Invariants of wreath products and subgroups of S_6

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Abstract Let G be a subgroup of S_6, the symmetric group of degree 6. For any field k, G acts naturally on the rational function field $k(x_1,\ldots,x_6)$ via k-automorphisms defined by $\sigma\cdot x_i = x_{\sigma(i)}$ for any $\sigma\in G$ and any $1\le i\le 6$. We prove the following theorem. The fixed field $k(x_1,\ldots,x_6)^G$ is rational (i.e., purely transcendental) over k, except possibly when G is isomorphic to $\mathrm{PSL}_2(\mathbb{F}_5)$, $\mathrm{PGL}_2(\mathbb{F}_5)$, or A_6. When G is isomorphic to $\mathrm{PSL}_2(\mathbb{F}_5)$ or $\mathrm{PGL}_2(\mathbb{F}_5)$, then $\mathbb{C}(x_1,\ldots,x_6)^G$ is $\mathbb{C}$-rational and $k(x_1,\ldots,x_6)^G$ is stably k-rational for any field k. The invariant theory of wreath products will be investigated also.

1. Introduction

Let k be a field. A finitely generated field extension L of k is called k-rational if L is purely transcendental over k; it is called stably k-rational if $L(y_1,y_2,\ldots,y_m)$ is k-rational where $y_1,\ldots,y_m$ are elements which are algebraically independent over L.

Let G be a subgroup of S_n where S_n is the symmetric group of degree n. For any field k, G acts naturally on the rational function field $k(x_1,\ldots,x_n)$ via k-automorphisms defined by $\sigma\cdot x_i = x_{\sigma(i)}$ for any $\sigma\in G$ and any $1\le i\le n$. Noether's [No] problem asks whether the fixed field $k(x_1,\ldots,x_n)^G := \{f\in k(x_1,\ldots,x_n) : \sigma(f)=f \text{ for all } \sigma\in G\}$ is k-rational (resp., stably k-rational). If G is embedded in S_N through the left regular representation (where $N=|G|$), then $k(x_1,\ldots,x_N)^G$ is nothing but $k(V_{\mathrm{reg}})^G$ where $\rho: G\to \mathrm{GL}(V_{\mathrm{reg}})$ is the regular representation of G, that is, $V_{\mathrm{reg}} = \bigoplus_{g\in G} k\cdot e_g$ is a k-vector space and $h\cdot e_g = e_{hg}$ for any $h,g\in G$. We will write $k(G) = k(V_{\mathrm{reg}})^G$ in the rest of the paper. The rationality problem of $k(G)$ is also called Noether's problem, for example, in the paper of Lenstra [Le].

If G is a transitive subgroup of S_n, then the G-field $k(x_1,\ldots,x_n)$ may be linearly embedded in the G-field $k(V_{\mathrm{reg}})$ by Lemma 1.5; thus $k(G)$ is rational over $k(x_1,\ldots,x_n)^G$ by Theorem 2.1. In particular, if $k(x_1,\ldots,x_n)^G$ is k-rational, then so is $k(G)$. In general, the rationality of $k(G)$ does not imply that of $k(x_1,\ldots,x_n)^G$, although there is no such an example.

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Noether's problem is related to the inverse Galois problem, to the existence of generic G-Galois extensions, and to the existence of versal G-torsors over k-rational field extensions. For a survey of this problem, see [GMS], [Sa], and [Sw].

This paper is a continuation of our paper [KW]. We recall the main results of [KW] first. Let k be any field, and let G be a subgroup of S_n acting naturally on $k(x_1,\ldots,x_n)$ by $\sigma\cdot x_i = x_{\sigma(i)}$ for any $\sigma\in G$ and any $1\le i\le n$.

THEOREM 1.1

(a) ([KW, Theorem 1.3]) *For any field k and any subgroup G of S_n, if $n\le 5$, then $k(x_1,\ldots,x_n)^G$ is k-rational.*

(b) ([KW, Theorem 1.4]) *Let k be any field, and let G be a transitive subgroup of S_7. If G is not isomorphic to the group $\mathrm{PSL}_2(\mathbb{F}_7)$ or the group A_7, then $k(x_1,\ldots,x_7)^G$ is k-rational.*

Moreover, when G is isomorphic to $\mathrm{PSL}_2(\mathbb{F}_7)$ and k is a field satisfying that $k\supset\mathbb{Q}(\sqrt{-7})$, then $k(x_1,\ldots,x_7)^G$ is also k-rational.

(c) ([KW, Theorem 1.5]) *Let k be any field, and let G be a transitive solvable subgroup of S_{11}. Then $k(x_1,\ldots,x_{11})^G$ is k-rational.*

We thank the referee who pointed out a result of Hoshi [Ho], which is related to some cases of the above theorem: if $p=7$ or 11 and $G=G_{pd}$ acts on $\mathbb{Q}(x_0,\ldots,x_{p-1})$ (see [KW, Definition 3.1]), then $\mathbb{Q}(x_0,\ldots,x_{p-1})^G$ is $\mathbb{Q}$-rational; similar results are valid when the base field $\mathbb{Q}$ is replaced by finite fields with certain assumptions. Moreover, further results may be found in an article in preparation by Hoshi and colleagues, when $13\le p\le 23$ or p is some larger prime number.

