Remarks on Hall algebras of triangulated categories

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Abstract In this paper, we introduce a motivic version of Toën's derived Hall algebra. Then we point out that the two kinds of Hall algebras in the sense of Toën and Kontsevich–Soibelman, respectively, are Drinfeld dual pairs, not only in the classical case (by counting over finite fields) but also in the motivic version. Consequently they are canonically isomorphic. All proofs, including that for the most important associative property, are deduced in a self-contained way by analyzing the symmetry properties around the octahedral axiom, a method we used previously.

1. Introduction

Let k be a finite field with q elements, and let $\mathcal{A}$ be a finitary k-category, that is, a (small) k-linear abelian category satisfying: (1) $\dim_k \operatorname{Hom}_{\mathcal{A}}(M,N) < \infty$; (2) $\dim_k \operatorname{Ext}^1_{\mathcal{A}}(M,N) < \infty$ for any $M, N \in \mathcal{A}$. The Hall algebra $\mathcal{H}(\mathcal{A})$ associated to a finitary category $\mathcal{A}$ was originally defined by Ringel [20] in order to realize a quantum group. In the simplest version, it is an associative algebra, which, as a $\mathbb{Q}$-vector space, has a basis consisting of the isomorphism classes $[X]$ for $X \in \mathcal{A}$ and has the multiplication $[X] * [Y] = \sum_{[L]} \mathrm{g}^L_{XY}[L]$, where $X, Y, L \in \mathcal{A}$ and $\mathrm{g}^L_{XY} = |\{M \subset L \mid M \simeq X \text{ and } L/M \simeq Y\}|$. The structure constant g^L_{XY} is called the *Hall number* and the algebra $\mathcal{H}(\mathcal{A})$ now is called Ringel–Hall algebra. The Ringel–Hall algebras have been developed into many variants (see [9]) as a framework involving the categorification and the geometrization of Lie algebras and quantum groups in the past two decades (e.g., see [18], [8], [9], [20], [13], [19], [14], [16], [17]). If $\mathcal{A} = \operatorname{mod}\Lambda$ for a hereditary finitary k-algebra Λ, then there exists a comultiplication $\delta : \mathcal{H}(\mathcal{A}) \to \mathcal{H}(\mathcal{A}) \otimes \mathcal{H}(\mathcal{A})$ constructed by Green [5]. Burban and Schiffmann in [3] and [2] also studied the (topological) comultiplication of $\mathcal{H}(\operatorname{Coh}\mathbb{X})$ for some curves $\mathbb{X}$. The comultiplication by Green naturally induces a new multiplication on $\mathcal{H}(\mathcal{A})$. We call this new algebra structure the Drinfeld dual of $\mathcal{H}(\mathcal{A})$ (see Section 2).

Toën [23] introduced derived Hall algebras associated to derived categories (see [11] for lattice algebras associated to derived categories of heredity cate-

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gories). In [24], the notion was extended for triangulated categories with some homological finiteness conditions, and a new proof for the associativity of derived Hall algebras was given. From the viewpoint of associativity, derived Hall algebras generalize Ringel–Hall algebras. The study of the theory of derived Hall algebras is meaningful. It is applied to characterize refined Donaldson–Thomas invariants via constructing an integration map from the derived Hall algebra of a 3-Calabi–Yau category to a quantum torus (see [10], [12]). Recently, Hernandez and Leclerc [6] defined the monoidal categorification of a derived Hall algebra. One may hope to construct an analogue of the comultiplication of Ringel–Hall algebras. In Sections 2 and 3 of this article, we define a map over derived Hall algebras analogous to comultiplications of Ringel–Hall algebras. In general, the map does not provide an algebra homomorphism even for the derived category of a hereditary algebra. However, it induces a new multiplication structure on a derived Hall algebra. Then we can write down the Drinfeld dual of a derived Hall algebra. The Drinfeld dual coincides with the finite-field version of the motivic Hall algebras introduced by Kontsevich and Soibelman [12]. In Section 4, we point out that the method of [24] provides two symmetries associated with the octahedral axiom and they are equivalent. The first symmetry implies the associativity of the derived Hall algebra in the sense of Toën; the second symmetry implies the associativity of its Drinfeld dual. In Section 5, we construct the motivic version of a derived Hall algebra and show that it is associative and isomorphic to the Kontsevich–Soibelman motivic Hall algebra.

2. The Drinfeld dual of an algebra

Let A be an associative algebra over $\mathbb{Q}$ such that, as a space, it has a basis $\{u_\alpha\}_{\alpha\in\mathcal{P}}$ and the multiplication is given by

$$u_\alpha\cdot u_\beta=\sum_{\lambda\in\mathcal{P}}g_{\alpha\beta}^{\lambda}u_\lambda,$$

where $g_{\alpha\beta}^{\lambda}\in\mathbb{Q}$ are the structure constants. We denote by $A\widehat{\otimes}A$ the $\mathbb{Q}$-space of all formal (possibly infinite) linear combinations $\sum_{\alpha,\beta\in\mathcal{P}}c_{\alpha,\beta}u_\alpha\otimes u_\beta$ with $c_{\alpha,\beta}\in\mathbb{Q}$, which can be viewed as the completion of $A\otimes A$. Assume that there exists a linear map $\delta: A\to A\widehat{\otimes}A$ defined by

$$\delta(u_\lambda)=\sum_{\alpha,\beta}h_{\lambda}^{\alpha\beta}u_\alpha\otimes u_\beta$$

satisfying that, for fixed $\alpha,\beta\in\mathcal{P}$, there are only finitely many λ's such that $h_{\lambda}^{\alpha\beta}\neq 0$.

Consider a nondegenerate symmetric bilinear form $(\bullet,\bullet): A\times A\to\mathbb{Q}$ such that

$$(u_\alpha,u_\beta)=\delta_{\alpha,\beta}t_\alpha$$

for some nonzero $t_\alpha\in\mathbb{Q}$. Then the bilinear form naturally induces the following two bilinear forms:

$$f_1:(A\otimes A)\times(A\widehat{\otimes}A)\to\mathbb{Q}\qquad\text{and}\qquad f_2:(A\widehat{\otimes}A)\times(A\otimes A)\to\mathbb{Q}.$$

The first bilinear form is defined by

$$f_1\Big(\sum_{\alpha,\beta\in\mathcal{P}} c_{\alpha,\beta}u_\alpha\otimes u_\beta, \sum_{\alpha',\beta'\in\mathcal{P}} d_{\alpha',\beta'}u_{\alpha'}\otimes u_{\beta'}\Big)=\sum_{\alpha,\beta\in\mathcal{P}} c_{\alpha,\beta}d_{\alpha,\beta}t_\alpha t_\beta.$$

Note that the sum of the right-hand side is induced by the first sum of the left-hand side, which is a finite sum. The map f_2 is defined similarly.

PROPOSITION 2.1

For any $a,b,c\in A$, the equality $(a,bc)=(\delta(a),b\otimes c)$ holds if and only if, for any u_β, u_γ, and u_α, we have $h_\alpha^{\beta\gamma}t_\beta t_\gamma=g_{\beta\gamma}^\alpha t_\alpha$.

Proof

The proof is the same as [21, Proposition 7.1]. It is enough to consider the case in which a, b, and c are three basis elements denoted by u_α, u_β, and u_γ, respectively. By definition,

$$(a,bc)=(u_\alpha,u_\beta u_\gamma)=\Big(u_\alpha,\sum_{\lambda\in\mathcal{P}} g_{\beta\gamma}^\lambda u_\lambda\Big)=g_{\beta\gamma}^\alpha t_\alpha$$

and

$$\big(\delta(a),b\otimes c\big)=\Big(\sum_{\xi,\tau\in\mathcal{P}} h_\alpha^{\xi\tau}u_\xi\otimes u_\tau, u_\beta\otimes u_\gamma\Big)=h_\alpha^{\beta\gamma}t_\beta t_\gamma.$$

The proposition follows. □

Let A^{Dr} be a space over $\mathbb{Q}$ with the basis $\{v_\alpha\}_{\alpha\in\mathcal{P}}$. We define the multiplication by setting

$$v_\alpha * v_\beta=\sum_{\lambda\in\mathcal{P}} h_\lambda^{\alpha\beta}v_\lambda.$$

THEOREM 2.2

Assume that $(a,bc)=(\delta(a),b\otimes c)$ for any $a,b,c\in A$. Then there exists an isomorphism

$$\Phi: A^{Dr}\to A$$

by sending v_α to $t_\alpha^{-1}u_\alpha$.

Proof

It is clear that the map Φ is an isomorphism of vector spaces. For any $\alpha,\beta\in\mathcal{P}$, $\Phi(v_\alpha*v_\beta)=\sum_{\lambda\in\mathcal{P}} h_\lambda^{\alpha\beta}t_\lambda^{-1}u_\lambda$. Also, we have that $\Phi(v_\alpha)\cdot\Phi(v_\beta)=t_\alpha^{-1}t_\beta^{-1}u_\alpha\cdot u_\beta=\sum_{\alpha,\beta\in\mathcal{P}} t_\alpha^{-1}t_\beta^{-1}g_{\alpha\beta}^\lambda u_\lambda$. Proposition 2.1 concludes that Φ is an algebra homomorphism, that is,

$$\Phi(v_\alpha*v_\beta)=\Phi(v_\alpha)\cdot\Phi(v_\beta).$$ □

As a corollary, the algebra A^{Dr} is also an associative algebra and is called the Drinfeld dual of A. The first canonical example comes from Ringel–Hall algebra. We recall its definition from [20] and [22].

