choose compactly supported smooth functions f_n with $\mathrm{supp}(f_n) \subset E$, $\|f_n\|_\infty \le 1$, and $f_n = 1$ on K_n. By [RS, Theorem VIII.5(d)] we have $f_n(B)v \to v = P_B(E)v$. Choose $v' \in \mathcal{H}^\infty$ with $\|v' - v\| < \varepsilon$. Then

$$\begin{aligned}\big\|f_n(B)v' - v\big\| &\le \big\|f_n(B)v' - f_n(B)v\big\| + \big\|f_n(B)v - v\big\| \\ &\le \|v' - v\| + \big\|f_n(B)v - v\big\| < \varepsilon\end{aligned}$$

for n large enough. As $f_n(B)v' \in \mathcal{H}^\infty \cap P_B(E)\mathcal{H}$ by Remark 2.5, the assertion follows. □

Appendix

A.1 Continuous mappings between locally convex spaces

DEFINITION A.1

Let E and F be locally convex spaces. We denote by $\mathrm{Hom}(E,F)$ the space of continuous linear maps from E to F and write $\mathrm{End}(E) := \mathrm{Hom}(E,E)$. A subset $Y \subset \mathrm{Hom}(E,F)$ is called *equicontinuous* if for every open 0-neighborhood U in F there exists a 0-neighborhood W in E such that $T(W) \subset U$ holds for every $T \in Y$.

PROPOSITION A.2 ([B2, II.1.4, PROPOSITION 4])

For $Y \subset \mathrm{Hom}(E,F)$ the following conditions are equivalent.

(a) *Y is equicontinuous.*

(b) *For every continuous seminorm p on F there exists a continuous seminorm q on E such that $p(Tx) \le q(x)$ holds for all $T \in Y$ and $x \in E$.*

PROPOSITION A.3 ([B2, II.1.4, PROPOSITION 4])

Let $m : E^n \to F$ be an n-linear map. Then m is continuous if and only if for every continuous seminorm p on F there exists a continuous seminorm q on E such that

$$p\big(m(x_1,\dots,x_n)\big) \le q(x_1)\cdots q(x_n)$$

holds for all $x_1,\dots,x_n \in E$.

A.2 Differentiation under the integral sign

LEMMA A.4

Let (Ω,Σ,μ) be a measure space, let E be a locally convex space, and let $W \subset E$ be an open subset. Let $f : \Omega \times W \to \mathbb{C}$ be a map such that $f_t := f(t,\cdot) \in C^\infty(W,\mathbb{C})$ for all $t \in \Omega$ and $f(\cdot,x) \in L^1(\Omega,\mu)$ for all $x \in W$. Assume that, for every $x_0 \in W$ and $k \in \mathbb{N}$, there exist open subsets $U_{x_0,k}$ of W and $V_{x_0,k}$ of E with $x_0 \in U_{x_0,k}$ and $0 \in V_{x_0,k}$ and a function $g_{x_0,k} \in L^1(\Omega,\mu)$ such that

$$\sup\big\{\big|\mathrm{d}^k f_t(x)(h'_1,\dots,h'_k)\big| : x \in U_{x_0,k}, h'_1,\dots,h'_k \in V_{x_0,k}\big\} \le g_{x_0,k}(t) \tag{13}$$

for all $t \in \Omega$. Then $F(\cdot) := \int_\Omega f(t,\cdot)\,d\mu(t)$ defines a smooth function on W with derivatives given by

$$\mathrm{d}^k F(x)(h_1,\dots,h_k) = \int_\Omega \mathrm{d}^k f_t(x)(h_1,\dots,h_k)\,d\mu(t).$$

Proof

We first show that F is C^1. Let $x_0 \in W$, let $h_1 \in E$, and let $U_{x_0,1}, V_{x_0,1}$ have the stated properties, where we assume without loss of generality that $U_{x_0,1}$ is

convex. Since $\mathrm{d}f_t(x)(h)$ is linear in h we may (by scaling of $V_{x_0,1}$) further assume that $h_1 \in V_{x_0,1}$. Let $t_n \to 0$ with $x_0 + t_n h_1 \in U_{x_0,1}$ for all n. Then we estimate

$$\Big|\frac{F(x_0+t_nh_1)-F(x_0)}{t_n} - \int_\Omega \mathrm{d}f_t(x_0)(h_1)\,d\mu(t)\Big| \le \int_\Omega h_n(t)\,d\mu(t),$$

where

$$h_n(t) := \Big|\frac{f_t(x_0+t_nh_1)-f_t(x_0)}{t_n} - \mathrm{d}f_t(x_0)(h_1)\Big|.$$