What we will prove in this paper is the case $G\subset S_6$. Specifically, we will establish the following theorem.

THEOREM 1.2

Let k be any field, and let G be any subgroup of S_6. Then $k(x_1,\ldots,x_6)^G$ is k-rational, except when G is isomorphic to the group A_6, $\mathrm{PSL}_2(\mathbb{F}_5)$, or $\mathrm{PGL}_2(\mathbb{F}_5)$.

If G is conjugate to the group $\mathrm{PSL}_2(\mathbb{F}_5)$ or $\mathrm{PGL}_2(\mathbb{F}_5)$ embedded in S_6, then $\mathbb{C}(x_1,\ldots,x_6)^G$ is $\mathbb{C}$-rational and $k(x_1,\ldots,x_6)^G$ is stably k-rational for any field k.

First of all, note that we do not know whether $k(x_1,\ldots,x_6)^{A_6}$ is k-rational or not. A second remark is that, as an abstract group, $\mathrm{PSL}_2(\mathbb{F}_5)$ (resp., $\mathrm{PGL}_2(\mathbb{F}_5)$) is isomorphic to A_5 (resp., S_5). However, the group $\mathrm{PGL}_2(\mathbb{F}_5)$ embedded in S_6 as a transitive subgroup (see the third paragraph of Section 4) provides a 6-dimensional reducible representation of S_5.

Note that another rationality problem related to Theorems 1.1 and 1.2 was posed in [HT1] and [HT2], namely, a subextension $K/\mathbb{Q}$ of $\mathbb{Q}(x_1,\ldots,x_n)/\mathbb{Q}$ was considered with $n=5$ or 6. In fact, K is the so-called field of cross ratios and is rational over $\mathbb{Q}$ with dimension $n-3$. The question is to study whether the fixed field K^G is $\mathbb{Q}$-rational when G is a transitive subgroup of S_n. However, it is not clear whether $\mathbb{Q}(x_1,\ldots,x_n)^G$ is rational over K^G. See [HT2, p. 192] and [Ts, Theorem B] when $G=S_n$.

We would like to point out two distinct aspects between Theorems 1.1 and 1.2 and the known results of Noether's problem for non-abelian groups (in the direction of affirmative answers). The first distinction is that, in most previous results (except those in [Ma], [CHK], [Ka2]), the assumption on the existence of roots of unity is required, while the main part of Theorems 1.1 and 1.2 works on any field k. The second distinction is that most previous results dealt with the rationality of $k(G)$, which is implied by that of $k(x_1,\ldots,x_n)^G$ (see the third paragraph of this section).

Since many transitive subgroups of S_6 are of the forms of wreath products $H\wr G$ where $H\subset S_2$, $G\subset S_3$ or $H\subset S_3$, $G\subset S_2$ (see Section 4), we embark on a study of the invariant theory of wreath products in Section 2 before the proof of Theorem 1.2. Here is a convenient criterion for group actions of wreath products.

THEOREM 1.3

Let k be any field, let $G\subset S_m$, and let $H\subset S_n$. Then the wreath product $\widetilde{G}:=H\wr G$ can be regarded as a subgroup of S_{mn}. If $k(x_1,\ldots,x_m)^G$ and $k(y_1,\ldots,y_n)^H$ are k-rational, then $k(z_1,\ldots,z_{mn})^{\widetilde{G}}$ is also k-rational.

An application of the above theorem is the following theorem of Tsunogai [Ts]. Note that our proof is different from the proof of Tsunogai.

THEOREM 1.4 (TSUNOGAI)

Let k be any field, let p be a prime number, and let C_p be the cyclic group of order p. For any integer $n\geq 2$, let P be a p-Sylow subgroup of S_n. If $k(C_p)$ is k-rational, then $k(x_1,\ldots,x_n)^P$ is also k-rational.

Note that Theorem 1.3 was obtained by Kuyk [Kuy1] under certain extra assumptions. For details, see the remark before Theorem 3.6.

The following lemma helps to clarify the relationship between the rationality of $k(x_1,\ldots,x_n)^G$ and that of $k(G)$ when G is a transitive subgroup of S_n.

LEMMA 1.5

Suppose that G is a transitive subgroup of S_n acting naturally on the rational function field $k(x_1,\ldots,x_n)$. Let $G\to\mathrm{GL}(V_{\mathrm{reg}})$ be the regular representation over a field k, and let $\{x(g):g\in G\}$ be a dual basis of V_{reg}. Then there is a G-equivariant embedding $\Phi:\bigoplus_{1\leq i\leq n}k\cdot x_i\to\bigoplus_{g\in G}k\cdot x(g)$. In particular, $k(G)$ is rational over $k(x_1,\ldots,x_n)^G$.

Proof

Note that $k(V_{\mathrm{reg}})=k(x(g):g\in G)$ with $h\cdot x(g)=x(hg)$ for any $h,g\in G$.