EXAMPLE 2.3

Let $\mathcal{A}$ be a small finitary abelian category and linear over some finite field k with q elements, and let $\mathcal{P}$ be the set of isomorphism classes of objects in $\mathcal{A}$. For $\alpha \in \mathcal{P}$, we take a representative $V_\alpha \in \mathcal{A}$. The Ringel–Hall algebra of $\mathcal{A}$ is a vector space $\mathcal{H}(\mathcal{A}) = \mathbb{Q}\mathcal{P} = \bigoplus_{\alpha\in\mathcal{P}} \mathbb{Q}u_\alpha$ with the multiplication

$$u_\alpha \cdot u_\beta = \sum_{\lambda\in\mathcal{P}} g_{\alpha\beta}^{\lambda} u_\lambda,$$

where $g_{\alpha\beta}^{\lambda} = |\{V \subseteq V_\lambda \mid V \cong V_\alpha, V_\lambda/V \cong V_\beta\}|$. Define the map (see [22, Section 1.4])

$$\delta : \mathcal{H}(\mathcal{A}) \to \mathcal{H}(\mathcal{A}) \widehat{\otimes} \mathcal{H}(\mathcal{A})$$

satisfying $\delta(u_\lambda) = \sum_{\alpha,\beta} h_\lambda^{\alpha\beta} u_\alpha \otimes u_\beta$ where $h_\lambda^{\alpha\beta} = |\operatorname{Ext}^1_\Lambda(V_\alpha, V_\beta)_{V_\lambda}| / |\operatorname{Hom}_\Lambda(V_\alpha, V_\beta)|$ (see [22]). For fixed $\alpha, \beta \in \mathcal{P}$, $\dim_k \operatorname{Ext}^1(V_\alpha, V_\beta) < \infty$. Then there are finitely many λ's such that $h_\lambda^{\alpha\beta} \neq 0$. The relation between $h_\lambda^{\alpha\beta}$ and $g_{\alpha\beta}^{\lambda}$ is given by the Riedtmann–Peng formula

$$h_\lambda^{\alpha\beta} = \frac{|\operatorname{Ext}^1_\Lambda(V_\alpha, V_\beta)_{V_\lambda}|}{|\operatorname{Hom}_\Lambda(V_\alpha, V_\beta)|} = g_{\alpha\beta}^{\lambda} a_\alpha a_\beta a_\lambda^{-1},$$

where $a_\alpha = |\operatorname{Aut}(V_\alpha)|$.

Define a symmetric bilinear form on $\mathcal{H}(\mathcal{A})$:

$$(u_\alpha, u_\beta) = \delta_{\alpha\beta} \frac{1}{|\operatorname{Aut}(V_\alpha)|} = \delta_{\alpha\beta} \frac{1}{a_\alpha}.$$

It induces bilinear forms $(\mathcal{H}(\mathcal{A}) \otimes \mathcal{H}(\mathcal{A})) \times (\mathcal{H}(\mathcal{A}) \widehat{\otimes} \mathcal{H}(\mathcal{A})) \to \mathbb{Q}$ and $(\mathcal{H}(\mathcal{A}) \widehat{\otimes} \mathcal{H}(\mathcal{A})) \times (\mathcal{H}(\mathcal{A}) \otimes \mathcal{H}(\mathcal{A})) \to \mathbb{Q}$ by setting

$$(a_1 \otimes a_2, b_1 \otimes b_2) = (a_1, b_1)(a_2, b_2).$$

Using Proposition 2.1 and the Riedtmann–Peng formula, we obtain

$$(a, bc) = \big(\delta(a), b \otimes c\big)$$

for any a, b, and c in $\mathcal{H}(\mathcal{A})$. The Drinfeld dual algebra of $\mathcal{H}(\mathcal{A})$ is a vector space $\mathcal{H}^{Dr}(\mathcal{A}) = \bigoplus_{\alpha\in\mathcal{P}} \mathbb{Q}v_\alpha$ with the multiplication

$$v_\alpha * v_\beta = \sum_{\lambda} h_\lambda^{\alpha\beta} v_\lambda.$$

Theorem 2.2 concludes an isomorphism $\Phi : \mathcal{H}^{Dr}(\mathcal{A}) \to \mathcal{H}(\mathcal{A})$ by setting $\Phi(v_\alpha) = a_\alpha u_\alpha$.

3. The derived Riedtmann–Peng formula

We recall some notations and results from [24]. Let k be a finite field with q elements, and let $\mathcal{C}$ be a (left) homologically finite k-additive triangulated category with the translation (or shift) functor $T = [1]$, that is, a finite k-additive triangulated category satisfying the following conditions (see [24]).

(1) the homomorphism space $\mathrm{Hom}(X,Y)$ for any two objects X and Y in $\mathcal{C}$ is a finite-dimensional k-space;

(2) the endomorphism ring $\mathrm{End}\,X$ for any indecomposable object X in $\mathcal{C}$ is a finite-dimensional local k-algebra;

(3) $\mathcal{C}$ is (left) locally homological finite; that is, $\sum_{i\geq 0}\dim_k \mathrm{Hom}(X[i],Y) < \infty$ for any X and Y in $\mathcal{C}$.

Note that the first two conditions imply the validity of the Krull–Schmidt theorem in $\mathcal{C}$, which means that any object in $\mathcal{C}$ can be uniquely decomposed into the direct sum of finitely many indecomposable objects up to isomorphism. For $X \in \mathcal{C}$, we denote by $[X]$ the isomorphism class of X.

For any X, Y, and Z in $\mathcal{C}$, we will use fg to denote the composition of morphisms $f: X \to Y$ and $g: Y \to Z$, and $|S|$ to denote the cardinality of a finite set S.

Given $X, Y, L \in \mathcal{C}$, put

$$W(X,Y;L) = \big\{(f,g,h) \in \mathrm{Hom}(X,L) \times \mathrm{Hom}(L,Y) \times \mathrm{Hom}\big(Y,X[1]\big) \;\big|\; X \xrightarrow{f} L \xrightarrow{g} Y \xrightarrow{h} X[1] \text{ is a triangle}\big\}.$$

There is a natural action of $\mathrm{Aut}\,X \times \mathrm{Aut}\,Y$ on $W(X,Y;L)$. The orbit of $(f,g,h) \in W(X,Y;L)$ is denoted by

$$(f,g,h)^\wedge := \big\{\big(af, gc^{-1}, ch\big(a[1]\big)^{-1}\big) \;\big|\; (a,c) \in \mathrm{Aut}\,X \times \mathrm{Aut}\,Y\big\}.$$

The orbit space is denoted by $V(X,Y;L) = \{(f,g,h)^\wedge \mid (f,g,h) \in W(X,Y;L)\}$. The radical of $\mathrm{Hom}(X,Y)$ is denoted by $\mathrm{radHom}(X,Y)$, which is the set

$$\big\{f \in \mathrm{Hom}(X,Y) \;\big|\; gfh \text{ is not an isomorphism for any } g: A \to X \text{ and } h: Y \to A \text{ with } A \in \mathcal{C} \text{ indecomposable}\big\}.$$

For any $L \xrightarrow{n} Z[1]$, there exist the decompositions $L = L_1(n) \oplus L_2(n)$, $Z[1] = Z_1[1](n) \oplus Z_2[1](n)$, and $b \in \mathrm{Aut}\,L$, $d \in \mathrm{Aut}\,Z$ such that $bn(d[1])^{-1} = \left(\begin{smallmatrix} n_{11} & 0 \\ 0 & n_{22}\end{smallmatrix}\right)$ and the induced map $n_{11}: L_1(n) \to Z_1[1](n)$ is an isomorphism and $n_{22}: L_2(n) \to Z_2[1](n)$ belongs to $\mathrm{radHom}(L_2(n), Z_2[1](n))$. The above decomposition only depends on the equivalence class of n up to an isomorphism. Let $\alpha = (l,m,n)^\wedge \in V(Z,L;M)$. If $\alpha = (l,m,n)^\wedge = (l',m',n')^\wedge$, then $L_1(n) = L_1(n')$. We can substitute $L_1(\alpha)$ for $L_1(n)$. To emphasize that n is related to α, we write $n(\alpha)$ as a substitute for n.

Denote by $\mathrm{Hom}(X,Y)_Z$ the subset of $\mathrm{Hom}(X,Y)$ consisting of the morphisms whose mapping cones are isomorphic to Z. For $X, Y \in \mathcal{C}$, we set

$$\{X,Y\} := \prod_{i>0} \big|\mathrm{Hom}\big(X[i],Y\big)\big|^{(-1)^i}.$$

By checking the stable subgroups of automorphism groups, we have the following proposition.

PROPOSITION 3.1 ([24, PROPOSITION 2.5])
For any M, L, and Z in $\mathcal{C}$, we have the following equalities:

$$\frac{|\operatorname{Hom}(M,L)_{Z[1]}|}{|\operatorname{Aut} L|}\cdot\frac{\{M,L\}}{\{Z,L\}\cdot\{L,L\}}=\sum_{\alpha\in V(Z,L;M)}\frac{|\operatorname{End} L_1(\alpha)|}{|\operatorname{Aut} L_1(\alpha)|},$$
$$\frac{|\operatorname{Hom}(Z,M)_{L}|}{|\operatorname{Aut} Z|}\cdot\frac{\{Z,M\}}{\{Z,L\}\cdot\{Z,Z\}}=\sum_{\alpha\in V(Z,L;M)}\frac{|\operatorname{End} L_1(\alpha)|}{|\operatorname{Aut} L_1(\alpha)|}.$$

Using this proposition, one can easily deduce the following corollary.

COROLLARY 3.2
For any X, Y, and L in $\mathcal{C}$, we have that

$$\frac{|\operatorname{Hom}(Y,X[1])_{L[1]}|}{|\operatorname{Aut} X|}\cdot\frac{\{Y,X[1]\}}{\{X,X\}}=\frac{|\operatorname{Hom}(L,Y)_{X[1]}|}{|\operatorname{Aut} L|}\cdot\frac{\{L,Y\}}{\{L,L\}}$$

and

$$\frac{|\operatorname{Hom}(Y[-1],X)_{L}|}{|\operatorname{Aut} Y|}\cdot\frac{\{Y[-1],X\}}{\{Y,Y\}}=\frac{|\operatorname{Hom}(X,L)_{Y}|}{|\operatorname{Aut} L|}\cdot\frac{\{X,L\}}{\{L,L\}}.$$

Let $\mathcal{A}$ be a finitary abelian category, and let $X,Y,L\in\mathcal{A}$. Define

$$E(X,Y;L)=\big\{(f,g)\in\operatorname{Hom}(X,L)\times\operatorname{Hom}(L,Y)\ \big|\ 0\to X\xrightarrow{f}L\xrightarrow{g}Y\to 0 \text{ is an exact sequence}\big\}.$$

The group $\operatorname{Aut} X\times\operatorname{Aut} Y$ acts freely on $E(X,Y;L)$ and the orbit of $(f,g)\in E(X,Y;L)$ is denoted by $(f,g)^\wedge:=\{(af,gc^{-1})\mid(a,c)\in\operatorname{Aut} X\times\operatorname{Aut} Y\}$. If the orbit space is denoted by $O(X,Y;L)=\{(f,g)^\wedge\mid(f,g)\in E(X,Y;L)\}$, then the *Hall number* $\mathrm{g}_{XY}^L=|O(X,Y;L)|$. It is easy to see that

$$\mathrm{g}_{XY}^L=\frac{|\mathcal{M}(X,L)_Y|}{|\operatorname{Aut} X|}=\frac{|\mathcal{M}(L,Y)_X|}{|\operatorname{Aut} Y|},$$

where $\mathcal{M}(X,L)_Y$ is the subset of $\operatorname{Hom}(X,L)$ consisting of monomorphisms $f:X\hookrightarrow L$ whose cokernels $\operatorname{Coker}(f)$ are isomorphic to Y and $\mathcal{M}(L,Y)_X$ is dually defined.