The relation from (13) yields that

$$h_n(t) = \Big|\int_0^1 \mathrm{d}f_t(x_0+st_nh_1)(h_1) - \mathrm{d}f_t(x_0)(h_1)\,ds\Big| \le 2g_{x_0,1}(t).$$

As $h_n(t) \to 0$ for all $t \in \Omega$, the dominated convergence theorem entails $\int_\Omega h_n(t)\,d\mu(t) \to 0$. Thus F is differentiable and $\mathrm{d}F(x)(h) = \int_\Omega \mathrm{d}f_t(x)(h)\,d\mu(t)$. Now let $x_0 \in W$, let $h_1 \in E$, and let $U_{x_0,1}, U_{x_0,2}, V_{x_0,1}, V_{x_0,2}$ have the stated properties, where we assume without loss of generality that $U := U_{x_0,1} \cap U_{x_0,2}$ is convex and $V := V_{x_0,1} \cap V_{x_0,2}$ is balanced. By scaling of $V_{x_0,1}$ and $V_{x_0,2}$, we may again assume that $h_1 \in V$. Let $\varepsilon > 0$, and set

$$\delta := \frac{\varepsilon}{(1 + \int_\Omega g_{x_0,1}\,d\mu(t) + \int_\Omega g_{x_0,2}\,d\mu(t))}.$$

For $x \in U \cap (x_0 + \delta\cdot V)$ and $h \in h_1 + \delta\cdot V$ we then have that

$$\begin{aligned}
&\big|\mathrm{d}F(x)(h) - \mathrm{d}F(x_0)(h_1)\big| \\
&\quad\le \int_\Omega \big|\mathrm{d}f_t(x)(h-h_1)\big|\,d\mu(t) \\
&\qquad + \int_\Omega \big|\mathrm{d}f_t(x)(h_1) - \mathrm{d}f_t(x_0)(h_1)\big|\,d\mu(t) \\
&\quad\le \delta\int_\Omega \big|\mathrm{d}f_t(x)\big(\delta^{-1}(h-h_1)\big)\big|\,d\mu(t) \\
&\qquad + \delta\int_\Omega\int_0^1 \big|\mathrm{d}^2 f_t\big(x_0+s(x-x_0)\big)\big(\delta^{-1}(x-x_0)\big)(h_1)\big|\,ds\,d\mu(t) \\
&\quad\le \delta\int_\Omega g_{x_0,1}(t)\,d\mu(t) + \delta\int_\Omega g_{x_0,2}(t)\,d\mu(t) < \varepsilon.
\end{aligned}$$

Since $U \cap (x_0 + \delta\cdot V)$ is an open x_0-neighborhood and $h_1 + \delta\cdot V$ is an open h_1-neighborhood, we conclude that $\mathrm{d}F$ is continuous. Hence F is continuously differentiable and therefore C^1.

We now argue by induction on k and assume that F is C^k, $k \ge 1$, with derivatives as stated. We must show that d^kF is C^1 with the appropriate derivative. Applying the C^1-case to $\mathrm{d}^kF(\cdot)(h_1,\dots,h_k)$ for fixed $h_1,\dots,h_k$ yields that $\mathrm{d}^kF(\cdot)(h_1,\dots,h_k)$ is differentiable with derivative

$$\begin{aligned}
\mathrm{d}^{k+1}F(x)(h_1,\dots,h_{k+1}) &= \mathrm{d}\big(\mathrm{d}^kF(\cdot)(h_1,\dots,h_k)\big)(x)(h_{k+1}) \\
&= \int_\Omega \mathrm{d}^{k+1}f_t(x)(h_1,\dots,h_{k+1})\,d\mu(t).
\end{aligned}$$

This map is continuous in $(x, h_1, \dots, h_{k+1})$, which may be shown by an analogous argument as for the C^1-case using (13). From here we conclude that $\mathrm{d}^k F$ is C^1. □

A.3 Arveson spectral theory for polynomially bounded actions

Let V be a complete complex locally convex space, and let $\alpha : \mathbb{R} \to \mathrm{GL}(V)$, $t \mapsto \alpha_t$ be a strongly continuous representation. Assume that α is polynomially bounded (see Definition 2.2(b)).