The equality in Corollary 3.2 can be considered as a generalization of the Riedtmann–Peng formula in abelian categories to homologically finite triangulated categories. Indeed, assume that $\mathcal{C}=\mathcal{D}^b(\mathcal{A})$ for a finitary abelian category $\mathcal{A}$ and X, Y, and $L\in\mathcal{A}$. Then one can obtain

$$\operatorname{Hom}\big(Y,X[1]\big)_{L[1]}=\operatorname{Ext}^1(Y,X)_L,\qquad \big\{Y,X[1]\big\}=\big|\operatorname{Hom}_{\mathcal{A}}(Y,X)\big|^{-1},$$

where $\operatorname{Ext}^1(X,Y)_L$ is the set of equivalence classes of extensions of Y by X with the middle term isomorphic to L and

$$g_{XY}^L=\frac{|\operatorname{Hom}(L,Y)_{X[1]}|}{|\operatorname{Aut} Y|},\qquad \{X,X\}=\{L,L\}=\{L,Y\}=0.$$

Under the assumption, Corollary 3.2 is reduced to the Riedtmann–Peng formula (see [19], [15])

$$\frac{|\operatorname{Ext}^1(Y,X)_L|}{|\operatorname{Hom}_{\mathcal{A}}(Y,X)|} = g_{XY}^L \cdot |\operatorname{Aut} X| \cdot |\operatorname{Aut} Y| \cdot |\operatorname{Aut} L|^{-1}.$$

For any X, Y, and $L \in \mathcal{C}$, set

$$F_{XY}^L = \frac{|\operatorname{Hom}(L,Y)_{X[1]}|}{|\operatorname{Aut} Y|} \cdot \frac{\{L,Y\}}{\{Y,Y\}} = \frac{|\operatorname{Hom}(X,L)_Y|}{|\operatorname{Aut} X|} \cdot \frac{\{X,L\}}{\{X,X\}}.$$

THEOREM 3.3 ([23], [24])

Let $\mathcal{H}(\mathcal{C})$ be the vector space over $\mathbb{Q}$ with the basis $\{u_{[X]} \mid X \in \mathcal{C}\}$. Endowed with the multiplication defined by

$$u_{[X]} \cdot u_{[Y]} = \sum_{[L]} F_{XY}^L u_{[L]},$$

$\mathcal{H}(\mathcal{C})$ is an associative algebra with the unit $u_{[0]}$.

The algebra $\mathcal{H}(\mathcal{C})$ is called the derived Hall algebra when $\mathcal{C}$ is a derived category. Here, we also use this name for a general left homologically finite triangulated category.

Now we define the Drinfeld dual of $\mathcal{H}(\mathcal{C})$. Set

$$h_L^{XY} = \left|\operatorname{Hom}_{\mathcal{C}}\big(Y,X[1]\big)_L\right| \cdot \{Y,X[1]\}.$$

Define a map $\delta : \mathcal{H}(\mathcal{C}) \to \mathcal{H}(\mathcal{C}) \widehat{\otimes} \mathcal{H}(\mathcal{C})$ by

$$\delta(u_{[L]}) = \sum_{[X],[Y]} h_L^{XY} u_{[X]} \otimes u_{[Y]}.$$

Define a symmetric bilinear form

$$(u_{[X]}, u_{[Y]}) = \delta_{[X],[Y]} \frac{1}{|\operatorname{Aut} X|\{X,X\}}.$$

It induces bilinear forms

$$\big(\mathcal{H}(\mathcal{C}) \widehat{\otimes} \mathcal{H}(\mathcal{C})\big) \times \big(\mathcal{H}(\mathcal{C}) \otimes \mathcal{H}(\mathcal{C})\big) \to \mathbb{Q} \qquad \text{and}$$
$$\big(\mathcal{H}(\mathcal{C}) \otimes \mathcal{H}(\mathcal{C})\big) \times \big(\mathcal{H}(\mathcal{C}) \widehat{\otimes} \mathcal{H}(\mathcal{C})\big) \to \mathbb{Q}$$

by setting

$$(a_1 \otimes a_2, b_1 \otimes b_2) = (a_1, b_1)(a_2, b_2).$$

Set $t_{[X]} = 1/(|\operatorname{Aut} X|\{X,X\})$. Then the derived Riedtmann–Peng formula in Corollary 3.2 can be written as

$$h_L^{XY} t_{[X]} t_{[Y]} = F_{XY}^L t_{[L]}$$

for any X, Y, and L in $\mathcal{C}$. Using Proposition 2.1, we have

$$(a, bc) = \big(\delta(a), b \otimes c\big), \quad \forall a, b, c \in \mathcal{H}(\mathcal{C}).$$

The Drinfeld dual algebra is a $\mathbb{Q}$-space $\mathcal{H}^{Dr}(\mathcal{C})$ with the basis $\{v_{[X]} \mid X \in \mathcal{C}\}$ and the multiplication

$$v_{[X]} * v_{[Y]} = \sum_{[L]} h_L^{XY} v_{[L]} = \{Y, X[1]\} \cdot \sum_{[L]} \big|\mathrm{Hom}\big(Y, X[1]\big)_{L[1]}\big| v_{[L]}$$
$$= \{Y[-1], X\} \cdot \sum_{[L]} \big|\mathrm{Hom}\big(Y[-1], X\big)_L\big| v_{[L]}.$$

Kontsevich and Soibelman [12] defined the motivic Hall algebra for an ind-constructible triangulated A_∞-category. We can define the finite-field version of a motivic Hall algebra for a homologically finite k-additive triangulated category, which is just $\mathcal{H}^{Dr}(\mathcal{C})$. Following Theorem 2.2, we have the immediate result.

COROLLARY 3.4 ([12, PROPOSITION 6.12])

The map $\Phi : \mathcal{H}^{Dr}(\mathcal{C}) \to \mathcal{H}(\mathcal{C})$ *by* $\Phi(v_{[X]}) = |\operatorname{Aut} X| \cdot \{X, X\} \cdot u_{[X]}$ *for any* $X \in \mathcal{C}$ *is an algebraic isomorphism between* $\mathcal{H}^{Dr}(\mathcal{C})$ *and* $\mathcal{H}(\mathcal{C})$.

Then Theorem 3.3 implies that the algebra $\mathcal{H}^{Dr}(\mathcal{C})$ is an associative algebra.

To introduce the extended twisted derived Hall algebra $\mathcal{H}_{et}(\mathcal{C})$ of $\mathcal{H}(\mathcal{C})$, we need more assumptions. Assume that $\mathcal{C}$ is homologically finite; that is,

$$\sum_{i \in \mathbb{Z}} \dim_k \mathrm{Hom}\big(X[i], Y\big) < \infty$$

for any X and Y in $\mathcal{C}$. For example, the derived category $\mathcal{C} = \mathcal{D}^b(\mathcal{A})$ for a small finitary abelian category $\mathcal{A}$ is homologically finite. Let $K(\mathcal{C})$ be the Grothendieck group of $\mathcal{C}$. One can define a bilinear form $\langle \bullet, \bullet \rangle : K(\mathcal{C}) \times K(\mathcal{C}) \to \mathbb{Z}$ by setting

$$\big\langle [X], [Y] \big\rangle = \sum_{i \in \mathbb{Z}} (-1)^i \dim_k \mathrm{Hom}_{\mathcal{C}}\big(X, Y[i]\big)$$

for $X, Y \in \mathcal{C}$. It induces the symmetric bilinear form $(\bullet, \bullet)$ on $K(\mathcal{C})$ by defining $([X], [Y]) = \langle [X], [Y] \rangle + \langle [Y], [X] \rangle$. For convenience, for any object $X \in \mathcal{C}$, we use the same notation $[X]$ for the isomorphism class and its image in $K(\mathcal{C})$. It is easy to check that this bilinear form is well defined. Set $v = \sqrt{q}$. Then $\mathcal{H}_{et}(\mathcal{C})$ is given by the $\mathbb{Q}(v)$-space with basis $\{K_\alpha u_{[X]} \mid \alpha \in K(\mathcal{C}), X \in \mathcal{C}\}$ and the multiplication defined by

$$(K_\alpha u_{[X]}) \circ (K_\beta u_{[Y]}) = v^{\langle [X],[Y] \rangle - (\beta, [X])} K_{\alpha+\beta} u_{[X]} \cdot u_{[Y]}.$$

Note that $K_0 = u_0 = 1$.

PROPOSITION 3.5

The algebra $H_{et}(\mathcal{C})$ *is associative.*

Proof

By definition, we have

$$\big[(K_\alpha u_{[X]}) \circ (K_\beta u_{[Y]})\big] \circ (K_\gamma u_{[Z]})$$
$$= \big[v^{\langle [X],[Y] \rangle - (\beta, [X])} K_{\alpha+\beta} u_{[X]} \cdot u_{[Y]}\big] \circ (K_\gamma u_{[Z]})$$
$$= v^{\langle [X],[Y] \rangle + \langle [L],[Z] \rangle - (\beta,[X]) - (\gamma,[L])} K_{\alpha+\beta+\gamma} \sum_{[L]} F_{XY}^L u_{[L]} \cdot u_{[Z]}$$

$$= v^{\langle [X],[Y]\rangle + \langle [X]+[Y],[Z]\rangle - (\beta,[X]) - (\gamma,[X]+[Y])} K_{\alpha+\beta+\gamma}(u_{[X]} \cdot u_{[Y]}) \cdot u_{[Z]}$$
$$= v^{\langle [X],[Y]+[Z]\rangle + \langle [Y],[Z]\rangle - (\beta+\gamma,[X]) - (\gamma,[Y])} K_{\alpha+\beta+\gamma} u_{[X]} \cdot (u_{[Y]} \cdot u_{[Z]})$$
$$= (K_\alpha u_{[X]}) \circ \big[(K_\beta u_{[Y]})\big] \circ (K_\gamma u_{[Z]})]. \qquad \square$$