DEFINITION A.5

(a) We define

$$\alpha_f(v) := \int_{\mathbb{R}} f(t)\alpha_t(v)\, dt \quad \text{for } v \in V, f \in \mathcal{S}(\mathbb{R}). \tag{14}$$

Then $\alpha_f \in \mathrm{End}(V)$, and this yields a representation of the convolution algebra $(\mathcal{S}(\mathbb{R}), *)$ on V. We define the *spectrum* of an element $v \in V$ by

$$\mathrm{Spec}_\alpha(v; \mathcal{S}) := \big\{ y \in \mathbb{R} : \big(\forall f \in \mathcal{S}(\mathbb{R})\big) \alpha_f v = 0 \Rightarrow \widehat{f}(y) = 0 \big\},$$

which is the hull of the annihilator ideal of v. For a subset $E \subseteq \mathbb{R}$, we now define the corresponding *Arveson spectral subspace*

$$V(E; \mathcal{S}) := \big\{ v \in V : \ \mathrm{Spec}_\alpha(v; \mathcal{S}) \subseteq \overline{E} \big\}.$$

(b) If α is equicontinuous, then (14) exists for all $f \in L^1(\mathbb{R})$ and we can define $\mathrm{Spec}_\alpha(v)$ and $V(E)$ as above with $\mathcal{S}(\mathbb{R})$ replaced by $L^1(\mathbb{R})$ (see [N4, Definition A.5(b)]).

We now want to transfer some results of [N4, Appendix A.2] to the case when α is polynomially bounded. We first need a technical lemma.

DEFINITION A.6

For an ideal $I \subseteq \mathcal{S}(\mathbb{R})$ we define its hull by

$$h(I) := \big\{ x \in \mathbb{R} : \widehat{f}(x) = 0 \text{ for all } f \in I \big\},$$

and for a subset $E \subseteq \mathbb{R}$ we define

$$I_0(E) := \big\{ f \in \mathcal{S}(\mathbb{R}) : \mathrm{supp}(\widehat{f}) \cap E = \emptyset \big\},$$

which is an ideal in $\mathcal{S}(\mathbb{R})$.

LEMMA A.7

(a) *$h(I_0(E)) = E$ for $E \subseteq \mathbb{R}$ closed.*
(b) *$I_0(h(I)) \subseteq I$ for every closed ideal $I \subseteq \mathcal{S}(\mathbb{R})$.*

Proof

(a) We obviously have $E \subseteq h(I_0(E))$. For $y \in \mathbb{R} \backslash E$ we find a compactly supported smooth function f which is nonzero at y and supported in a compact

neighborhood of y intersecting E trivially. Then $f = \widehat{h}$ with $h \in I_0(E)$ shows that $y \notin h(I_0(E))$, and thus $h(I_0(E)) = E$.

(b) For $F \subseteq \mathbb{R}$ closed, set

$$I_c(F) := \big\{ f \in \mathcal{S}(\mathbb{R}) : \operatorname{supp}(\widehat{f}) \text{ is compact and } \operatorname{supp}(\widehat{f}) \cap F = \emptyset \big\}.$$

Note that $I_c(F) \subseteq I_0(F)$ is dense with respect to the Fréchet topology of $\mathcal{S}(\mathbb{R})$. Thus it suffices to show that $I_c(h(I)) \subseteq I$ for every ideal $I \subseteq \mathcal{S}(\mathbb{R})$. We consider the Fourier-transformed ideal $\widehat{I} := \{\widehat{f} : f \in I\}$ in $\mathcal{S}(\mathbb{R})$ with pointwise multiplication. Let $f \in \mathcal{S}(\mathbb{R})$ be a compactly supported function which vanishes on a neighborhood of $h(I)$. We must show that $f \in \widehat{I}$. Choose a compact neighborhood K of $\operatorname{supp}(f)$ which is also disjoint from $h(I)$. Since $h(I) \cap K = \emptyset$, for every $p \in K$ there is an $f_p \in \widehat{I}$ with $f_p(p) \neq 0$. A standard compactness argument yields $f_{p_1}, \ldots, f_{p_k} \in \widehat{I}$ such that the sets $\{t \in \mathbb{R} : f_{p_j}(t) \neq 0\}$, $1 \leq j \leq k$, cover K. Set $g := |f_{p_1}|^2 + \cdots + |f_{p_k}|^2$, and note that $g \in \widehat{I}$. Furthermore, $g(t) > 0$ for every $t \in K$. Now by the smooth Urysohn lemma we can choose a compactly supported smooth function $h : \mathbb{R} \to \mathbb{R}$ such that $\operatorname{supp}(h) \subseteq \operatorname{int}(K)$ and $h|_{\operatorname{supp}(f)} = 1$. Then $g_2 := (h/g)g \in \widehat{I}$ and $g_2|_{\operatorname{supp}(f)} = 1$. Thus $f = fg_2 \in \widehat{I}$. □