Dually, one can define $\mathcal{H}^-_{et}(\mathcal{C})$ with the basis $\{K_\alpha u^-_{[X]} \mid \alpha \in K(\mathcal{C}), X \in \mathcal{C}\}$ and the multiplication

$$(K_\alpha u^-_{[X]}) \circ (K_\beta u^-_{[Y]}) = v^{\langle [X],[Y]\rangle + (\beta,[X])} K_{\alpha+\beta} u^-_{[X]} \cdot u^-_{[Y]},$$

where

$$u^-_{[X]} \cdot u^-_{[Y]} = \sum_{[L]} F^L_{XY} u^-_{[L]}.$$

There exist a map $\delta : \mathcal{H}_{et}(\mathcal{C}) \to \mathcal{H}_{et}(\mathcal{C}) \widehat{\otimes} \mathcal{H}_{et}(\mathcal{C})$ by setting

$$\delta(K_\gamma u_{[L]}) = \sum_{[X],[Y]} v^{\langle [X],[Y]\rangle} h^{XY}_L K_\gamma u_{[X]} K_{[Y]} \otimes K_\gamma u_{[Y]}$$

and a bilinear form on $\mathcal{H}_{et}(\mathcal{C}) \times \mathcal{H}^-_{et}(\mathcal{C})$ defined by

$$(K_\alpha u_{[X]}, K_\beta u^-_{[Y]}) = v^{-(\alpha,\beta) - (\beta,[X]) + (\alpha,[Y])} \delta_{[X],[Y]} \frac{1}{|\operatorname{Aut} X| \{X,X\}}.$$

It naturally induces a bilinear form

$$\big(\mathcal{H}_{et}(\mathcal{C}) \widehat{\otimes} \mathcal{H}_{et}(\mathcal{C})\big) \times \big(\mathcal{H}^-_{et}(\mathcal{C}) \otimes \mathcal{H}^-_{et}(\mathcal{C})\big) \to \mathbb{Q}(v).$$

PROPOSITION 3.6

For any $a \in \mathcal{H}_{et}(\mathcal{C})$ and $b, c \in \mathcal{H}^-_{et}(\mathcal{C})$, we have

$$(a, bc) = \big(\delta(a), b \otimes c\big).$$

Proof

It is enough to consider the case when $a = K_\gamma u_{[L]}$, $b = K_\alpha u^-_{[X]}$, and $c = K_\beta u^-_{[Y]}$. By definition,

$$\big(K_\gamma u_{[L]}, (K_\alpha u^-_{[X]}) \circ (K_\beta u^-_{[Y]})\big)$$
$$= v^{\langle [X],[Y]\rangle + (\beta,[X])} (K_\gamma u_{[L]}, K_{\alpha+\beta} u^-_{[X]} \cdot u^-_{[Y]})$$
$$= v^{\langle [X],[Y]\rangle + (\beta,[X])} F^L_{XY} (K_\gamma u_{[L]}, K_{\alpha+\beta} u^-_{[L]})$$
$$= v^{\langle [X],[Y]\rangle + (\beta,[X]) - (\gamma,\alpha+\beta) - (\alpha+\beta,L) + (\gamma,[L])} F^L_{XY} \cdot t_{[L]}$$
$$= v^{\langle [X],[Y]\rangle - (\gamma,\alpha+\beta-[X]-[Y]) - (\alpha,[X]+[Y]) - (\beta,[Y])} F^L_{XY} \cdot t_{[L]}.$$

Similarly, by definition, we have

$$\big(\delta(K_\gamma u_{[L]}), (K_\alpha u^-_{[X]}) \otimes (K_\beta u^-_{[Y]})\big)$$
$$= h^{XY}_L v^{\langle [X],[Y]\rangle} \big((K_\gamma u_{[X]} K_{[Y]}) \otimes (K_\gamma u_{[Y]}), (K_\alpha u^-_{[X]}) \otimes (K_\beta u^-_{[Y]})\big)$$

$$= h_L^{XY} v^{\langle [X],[Y]\rangle - ([X],[Y])} (K_{\gamma+[Y]} u_{[X]}, K_\alpha u^-_{[X]}) \cdot (K_\gamma u_{[Y]}, K_\beta u^-_{[Y]})$$
$$= h_L^{XY} v^{\langle [X],[Y]\rangle - ([X],[Y]) - (\gamma+[Y],\alpha) - (\alpha,[X]) + (\gamma+[Y],[X]) - (\gamma,\beta) - (\beta,[Y]) + (\gamma,[Y])}$$
$$\times t_{[X]} t_{[Y]}$$
$$= v^{\langle [X],[Y]\rangle + (\beta,[X]) - (\gamma,\alpha+\beta) - (\alpha+\beta,L) + (\gamma,[L])} h_L^{XY} t_{[X]} t_{[Y]}.$$

This completes the proof. □

PROPOSITION 3.7
Let $\mathcal{H}_{et}^{Dr}$ be the $\mathbb{Q}(v)$-space with the basis $\{K_\alpha \theta_{[X]} \mid \alpha \in K(\mathcal{C}), X \in \mathcal{C}\}$ and the multiplication given by

$$K_\alpha \theta_{[X]} * K_\beta \theta_{[Y]} = v^{(\beta,[X]) - 2(\alpha,\beta)} \sum_{[L]} v^{\langle [X],[Y]\rangle} h_L^{XY} K_{\alpha+\beta} \theta_{[L]}.$$

Then the map $\Phi : \mathcal{H}_{et}^{Dr} \to \mathcal{H}_{et}^-$ defined by $\Phi(K_\alpha \theta_{[X]}) = v^{(\alpha,\alpha)} t_{[X]}^{-1} K_\alpha u^-_{[X]}$ is an isomorphism of algebras.

By this proposition, we can view $\mathcal{H}_{et}^{Dr}$ as the Drinfeld double of $\mathcal{H}_{et}^-$.

4. Two symmetries

Consider the following commutative diagram in $\mathcal{C}$, which is a pushout and a pullback in the same time:

$$\begin{tikzcd} L' \arrow[r, "f'"] \arrow[d, "m'"] & M \arrow[d, "m"] \\ X \arrow[r, "f"] & L \end{tikzcd}$$

Applying the octahedral axiom, one can obtain the following commutative diagram:

$$\begin{tikzcd} Z \arrow[r, equal] \arrow[d, dotted, "l'"] & Z \arrow[d, "l"] & & \\ L' \arrow[r, "f'"] \arrow[d, "m'"] & M \arrow[r, dotted, "g'"] \arrow[d, "m"] & Y \arrow[r, dotted, "h'"] \arrow[d, equal] & L'[1] \arrow[d, "m'[1]"] \\ X \arrow[r, "f"] \arrow[d, dotted, "n'"] & L \arrow[r, "g"] \arrow[d, "n"] & Y \arrow[r, "h"] & X[1] \\ Z[1] \arrow[r, equal] & Z[1] & & \end{tikzcd} \tag{4.1}$$

with rows and columns being distinguished triangles, and a distinguished triangle

$$L' \xrightarrow{(f' \ \ -m')} M \oplus X \xrightarrow{\binom{m}{f}} L \xrightarrow{\theta} L'[1]. \tag{4.2}$$

The above triangle induces two sets

$$\begin{aligned}
&\mathrm{Hom}(M \oplus X, L)^{Y,Z[1]}_{L'[1]} \\
&:= \left\{ \begin{pmatrix} m \\ f \end{pmatrix} \in \mathrm{Hom}(M \oplus X, L) \,\middle|\, \right. \\
&\qquad \left. \mathrm{Cone}(f) \simeq Y, \mathrm{Cone}(m) \simeq Z[1], \text{ and } \mathrm{Cone}\begin{pmatrix} m \\ f \end{pmatrix} \simeq L'[1] \right\}
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Hom}(L', M \oplus X)^{Y,Z[1]}_{L} \\
&:= \big\{ (f', -m') \in \mathrm{Hom}(L', M \oplus X) \,\big| \\
&\qquad \mathrm{Cone}(f') \simeq Y, \mathrm{Cone}(m') \simeq Z[1], \text{ and } \mathrm{Cone}(f', -m') \simeq L \big\}.
\end{aligned}$$

SYMMETRY I

The orbit space of $\mathrm{Hom}(M \oplus X, L)^{Y,Z[1]}_{L'[1]}$ *under the action of* $\mathrm{Aut}\, L$ *and the orbit space of* $\mathrm{Hom}(L', M \oplus X)^{Y,Z[1]}_{L}$ *under the action of* $\mathrm{Aut}\, L'$ *coincide. More explicitly, the symmetry implies the identity:*

$$\begin{aligned}
&\frac{|\,\mathrm{Hom}(M \oplus X, L)^{Y,Z[1]}_{L'[1]}|}{|\,\mathrm{Aut}\, L|} \frac{\{M \oplus X, L\}}{\{L', L\}\{L, L\}} \\
&= \frac{|\,\mathrm{Hom}(L', M \oplus X)^{Y,Z[1]}_{L}|}{|\,\mathrm{Aut}\, L'|} \frac{\{L', M \oplus X\}}{\{L', L\}\{L', L'\}}.
\end{aligned}$$

Proof

The equality is a direct application of Proposition 3.1 to the triangle (4.2). □

Roughly speaking, Symmetry I compares

$$L' \xrightarrow{(f' \ \ -m')} M \oplus X \qquad \text{and} \qquad M \oplus X \xrightarrow{\binom{m}{f}} L$$

in the triangle (4.2).

The diagram (4.1) induces a *new symmetry* which compares

$$L' \xrightarrow{f'} M \xrightarrow{m} L \qquad \text{and} \qquad L' \xrightarrow{m'} X \xrightarrow{f} L.$$

Using the derived Riedtmann–Peng formula (see Corollary 3.2), we have

$$\frac{|\,\mathrm{Hom}(Y[-1], L')_M|}{|\,\mathrm{Aut}\, Y|} \cdot \frac{\{Y[-1], L'\}}{\{Y, Y\}} = \frac{|\,\mathrm{Hom}(L', M)_Y|}{|\,\mathrm{Aut}\, M|} \cdot \frac{\{L', M\}}{\{M, M\}},$$

$$\frac{|\operatorname{Hom}(L,Z[1])_{M[1]}|}{|\operatorname{Aut} Z|}\cdot\frac{\{L,Z[1]\}}{\{Z,Z\}}=\frac{|\operatorname{Hom}(M,L)_{Z[1]}|}{|\operatorname{Aut} M|}\cdot\frac{\{M,L\}}{\{M,M\}},$$
$$\frac{|\operatorname{Hom}(X,Z[1])_{L'[1]}|}{|\operatorname{Aut} Z|}\cdot\frac{\{X,Z[1]\}}{\{Z,Z\}}=\frac{|\operatorname{Hom}(L',X)_{Z[1]}|}{|\operatorname{Aut} L'|}\cdot\frac{\{L',X\}}{\{L',L'\}},$$

and

$$\frac{|\operatorname{Hom}(Y[-1],X)_{L}|}{|\operatorname{Aut} Y|}\cdot\frac{\{Y[-1],X\}}{\{Y,Y\}}=\frac{|\operatorname{Hom}(X,L)_{Y}|}{|\operatorname{Aut} L|}\cdot\frac{\{X,L\}}{\{L,L\}}.$$

Hence, one can convert to compare

$$Y\xrightarrow{h'}L'[1],\qquad L\xrightarrow{n}Z[1]\qquad\text{and}\qquad X\xrightarrow{n'}Z[1],\qquad Y\xrightarrow{h}X[1]$$

in (4.1). To describe the second symmetry, we need to introduce some notations. Fix X, Y, Z, M, L, and L', define

$$\begin{aligned}\mathcal{D}_{L,L'}=\big\{(m,f,h,n)\in{}&\operatorname{Hom}(M,L)\times\operatorname{Hom}(X,L)\\&\times\operatorname{Hom}\big(Y,X[1]\big)\times\operatorname{Hom}\big(L,Z[1]\big)\mid\\&(m,f,h,n)\text{ induces a diagram with the form of (4.1)}\big\},\end{aligned}$$

and define

$$\begin{aligned}\mathcal{D}_{L',L}=\big\{(f',m',h',n')\in{}&\operatorname{Hom}(L',X)\times\operatorname{Hom}(L',M)\\&\times\operatorname{Hom}\big(Y,L'[1]\big)\times\operatorname{Hom}\big(X,Z[1]\big)\mid\\&(m',f',h',n')\text{ induces a diagram with the form of (4.1)}\big\}.\end{aligned}$$

Here, "(m,f,h,n) induces a diagram with the form of (4.1)" means that there exist morphisms m', f', h', n', g, g', l, l' such that all morphisms constitute a diagram as in (4.1). The crucial point is that the following diagram

$$\begin{tikzcd} L' \arrow[r, "f'"] \arrow[d, "m'"] & M \arrow[d, "m"] \\ X \arrow[r, "f"] & L \end{tikzcd}$$

is both a pushout and a pullback and rows and columns in (4.1) are distinguished triangles. Note that the pair (f',m') is uniquely determined by (m,f,h,n) up to isomorphisms as required, so the above notation is well defined.

There exist natural projections

$$p:\mathcal{D}_{L,L'}\to\operatorname{Hom}\big(Y,X[1]\big)\times\operatorname{Hom}\big(L,Z[1]\big),$$
$$i_1:\operatorname{Hom}\big(Y,X[1]\big)\times\operatorname{Hom}\big(L,Z[1]\big)\to\operatorname{Hom}\big(Y,X[1]\big),$$

and

$$i_2:\operatorname{Hom}\big(Y,X[1]\big)\times\operatorname{Hom}\big(L,Z[1]\big)\to\operatorname{Hom}\big(L,Z[1]\big).$$

The image of $i_1 \circ p$ is denoted by $\mathrm{Hom}(Y, X[1])^{L'}_{L[1]}$, and given $h \in \mathrm{Hom}(Y, X[1])^{L'}_{L[1]}$, define $\mathrm{Hom}(L, Z[1])^{h,L'}_{M[1]}$ to be $i_2 \circ p^{-1} \circ i_1^{-1}(h)$. It is clear that

$$\mathrm{Hom}\big(Y, X[1]\big)_{L[1]} = \bigsqcup_{[L']} \mathrm{Hom}\big(Y, X[1]\big)^{L'}_{L[1]}.$$

Similarly, there exist projections

$$q : \mathcal{D}_{L',L} \to \mathrm{Hom}\big(Y, L'[1]\big) \times \mathrm{Hom}\big(X, Z[1]\big),$$
$$j_1 : \mathrm{Hom}\big(Y, L'[1]\big) \times \mathrm{Hom}\big(X, Z[1]\big) \to \mathrm{Hom}\big(Y, L'[1]\big),$$

and

$$j_2 : \mathrm{Hom}\big(Y, L'[1]\big) \times \mathrm{Hom}\big(X, Z[1]\big) \to \mathrm{Hom}\big(X, Z[1]\big).$$

The image of $j_1 \circ q$ is denoted by $\mathrm{Hom}(X, Z[1])^{L}_{L'[1]}$, and for any $n' \in \mathrm{Hom}(X, Z[1])_{L'[1]}$, we denote $j_2 \circ p^{-1} \circ j_1^{-1}(n')$ by $\mathrm{Hom}(Y, L'[1])^{n',L}_{M[1]}$.

SYMMETRY II

- *Fix* $h \in \mathrm{Hom}(Y, X[1])^{L'}_{L[1]}$*; then there exists a surjective map*

$$f_* : \mathrm{Hom}\big(L, Z[1]\big)^{h,L'}_{M[1]} \to \mathrm{Hom}\big(X, Z[1]\big)^{L}_{L'[1]}$$

such that the cardinality of any fiber is

$$\big|(f^*)^{-1}\big| := \big|\mathrm{Hom}\big(Y, Z[1]\big)\big| \cdot \big\{X \oplus Y, Z[1]\big\} \cdot \big\{L, Z[1]\big\}^{-1}.$$

- *Fix* $n' \in \mathrm{Hom}(X, Z[1])^{L}_{L'[1]}$*; then there exists a surjective map*

$$(m')_* : \mathrm{Hom}\big(Y, L'[1]\big)^{n',L}_{M[1]} \to \mathrm{Hom}\big(Y, X[1]\big)^{L'}_{L[1]}$$

such that the cardinality of any fiber is

$$\big|(m')^{-1}\big| := \big|\mathrm{Hom}\big(Y, Z[1]\big)\big| \cdot \big\{Y, X[1] \oplus Z[1]\big\} \cdot \big\{Y, L'[1]\big\}^{-1}.$$

- $|(f_*)^{-1}| \cdot \{Y, X[1]\} \cdot \{L, Z[1]\} = |(m')_*^{-1}| \cdot \{X, Z[1]\} \cdot \{Y, L'[1]\}$.

Proof
Given $h \in \mathrm{Hom}(Y, X[1])^{L'}_{L[1]}$, there exists a triangle

$$\alpha : X \xrightarrow{f} L \xrightarrow{g} Y \xrightarrow{h} X[1].$$

Applying the functor $\mathrm{Hom}(\bullet, Z[1])$ on the triangle, one can obtain a long exact sequence

$$\cdots \longrightarrow \mathrm{Hom}\big(Y, Z[1]\big) \xrightarrow{u} \mathrm{Hom}\big(L, Z[1]\big) \xrightarrow{v} \mathrm{Hom}\big(X, Z[1]\big) \longrightarrow \cdots.$$

Then the cardinality of $\mathrm{Im}(u)$ is $|\mathrm{Hom}(Y, Z[1])| \cdot \{X \oplus Y, Z[1]\} \cdot \{L, Z[1]\}^{-1}$. The map f_* is the restriction of v to $\mathrm{Hom}(L, Z[1])^{h,L'}_{M[1]}$. By definition, f_* is epic and the fiber is isomorphic to $\mathrm{Ker}(v) = \mathrm{Im}(u)$. Thus the first statement is obtained.

The second statement can be proved in the same way. The third statement is a direct confirmation. □

Note that there also exist projections

$$p_{12}: \mathcal{D}_{L,L'} \to \mathrm{Hom}(M \oplus X, L)$$

with the image $\mathrm{Hom}(M \oplus X, L)_{L'[1]}^{Y,Z[1]}$ and

$$q_{12}: \mathcal{D}_{L',L} \to \mathrm{Hom}(L', M \oplus X)$$

with the image $\mathrm{Hom}(L', M \oplus X)_{L}^{Y,Z[1]}$. Symmetry I characterizes the relation between $\mathrm{Im}\, p_{12}$ and $\mathrm{Im}\, q_{12}$. Meanwhile, Symmetry II characterizes the relation between $\mathrm{Im}\, p$ and $\mathrm{Im}\, q$. The relation between $\mathrm{Im}\, p_{12}$ and $\mathrm{Im}\, p$ ($\mathrm{Im}\, q_{12}$ and $\mathrm{Im}\, q$) is implicitly shown by the derived Riedtmann–Peng formula (see Corollary 3.2). More explicitly, consider the projections

$$t_1: \mathrm{Hom}(M \oplus X, L) \to \mathrm{Hom}(X, L),$$

$$t_2: \mathrm{Hom}(M \oplus X, L) \to \mathrm{Hom}(M, L),$$

$$s_1: \mathrm{Hom}(L', M \oplus X) \to \mathrm{Hom}(L', X),$$

and

$$s_2: \mathrm{Hom}(L', M \oplus X) \to \mathrm{Hom}(L', M).$$

Using Corollary 3.2, we obtain that

$$\frac{|\mathrm{Hom}(Y, X[1])_{L[1]}^{L'}|}{|\mathrm{Aut}\, Y|} \cdot \frac{\{Y, X[1]\}}{\{Y, Y\}} = \frac{|\mathrm{Im}\, t_1 \circ p_{12}|}{|\mathrm{Aut}\, L|} \cdot \frac{\{X, L\}}{\{L, L\}}.$$

Given a triangle $\alpha: X \xrightarrow{f} L \xrightarrow{g} Y \xrightarrow{h} X[1]$ with $h \in \mathrm{Hom}(Y, X[1])_{L[1]}^{L'}$, applying Corollary 3.2 again, we have

$$\frac{|\mathrm{Hom}(L, Z[1])_{M[1]}^{h,L'}|}{|\mathrm{Aut}\, Z|} \cdot \frac{\{L, Z[1]\}}{\{Z, Z\}} = \frac{|t_2 \circ p_{12}^{-1} \circ t_1^{-1}(f)|}{|\mathrm{Aut}\, M|} \cdot \frac{\{M, L\}}{\{M, M\}}.$$

In the same way, we have

$$\frac{|\mathrm{Hom}(X, Z[1])_{L'[1]}^{L}|}{|\mathrm{Aut}\, Z|} \cdot \frac{\{X, Z[1]\}}{\{Z, Z\}} = \frac{|\mathrm{Im}\, s_1 \circ q_{12}|}{|\mathrm{Aut}\, L'|} \cdot \frac{\{L', X\}}{\{L', L'\}},$$

and

$$\frac{|\mathrm{Hom}(Y, L'[1])_{M[1]}^{n',L}|}{|\mathrm{Aut}\, Y|} \cdot \frac{\{Y, L'[1]\}}{\{Y, Y\}} = \frac{|s_2 \circ q_{12}^{-1} \circ s_1^{-1}(f)|}{|\mathrm{Aut}\, M|} \cdot \frac{\{L', M\}}{\{M, M\}}.$$

The above four identities induce the equivalence of Symmetry I and Symmetry II.