PROPOSITION A.8

For each subset $E \subseteq \mathbb{R}$, we have that

$$V(E;\mathcal{S}) = \big\{ v \in V : \ \alpha_f(v) = 0 \text{ for all } f \in \mathcal{S}(\mathbb{R}) \text{ with } \operatorname{supp}(\widehat{f}) \cap \overline{E} = \emptyset \big\}. \tag{15}$$

Proof

Assume without loss of generality that $E \subseteq \mathbb{R}$ is closed. For $v \in V$ we denote by $\mathcal{S}(\mathbb{R})_v = \{f \in \mathcal{S}(\mathbb{R}) : \alpha_f(v) = 0\}$ the annihilator ideal of v, which is closed in $\mathcal{S}(\mathbb{R})$. Note that $\operatorname{Spec}_\alpha(v;\mathcal{S}) = h(\mathcal{S}(\mathbb{R})_v)$, and note that the right-hand side of (15) equals

$$M := \big\{ v \in V : I_0(E) \subseteq \mathcal{S}(\mathbb{R})_v \big\}.$$

For $v \in V(E;\mathcal{S})$, we have $h(\mathcal{S}(\mathbb{R})_v) \subseteq E$ and therefore $I_0(E) \subseteq I_0(h(\mathcal{S}(\mathbb{R})_v)) \subseteq \mathcal{S}(\mathbb{R})_v$ by Lemma A.7(b), which implies that $v \in M$. For $v \in M$, we have $\operatorname{Spec}_\alpha(v;\mathcal{S}) = h(\mathcal{S}(\mathbb{R})_v) \subseteq h(I_0(E)) = E$ by Lemma A.7(a), so that $v \in V(E;\mathcal{S})$. Hence $V(E;\mathcal{S}) = M$. □

The preceding proposition shows, in particular, that $V(E;\mathcal{S})$ is a closed subspace of V. The following lemma shows that $\operatorname{Spec}_\alpha(v;\mathcal{S})$ and $V(E;\mathcal{S})$ are natural generalizations of $\operatorname{Spec}_\alpha(v)$ and the Arveson spectral subspace $V(E)$, respectively, to the case when α is polynomially bounded.

LEMMA A.9

Assume that α is equicontinuous. Then $V(E) = V(E;\mathcal{S})$ for all $E \subseteq \mathbb{R}$ and $\operatorname{Spec}_\alpha(v) = \operatorname{Spec}_\alpha(v;\mathcal{S})$ for all $v \in V$.

Proof
Let $E \subseteq \mathbb{R}$, and assume without loss of generality that E is closed. We have that $V(E;\mathcal{S}) \subseteq V(E)$ as $\mathrm{Spec}_\alpha(v) \subseteq \mathrm{Spec}_\alpha(v;\mathcal{S})$ and

$$V(E) = \big\{v \in V : \ \alpha_f(v) = 0 \text{ for all } f \in L^1(\mathbb{R}) \text{ with } \mathrm{supp}(\widehat{f}) \cap E = \emptyset\big\}$$

by [N4, Remark A.6]. With Proposition A.8 we thus obtain $V(E) \subseteq V(E;\mathcal{S})$ and conclude that $V(E) = V(E;\mathcal{S})$. Let $v \in V$, and let $F := \mathrm{Spec}_\alpha(v)$. Since F is closed, this implies that $v \in V(F) = V(F;\mathcal{S})$, so that $\mathrm{Spec}_\alpha(v;\mathcal{S}) \subseteq F$ yields $F = \mathrm{Spec}_\alpha(v;\mathcal{S})$. □

The following proposition is a version of [N4, Proposition A.14] for polynomially bounded representations of $\mathbb{R}$.