Here, we sketch the proof of Theorem 3.3. Proving $u_{[Z]} * (u_{[X]} * u_{[Y]}) = (u_{[Z]} * u_{[X]}) * u_{[Y]}$ is equivalent to proving that

$$\sum_{[L]} F_{XY}^{L} F_{ZL}^{M} = \sum_{[L']} F_{ZX}^{L'} F_{L'Y}^{M}.$$

We can check directly that the left-hand side equals the following:

$$\frac{1}{|\operatorname{Aut} X|\cdot\{X,X\}}\sum_{[L]}\sum_{[L']}\frac{|\operatorname{Hom}(M\oplus X,L)_{L'[1]}^{Y,Z[1]}|}{|\operatorname{Aut} L|}\cdot\frac{\{M\oplus X,L\}}{\{L,L\}},$$

and that the right-hand side equals

$$\frac{1}{|\operatorname{Aut} X|\cdot\{X,X\}}\sum_{[L']}\sum_{[L]}\frac{|\operatorname{Hom}(L',M\oplus X)_{L}^{Y,Z[1]}|}{|\operatorname{Aut} L'|}\cdot\frac{\{L',M\oplus X\}}{\{L',L'\}}.$$

Symmetry I naturally gives that the left-hand and right-hand sides are equal, and then the proof of Theorem 3.3 is obtained immediately.

By using Symmetry II, we can provide a direct proof of the associativity of $\mathcal{H}^{Dr}(\mathcal{C})$.

THEOREM 4.1 ([12, PROPOSITION 6.10])
The algebra $\mathcal{H}^{Dr}(\mathcal{C})$ is associative.

Proof
For any X, Y, and Z in $\mathcal{C}$, we need to prove that

$$v_{[Z]}*(v_{[X]}*v_{[Y]})=(v_{[Z]}*v_{[X]})*v_{[Y]}.$$

By the definition of the multiplication, it is equivalent to proving that

$$\sum_{[L]}\{Y,X[1]\}\{L,Z[1]\}\big|\operatorname{Hom}\big(Y,X[1]\big)_{L[1]}\big|\cdot\big|\operatorname{Hom}\big(L,Z[1]\big)_{M[1]}\big|$$
$$=\sum_{[L']}\{X,Z[1]\}\{Y,L'[1]\}\big|\operatorname{Hom}\big(X,Z[1]\big)_{L'[1]}\big|\cdot\big|\operatorname{Hom}\big(Y,L'[1]\big)_{M[1]}\big|.$$

Following the first statement of Symmetry II, the left-hand side is equal to

$$\sum_{[L],[L']}\{Y,X[1]\}\{L,Z[1]\}\big|\operatorname{Hom}\big(Y,X[1]\big)_{L[1]}^{L'}\big|\cdot\big|\operatorname{Hom}\big(X,Z[1]\big)_{L'[1]}^{L}\big|$$
$$\cdot\big|\operatorname{Hom}\big(Y,Z[1]\big)\big|\cdot\big\{X\oplus Y,Z[1]\big\}\cdot\big\{L,Z[1]\big\}^{-1}.$$

Following the second statement of Symmetry II, the right-hand side is equal to

$$\sum_{[L'],[L]}\{X,Z[1]\}\{Y,L'[1]\}\big|\operatorname{Hom}\big(X,Z[1]\big)_{L'[1]}^{L}\big|\cdot\big|\operatorname{Hom}\big(Y,X[1]\big)_{L[1]}^{L'}\big|$$
$$\cdot\big|\operatorname{Hom}\big(Y,Z[1]\big)\big|\cdot\big\{Y,X[1]\oplus Z[1]\big\}\cdot\big\{Y,L'[1]\big\}^{-1}.$$

The equality of the left-hand and right-hand sides is just the third statement of Symmetry II. □

5. Motivic Hall algebras

Let $\mathbb{K}$ be an algebraically closed field. An ind-constructible set is a countable union of nonintersecting constructible sets.

EXAMPLE 5.1 ([7], [25])

Let $\mathbb{C}$ be the complex field, and let A be a finite-dimensional algebra $\mathbb{C}Q/I$ with indecomposable projective modules P_i, $i=1,\dots,l$. Given a projective complex $P^\bullet=(P^i,\partial_i)_{i\in\mathbb{Z}}$ with $P^i=\bigoplus_{j=1}^{l}e^i_jP_j$. We denote by $\underline{e}^i$ the vector $(e^i_1,e^i_2,\dots,e^i_l)$. The sequence denoted by $\underline{\mathbf{e}}=\underline{\mathbf{e}}(P^\bullet)=(\underline{e}^i)_{i\in\mathbb{Z}}$ is called the projective dimension sequence of $P^\bullet$. We assume that only finitely many $\underline{e}^i$ in $\underline{\mathbf{e}}$ are nonzero. Define $\mathcal{P}(A,\underline{\mathbf{e}})$ to be the subset of

$$\prod_{i\in\mathbb{Z}}\mathrm{Hom}_A(P^i,P^{i+1})=\prod_{i\in\mathbb{Z}}\mathrm{Hom}_A\Big(\bigoplus_{j=1}^{l}e^i_jP_j,\bigoplus_{j=1}^{l}e^{i+1}_jP_j\Big),$$

which consists of elements $(\partial_i:P^i\to P^{i+1})_{i\in\mathbb{Z}}$ such that $\partial_{i+1}\partial_i=0$ for all $i\in\mathbb{Z}$. It is an affine variety with a natural action of the algebraic group $G_{\underline{\mathbf{e}}}=\prod_{i\in\mathbb{Z}}\mathrm{Aut}_A(P^i)$. Let $K_0(\mathcal{D}^b(A))$, or simply K_0, be the Grothendieck group of the derived category $\mathcal{D}^b(A)$, and let $\underline{\dim}:\mathcal{D}^b(A)\to K_0(\mathcal{D}^b(A))$ be the canonical surjection. It induces a canonical surjection from the abelian group of dimension vector sequences to K_0; we still denote it by $\underline{\dim}$. Given $\mathbf{d}\in K_0$, the set

$$\mathcal{P}(A,\mathbf{d})=\bigsqcup_{\underline{\mathbf{e}}\in\underline{\dim}^{-1}(\mathbf{d})}\mathcal{P}(A,\underline{\mathbf{e}})$$

is an ind-constructible set.

We recall the notion of motivic invariants of quasiprojective varieties in [9] (see also [1]). Suppose that Λ is a commutative $\mathbb{Q}$-algebra with identity 1. Let $\Upsilon:$ $\{$isomorphism classes $[X]$ of quasiprojective $\mathbb{K}$-varieties $X\}\to\Lambda$ satisfy that

(1) $\Upsilon([X])=\Upsilon([Z])+\Upsilon([U])$ for a closed subvariety $Z\subseteq X$ and $U=X\setminus Z$;
(2) $\Upsilon([X\times Y])=\Upsilon([X])\Upsilon([Y])$;
(3) if we write $\Upsilon([\mathbb{K}])=\mathbb{L}$, then $\mathbb{L}$ and $\mathbb{L}^k-1$ for $k=1,2,\dots$ are invertible in Λ.

Let X be a constructible set over $\mathbb{K}$, and let G be an affine algebraic group acting on X. Then $(X,G)=(X,G,\alpha)$ is called a constructible stack (see [12, Section 4.2]), where α is an action of G on X. In [12, Section 4.2], the authors defined the 2-category of constructible stacks. Define $\mathrm{Mot}_{st}((X,G))=\mathrm{Mot}_{st}((X,G),(\Upsilon,\Lambda))$ to be the Λ-module generated by equivalence classes of 1-morphisms of constructible stacks $[(Y,H)\to(X,G)]$ with the following relations:

(1) $[(Y_1,G_1)\sqcup(Y_2,G_2)\to(X,G)]=[(Y_1,G_1)\to(X,G)]+[(Y_2,G_2)\to(X,G)]$;
(2) $[(Y,H)\to(X,G)]=[(Z\times A^d_{\mathbb{K}},H)\to(X,G)]$ if $Y\to Z$ is an H-equivariant constructible vector bundle of rank d;
(3) if we let (Y,H) be a constructible stack, let U be a quasiprojective $\mathbb{K}$-variety with trivial action of H, and let $\pi:(Y,H)\times U\to(Y,H)$, then $[\rho:(Y,H)\times U\to(X,G)]=\Upsilon(U)[\rho\circ\pi:(Y,H)\to(X,G)]$.

Let us recall the definition of motivic Hall algebras from [12]. Let $\mathcal{C}$ be an ind-constructible triangulated A_∞-category over $\mathbb{K}$, and objects in $\mathcal{C}$ form an ind-constructible set $\mathfrak{Obj}(\mathcal{C}) = \bigsqcup_{i\in I} \mathcal{X}_i$ for countable constructible sets $\mathcal{X}_i$ with the action of an affine algebraic group G_i on $\mathcal{X}_i$. For $X, Y \in \mathfrak{Obj}(\mathcal{C})$, set

$$\{X,Y\} := \mathbb{L}^{\sum_{i>0}(-1)^i \dim_{\mathbb{C}} \operatorname{Hom}(X[i],Y)}.$$

For any $i, j \in I$, consider the maps $\Phi_1 : \mathcal{X}_i \times \mathcal{X}_j \to \Lambda$ sending (M,N) to

$$\Upsilon\big(\big[\operatorname{Hom}_{\mathcal{C}}(M,N)\big]\big) = \mathbb{L}^{\dim_{\mathbb{K}} \operatorname{Hom}_{\mathcal{C}}(M,N)},$$

$\Phi_2 : \mathcal{X}_i \times \mathcal{X}_j \to \Lambda$ mapping (M,N) to $\{M,N\}$, and $\Phi_3 : \mathcal{X}_i \to \Lambda$ sending M to $\Upsilon([\operatorname{Aut}(M)])$ (see Proposition 5.3). In the following, we assume that all three maps are constructible functions.

Consider the Λ-module

$$\mathcal{MH}(\mathcal{C}) = \bigoplus_{i\in I} \operatorname{Mot}_{st}\big((\mathcal{X}_i, G_i)\big).$$

One can endow the module with the multiplication

$$\big[\pi_1 : \mathcal{S}_1 \to \mathfrak{Obj}(\mathcal{C})\big] * \big[\pi_2 : \mathcal{S}_2 \to \mathfrak{Obj}(\mathcal{C})\big] = \sum_{n\in\mathbb{Z}} \mathbb{L}^{-n}\big[\pi : \mathcal{W}_n \to \mathfrak{Obj}(\mathcal{C})\big],$$

where $\pi_1(\mathcal{S}_1) \subseteq \mathcal{X}_i$, $\pi_2(\mathcal{S}_2) \subseteq \mathcal{X}_j$ for some $i, j \in I$, and

$$\mathcal{W}_n = \Big\{(s_1, s_2, \alpha) \,\Big|\, s_i \in \mathcal{S}_i, \alpha \in \operatorname{Hom}_{\mathcal{C}}\big(\pi_2(s_2), \pi_1(s_1)[1]\big),$$
$$\sum_{i>0}(-1)^i \dim_{\mathbb{C}} \operatorname{Hom}\big(\pi_2(s_2)[i], \pi_1(s_1)[1]\big) = -n\Big\}.$$

The map π sends (s_1, s_2, α) to $\operatorname{Cone}(\alpha)[-1]$. Here, for simplicity of notation, we write $[S \to \mathcal{X}_i]$ instead of $[(S, G_s) \to (\mathcal{X}_i, G_i)]$. The algebra $\mathcal{MH}(\mathcal{C})$ is called the motivic Hall algebra associated to $\mathcal{C}$.

For convenience, we use the integral notation for the right-hand term in the definition of the multiplication. Then the multiplication can be rewritten as

$$\begin{aligned}
&\big[\pi_1 : \mathcal{S}_1 \to \mathfrak{Obj}(\mathcal{C})\big] \cdot \big[\pi_2 : \mathcal{S}_2 \to \mathfrak{Obj}(\mathcal{C})\big] \\
&\quad := \int_{s_1\in\mathcal{S}_1, s_2\in\mathcal{S}_2} \big[\operatorname{Hom}_{\mathcal{C}}\big(\pi_2(s_2), \pi_1(s_1)[1]\big) \to \mathfrak{Obj}(\mathcal{C})\big] \cdot \big\{\pi_2(s_2), \pi_1(s_1)[1]\big\} \\
&\quad := \int_{s_1\in\mathcal{S}_1, s_2\in\mathcal{S}_2} \big\{\pi_2(s_2), \pi_1(s_1)[1]\big\} \cdot \int_{\alpha\in\operatorname{Hom}_{\mathcal{C}}(\pi_2(s_2),\pi_1(s_1)[1])_E} v_{[E]},
\end{aligned}$$

where $v_{[E]} := [\pi : pt \to \mathfrak{Obj}(\mathcal{C})]$ with $\pi(pt) = E$. Note that $\operatorname{Hom}_{\mathcal{C}}(\pi_2(s_2), \pi_1(s_1)[1])_E$ is a constructible set (see [4, Appendix]).

THEOREM 5.2 ([12, PROPOSITION 10])

With the above multiplication, $\mathcal{MH}(\mathcal{C})$ becomes an associative algebra.

Inspired by [12] and [24], the proof can be considered as a motivic version of Symmetry II.

Proof
By the reformulation of the definition of multiplication, the proof of the theorem is easily reduced to the case when $\mathcal{S}_i$ is just a point. Given X, Y, and $Z \in \mathfrak{Obj}(\mathcal{C})$, $v_{[Z]} * (v_{[X]} * v_{[Y]})$ is equal to

$$\mathcal{T}_1 := \int_{\alpha \in \mathrm{Hom}(Y,X[1])_{L[1]}} \int_{\beta \in \mathrm{Hom}(L,Z[1])_{M[1]}} \{Y, X[1]\} \cdot \{L, Z[1]\} \cdot v_{[M]}$$

and $(v_{[Z]} * v_{[X]}) * v_{[Y]}$ is equal to

$$\mathcal{T}_2 := \int_{\alpha' \in \mathrm{Hom}(X,Z[1])_{L'[1]}} \int_{\beta' \in \mathrm{Hom}(Y,L'[1])_{M[1]}} \{X, Z[1]\} \cdot \{Y, L'[1]\} \cdot v_{[M]}.$$

Using the notation in Section 3, we have

$$\mathcal{T}_1 = \int_{\alpha' \in \mathrm{Hom}(X,Z[1])^{L}_{L'[1]}, \alpha \in \mathrm{Hom}(Y,X[1])^{L'}_{L[1]}} \int_{\beta \in \mathrm{Hom}(L,Z[1])^{\alpha,L'}_{M[1]}} \{Y, X[1]\} \cdot \{L, Z[1]\} \cdot v_{[M]}$$

and

$$\mathcal{T}_2 = \int_{\alpha \in \mathrm{Hom}(Y,X[1])^{L'}_{L[1]}, \alpha' \in \mathrm{Hom}(X,Z[1])^{L}_{L'[1]}} \int_{\beta' \in \mathrm{Hom}(Y,L'[1])^{\alpha',L}_{M[1]}} \{X, Z[1]\} \cdot \{Y, L'[1]\} \cdot v_{[M]}.$$

As in the proof of Theorem 4.1, fix $\alpha \in \mathrm{Hom}(Y, X[1])^{L'}_{L[1]}$; by (4.1), there is a constructible bundle $\mathrm{Hom}(L, Z[1])^{\alpha,L'}_{M[1]} \to \mathrm{Hom}(X, Z[1])^{L}_{L'[1]}$ with fiber dimension

$$\dim \mathrm{Hom}\big(Y, Z[1]\big) + \dim\big\{X \oplus Y, Z[1]\big\} - \dim\big\{L, Z[1]\big\}.$$

Fix $\alpha' \in \mathrm{Hom}(X, Z[1])^{L}_{L'[1]}$; by (4.1), there is a constructible bundle

$$\mathrm{Hom}\big(Y, L'[1]\big)^{\alpha',L}_{M[1]} \to \mathrm{Hom}\big(Y, X[1]\big)^{L'}_{L[1]}$$

with fiber dimension

$$\dim \mathrm{Hom}\big(Y, Z[1]\big) + \dim\big\{Y, X[1] \oplus Z[1]\big\} - \dim\big\{Y, L'[1]\big\}.$$

Hence, we have $\mathcal{T}_1 = \mathcal{T}_2$. □

Given an indecomposable object $X \in \mathcal{C}$, $\Upsilon([\mathrm{End}_{\mathcal{C}}(X)]) = \mathbb{L}^{\dim_k \mathrm{End}_{\mathcal{C}}(X)}$, and

$$\Upsilon\big([\mathrm{Aut}\, X]\big) = \mathbb{L}^{\dim_k \mathrm{radEnd}\, X}(\mathbb{L}^{d(X)} - 1),$$

where $d(X) = \dim_k(\mathrm{End}\, X / \mathrm{radEnd}\, X)$. Given $n \in \mathbb{N}$, consider the morphism

$$\mathrm{Aut}(nX) \to \mathrm{GL}_n(\mathrm{End}\, X / \mathrm{radEnd}\, X);$$

the fiber is an affine space (consisting of matrices with elements belonging to $\mathrm{radEnd}\, X$) of dimension $n^2 \dim_k(\mathrm{radEnd}\, X)$. Hence, by [1, Lemma 2.6], we have

$$\Upsilon\big(\big[\mathrm{Aut}(nX)\big]\big) = \mathbb{L}^{n^2 \dim_k(\mathrm{radEnd}\, X) + (1/2)n(n-1)d(X)} \prod_{k=1}^{n} (\mathbb{L}^{kd(X)} - 1).$$

Generally, an object $X \in \mathcal{C}$ is isomorphic to $n_1X_1 \oplus n_2X_2 \oplus \cdots \oplus n_tX_t$, where $X_i \ncong X_j$ for $i, j = 1, \ldots, t$ and $i \neq j$. Consider the natural morphism

$$\mathrm{Aut}(X) \to \mathrm{Aut}(n_1X_1) \times \cdots \times \mathrm{Aut}(n_tX_t).$$

It is a vector bundle of dimension $\sum_{i\neq j} \mathrm{Hom}(n_iX_i, n_jX_j)$. Hence, we have the following result.

PROPOSITION 5.3

For $X \in \mathcal{C}$, $\Upsilon([\mathrm{Aut}\, X]) = \mathbb{L}^t \prod_{i=1}^{l} \prod_{j=1}^{s_i} (\mathbb{L}^{jd(X)} - 1)$ for some $t, l, s_1, \ldots, s_l \in \mathbb{N}$ and then is invertible in Λ.

We introduce some necessary notations. Let $\mathcal{W} = \bigsqcup_{n\in\mathbb{Z}} \mathcal{W}_n$. Then $\pi : \mathcal{W} \to \mathfrak{Obj}(\mathcal{C})$ (for simplicity, we use the same notation as $\pi : \mathcal{W}_n \to \mathfrak{Obj}(\mathcal{C})$) induces that $\pi(\mathcal{W}) \subseteq \mathfrak{Obj}(\mathcal{C})$. For $X \in \mathcal{C}$, if $\Upsilon([\mathrm{Aut}\, X]) = \mathbb{L}^t \prod_{i=1}^{l} \prod_{j=1}^{s_i} (\mathbb{L}^{jd(X)} - 1)$, then write $d_a(X) = (t, s_1, s_2, \ldots, s_l)$ and $\mathbb{L}^{d_a} = \mathbb{L}^t \prod_{i=1}^{l} \prod_{j=1}^{s_i} (\mathbb{L}^{jd(X)} - 1)$. For $X, Y, L \in \mathcal{C}$, write

$$d_{\{X,Y\}} = \sum_{i>0} (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}\big(X[i], Y\big)$$

and $d^* = d_{(X,Y,L)} = (d_a(X), d_{\{X,L\}}, d_{\{X,X\}})$. For a fixed triple $d^* = (d_a, l, m)$ with $d_a = (t, s_1, s_2, \ldots, s_l)$ and two pairs $[\pi_i : \mathcal{S}_i \to \mathfrak{Obj}(\mathcal{C})]$ for $i = 1, 2$, define

$$\mathcal{V}_{d^*} = \big\{(s_1, s_2, L, \beta) \,\big|\, L \in \pi(\mathcal{W}), \beta \in \mathrm{Hom}\big(\pi_1(s_1), L\big) \text{ with } \mathrm{Cone}(\beta) \cong \pi_2(s_2) \text{ and}$$
$$d_a\big(\mathrm{Aut}\big(\pi_1(s_1)\big)\big) = d_a, d_{\{\pi_1(s_1), \mathrm{Cone}(\alpha)\}} = l, d_{\{\pi_1(s_1), \pi_1(s_1)\}} = m\big\}.$$