PROPOSITION A.10
Assume that (α_j, V_j), $j = 1,2,3$, are continuous polynomially bounded representations of $\mathbb{R}$ on the complete complex locally convex spaces V_j, and assume that $\beta : V_1 \times V_2 \to V_3$ is a continuous equivariant bilinear map. Then we have, for closed subsets $E_1, E_2 \subseteq \mathbb{R}$, the relation

$$\beta\big(V_1(E_1;\mathcal{S}) \times V_2(E_2;\mathcal{S})\big) \subseteq V_3(E_1 + E_2;\mathcal{S}).$$

Proof
By Proposition A.8 the assertion can be proved (with trivial changes) as in [N4, Proposition A.14] once we know that [N4, Lemma A.13] (or equivalently [A, Proposition 2.2]) holds also in the polynomially bounded case. With Proposition A.8 and Lemma A.7, the proof of [A, Proposition 2.2] carries over to the polynomially bounded case. □

Acknowledgment. The authors thank the referees for the most careful reading of the manuscript, which led to numerous improvements in the presentation of the article.

References

[A] W. Arveson, *On groups of automorphisms of operator algebras*, J. Funct. Anal. **15** (1974), 217–243. MR 0348518.

[B1] N. Bourbaki, *Éléments de mathématique: Groupes et algèbres de Lie, chapitres 1, 4–6, 7–8*, Masson, Paris, 1982.

[B2] ———, *Topological Vector Spaces, Chapters 1–5*, Springer, Berlin, 1987. MR 0910295. DOI 10.1007/978-3-642-61715-7.

[Ha] R. S. Hamilton, *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), 65–222. MR 0656198. DOI 10.1090/S0273-0979-1982-15004-2.

[Hu] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Grad. Texts in Math. **9**, Springer, Berlin, 1972. MR 0323842.

[MN] S. Merigon and K.-H. Neeb, *Semibounded unitary representations of mapping groups with values in infinite dimensional Hermitian groups*, in preparation.

[N1] K.-H. Neeb, "Infinite dimensional groups and their representations" in *Infinite Dimensional Kähler Manifolds (Oberwolfach, 1995)*, DMV Sem. **31**, Birkhäuser, Basel, 2001, 131–178. MR 1853240.

[N2] ———, *Towards a Lie theory of locally convex groups*, Jpn. J. Math. **1** (2006), 291–468. MR 2261066. DOI 10.1007/s11537-006-0606-y.

[N3] ———, *On differentiable vectors for representations of infinite dimensional Lie groups*, J. Funct. Anal. **259** (2010), 2814–2855. MR 2719276. DOI 10.1016/j.jfa.2010.07.020.

[N4] ———, "Holomorphic realization of unitary representations of Banach–Lie groups" in *Lie Groups: Structure, Actions, and Representations*, Progr. Math. **306**, Birkhäuser, New York, 2013, 185–223. MR 3186693. DOI 10.1007/978-1-4614-7193-6_10.

[NS1] K.-H. Neeb and H. Salmasian, *Positive definite superfunctions and unitary representations of Lie supergroups*, Transform. Groups **18** (2013), 803–844. MR 3084335. DOI 10.1007/s00031-013-9228-7.

[NS2] ———, *Classification of positive energy representations of the Virasoro group*, Int. Math. Res. Not. IMRN, published electronically 8 November 2014. DOI 10.1093/imrn/rnu197.

[RS] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, I: Functional Analysis*, Academic Press, New York, 1980. MR 0751959.

[vN] J. van Neerven, *The Asymptotic Behaviour of Semigroups of Linear Operators*, Oper. Theory Adv. Appl. **88**, Birkhäuser, Basel, 1996. MR 1409370. DOI 10.1007/978-3-0348-9206-3.

Neeb: Department Mathematik, Friedrich-Alexander-Universität Erlangen-Nürnberg, Erlangen, Germany; karl-hermann.neeb@math.uni-erlangen.de

Salmasian: Department of Mathematics and Statistics, University of Ottawa, Ottawa, Ontario, Canada; hsalmasi@uottawa.ca

Zellner: Department Mathematik, Friedrich-Alexander-Universität Erlangen-Nürnberg, Erlangen, Germany; zellner@mi.uni-erlangen.de