Consider the Λ-module

$$\mathcal{MH}_T(\mathcal{C}) = \bigoplus_{i\in I} \mathrm{Mot}_{st}\big((\mathcal{X}_i, G_i)\big)$$

endowed with the multiplication

$$\begin{aligned}
&\big[\pi_1 : \mathcal{S}_1 \to \mathfrak{Obj}(\mathcal{C})\big] \cdot \big[\pi_2 : \mathcal{S}_2 \to \mathfrak{Obj}(\mathcal{C})\big] \\
&\quad = \sum_{d_a, l, m} \big[\psi : \mathcal{V}_{d^*} \to \mathfrak{Obj}(\mathcal{C})\big] \mathbb{L}^{-d_a} \mathbb{L}^{l-m} \\
&\quad := \int_{s_1 \in \mathcal{S}_1, s_2 \in \mathcal{S}_2} \Upsilon\big(\big[\mathrm{Aut}\big(\pi(s_1)\big)\big]\big)^{-1} \big\{\pi_1(s_1), \pi_1(s_1)\big\}^{-1} \\
&\qquad \times \int_{L\in\pi(\mathcal{W})} \big[\mathrm{Hom}\big(\pi_1(s_1), L\big) \to \mathfrak{Obj}(\mathcal{C})\big],
\end{aligned}$$

where $\pi_1(\mathcal{S}_1) \subseteq \mathcal{X}_i$, $\pi_2(\mathcal{S}_2) \subseteq \mathcal{X}_j$ for some $i, j \in I$, and $\psi(s_1, s_2, L, \beta) = L$. Then $\mathcal{MH}_T(\mathcal{C})$ is an Λ-algebra.

Given $Z, M \in \mathcal{C}$ and $l : Z \to M$, there is a unique distinguished triangle

$$Z \xrightarrow{l} M \xrightarrow{m} L \xrightarrow{n} Z[1],$$

where $L = \mathrm{Cone}(l)$ and $m = (\,1\ 0\,)$, $n = \binom{0}{1}$. Set

$$n\,\mathrm{Hom}\big(Z[1],L\big) = \big\{nt \,\big|\, t \in \mathrm{Hom}\big(Z[1],L\big)\big\}$$

and

$$\mathrm{Hom}\big(Z[1],L\big)n = \big\{tn \,\big|\, t \in \mathrm{Hom}\big(Z[1],L\big)\big\}.$$

It is easy to check that they are vector spaces.

LEMMA 5.4 ([24, LEMMA 2.4])
With the above notation, we have

$$\Upsilon\big(\big[n\,\mathrm{Hom}\big(Z[1],L\big)\big]\big) = \{M,L\}\{Z,L\}^{-1}\{L,L\}^{-1}$$

and

$$\Upsilon\big(\big[\mathrm{Hom}\big(Z[1],L\big)n\big]\big) = \{Z,M\}\{Z,L\}^{-1}\{Z,Z\}^{-1}.$$

Now we give the motivic version of the derived Riedtmann–Peng formula.

PROPOSITION 5.5
For $Z,L,M \in \mathcal{C}$, we have

$$\begin{aligned}&\big[\mathrm{Hom}_{\mathcal{C}}(Z,M)_L \to \mathfrak{Obj}(\mathcal{C})\big]\cdot \frac{\{Z,M\}}{\Upsilon([\mathrm{Aut}\,Z])\cdot\{Z,L\}\cdot\{Z,Z\}}\\ &\quad= \big[\mathrm{Hom}_{\mathcal{C}}(M,L)_{Z[1]} \to \mathfrak{Obj}(\mathcal{C})\big]\cdot \frac{\{M,L\}}{\Upsilon([\mathrm{Aut}\,L])\cdot\{Z,L\}\cdot\{L,L\}}.\end{aligned}$$

Proof
Define the constructible set

$$S_1 = \big\{(a_L,l,tn) \,\big|\, a_L \in \mathrm{Aut}\,L, l \in \mathrm{Hom}_{\mathcal{C}}(Z,M)_L, t \in \mathrm{Hom}_{\mathcal{C}}\big(Z[1],L\big)\big\}$$

and

$$\begin{aligned}S_2 = \big\{(a_Z,m,nt) \,\big|\,&\\ &a_Z \in \mathrm{Aut}\,Z, m \in \mathrm{Hom}_{\mathcal{C}}(M,L)_{Z[1]}, g_2 \in \mathrm{Aut}\,L, t \in \mathrm{Hom}_{\mathcal{C}}\big(Z[1],L\big)\big\}.\end{aligned}$$

Here, the projection $S_1 \to \mathrm{Aut}\,L \times \mathrm{Hom}_{\mathcal{C}}(Z,M)_L$ is a constructible bundle of dimension $\dim_k \mathrm{Hom}_{\mathcal{C}}(Z[1],L)n$. (The choice of n is irrelevant.) In the same way, S_2 is a constructible bundle of dimension $n\dim_k \mathrm{Hom}_{\mathcal{C}}(Z[1],L)$. Note that, given $g_1 \in \mathrm{Aut}\,Z$, $g_1 \circ l = l$ means that $g_1 = 1 + g_1'$ and $g_1' \circ l = 0$. However, $g_1' \circ l = 0$ is equivalent to saying that $g_1' = t \circ n$ for some $t \in \mathrm{Hom}(Z[1],L)$. Similarly, $g_2 \in \mathrm{Aut}\,L$ satisfies $mg_2 = m$ if and only if $g_2 \in 1 + n\,\mathrm{Hom}(Z[1],L)$. For any $l' \in \mathrm{Hom}_{\mathcal{C}}(Z,M)_L$, there exists a unique $a_Z \in \mathrm{Aut}\,Z$ such that $a_Z l = l'$. There is an isomorphism $S_1 \to S_2$ defined by sending (a_L,l',tn) to $(a_Z,a_L m,nt)$. Hence, we have that $[S_1 \to \mathfrak{Obj}(\mathcal{C})] = [S_2 \to \mathfrak{Obj}(\mathcal{C})]$. On the other hand, by the definition of motivic Hall algebras and Lemma 5.4, we have

$$\begin{aligned}&\big[S_1 \to \mathfrak{Obj}(\mathcal{C})\big]\\&\quad = \Upsilon\big([\operatorname{Aut} L]\big)\{Z,M\}\{Z,L\}^{-1}\{Z,Z\}^{-1} \cdot \big[\operatorname{Hom}_{\mathcal{C}}(Z,M)_L \to \mathfrak{Obj}(\mathcal{C})\big]\end{aligned}$$

and

$$\begin{aligned}&\big[S_2 \to \mathfrak{Obj}(\mathcal{C})\big]\\&\quad = \Upsilon\big([\operatorname{Aut} Z]\big)\{M,L\}\{Z,L\}^{-1}\{L,L\}^{-1} \cdot \big[\operatorname{Hom}_{\mathcal{C}}(M,L)_{Z[1]} \to \mathfrak{Obj}(\mathcal{C})\big].\end{aligned}$$

This concludes the proof of the proposition. □

THEOREM 5.6

With the above defined multiplication, $\mathcal{MH}_T(\mathcal{C})$ becomes an associative algebra.

The proof can be considered to be the motivic version of the proof for [24, Theorem 3.6].

Proof

Set $u_{[E]} := [\pi : pt \to \mathfrak{Obj}(\mathcal{C})]$ with $\pi(pt) = E$. Given three functions $[\pi_i : \mathcal{S}_i \to \mathfrak{Obj}(\mathcal{C})]$ for $i = 1,2,3$, we need to prove

$$[\pi_3] \cdot \big([\pi_1] \cdot [\pi_2]\big) = \big([\pi_3] \cdot [\pi_1]\big) \cdot [\pi_2].$$

By the reformulation of the definition of multiplication, the proof of the theorem is easily reduced to the case when $\mathcal{S}_i$ is just a point. Let $\pi_3(pt) = Z$, $\pi_2(pt) = Y$, and $\pi_1(pt) = X$. Set $t_{[X]} = \Upsilon^{-1}([\operatorname{Aut}(X)]) \cdot \{X,X\}$. Then $u_{[Z]} * (u_{[X]} * u_{[Y]})$ is equal to

$$\int_{L\in\pi(\mathcal{W}),L'\in\pi(\mathcal{W}')} \big[\operatorname{Hom}(M \oplus X, L)^{Y,Z[1]}_{L'[1]} \to \mathfrak{Obj}(\mathcal{C})\big] t_{[X]} t_{[L]} \{M \oplus X, L\},$$

where $\pi(\mathcal{W}')$ is the image of $\pi : \operatorname{Ext}^1(Z,X) \to \mathfrak{Obj}(\mathcal{C})$ by sending α to its middle term. Similarly, we have that $(u_{[Z]} * u_{[X]}) * u_{[Y]}$ is equal to

$$\int_{L\in\pi(\mathcal{W}),L'\in\pi(\mathcal{W}')} \big[\operatorname{Hom}(L', M \oplus X)^{Y,Z[1]}_{L} \to \mathfrak{Obj}(\mathcal{C})\big] t_{[X]} t_{[L']} \{L', M \oplus X\}.$$

Following Proposition 5.5, we have that $(u_{[Z]} * u_{[X]}) * u_{[Y]} = u_{[Z]} * (u_{[X]} * u_{[Y]})$. □

THEOREM 5.7

There exists an algebra isomorphism $\Phi : \mathcal{MH}(\mathcal{C}) \to \mathcal{MH}_T(\mathcal{C})$ defined by

$$\Phi\big(\big[\pi : \mathcal{S} \to \mathfrak{Obj}(\mathcal{C})\big]\big) = \int_{s\in\mathcal{S}} t_{[\pi(s)]} u_{[\pi(s)]},$$

where $t_{[\pi(s)]} = \Upsilon^{-1}([\operatorname{Aut}(\pi(s))]) \cdot \{\pi(s), \pi(s)\}$.

The proof is a direct application of Proposition 5.5.

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