Define $H=\{g\in G:g(1)=1\}$. Choose a coset decomposition $G=\bigcup_{1\leq i\leq n}g_iH$ such that, for any $g\in G$, $g\cdot g_iH=g_jH$ if and only if $g(i)=j$.

Define a k-linear map $\Phi : \bigoplus_{1\le i\le n} k\cdot g_iH \to \bigoplus_{g\in G} k\cdot x(g)$ by $\Phi(g_iH) = \sum_{h\in H} x(g_ih) \in \bigoplus_{g\in G} k\cdot x(g)$.

Note that Φ is a G-equivariant map. It is not difficult to show that Φ is injective.

Consider the action of G on the field $k(x_1,\dots,x_n)$. Identify the cosets g_iH with x_i. It follows that, via Φ, the G-field $k(x_1,\dots,x_n)$ is linearly embedded into $k(x(g) : g\in G)$. By applying Theorem 2.1(a), we find that $k(G)$ is rational over $k(x_1,\dots,x_n)^G$. □

REMARK

If G is a subgroup of S_n and it is possible to embed $\bigoplus_{1\le i\le n} k\cdot x_i$ into $\bigoplus_{g\in G} k\cdot x(g)$, then G is a transitive subgroup.

Suppose that $T_1,\dots,T_t$ are the G-orbits of the set $\{x_1,\dots,x_n\}$ with $t\ge 2$. Each T_i contributes a trivial representation of G, but the regular representation contains only one trivial representation. Thus it is impossible that such a G-embedding exists.

One may also consider the rationality problem of $k(x_1,\dots,x_8)^G$ where G is a transitive subgroup of S_8; some cases were studied previously in [CHK] and [HHR]. If k contains enough roots of unity, for example, $\zeta_8\in k$, then it is not very difficult to show that $k(x_1,\dots,x_8)^G$ is k-rational for many such groups G by standard methods and previously known results (except possibly when G is a non-abelian simple group). However, if k is any field and $g = \langle\sigma\rangle$ where $\sigma = (1,2,\dots,8)$, by Endo–Miyata's theorem, it is known that $k(x_1,\dots,x_8)^G$ is k-rational if and only if $k(\zeta_8)$ is cyclic over k or $\operatorname{char} k = 2$ (see [EM, Proposition 3.9] and [Le]; see also [Ka1, Theorem 1.8]). In fact, the $\operatorname{char} k = 2$ case follows from a result of Kuniyoshi [Ku] and Gaschütz [Ga]. The proof of Endo–Miyata's theorem is nontrivial; similar complicated situations may happen in other subgroups of S_8. For a recent investigation, see the article of Wang and Zhou [WZ].

We organize this article as follows. In Section 2, we list several rationality criteria which will be used later. A detailed discussion of wreath products will be given in Section 3. Our method is applicable not only in the Noether problem (i.e., the rational invariants), but also in the polynomial invariants (see Theorem 3.8). The proof of Theorem 1.2 will be given in Section 4.

Standing terminology. Throughout the paper, we denote by S_n, A_n, C_n, D_n the symmetric group of degree n, the alternating group of degree n, the cyclic group of order n, and the dihedral group of order $2n$, respectively. If k is any field, then $k(x_1,\dots,x_m)$ denotes the rational function field of m variables over k; this holds similarly for $k(y_1,\dots,y_n)$ and $k(z_1,\dots,z_l)$. When $\rho : G\to \mathrm{GL}(V)$ is a representation of G over a field k, $k(V)$ denotes the rational function field $k(x_1,\dots,x_n)$ with the induced action of G where $\{x_1,\dots,x_n\}$ is a basis of the dual space V^* of V. In particular, when $V = V_{\mathrm{reg}}$ is the regular representation space, denote by $\{x(g) : g\in G\}$ a dual basis of V_{reg}; then $k(V_{\mathrm{reg}}) = k(x(g) : g\in G)$ where $h\cdot x(g) = x(hg)$ for any $h, g\in G$. We will write $k(G) := k(V_{\mathrm{reg}})^G$. When G is a subgroup of S_n, we say that G acts naturally on the rational function field $k(x_1,\ldots,x_n)$ by k-automorphisms if $\sigma\cdot x_i = x_{\sigma(i)}$ for any $\sigma\in G$ and any $1\le i\le n$.

If σ is a k-automorphism of the rational function field $k(x_1,\ldots,x_n)$, then it is called a monomial automorphism if $\sigma(x_j) = b_j(\sigma)\prod_{1\le i\le n} x_i^{a_{ij}}$ where $(a_{ij})_{1\le i,j\le n}\in \mathrm{GL}_n(\mathbb{Z})$, and $b_j(\sigma)\in k^\times$. If $b_j(\sigma)=1$, then the automorphism σ is called purely monomial. The group action of a finite group G acting on $k(x_1,\ldots,x_n)$ is called a monomial action (resp., a purely monomial action) if σ acts on $k(x_1,\ldots,x_n)$ by a monomial (resp., purely monomial) k-automorphism for all $\sigma\in G$.

In discussing wreath products, we denote by X or Y any set without extra structures unless otherwise specified. The set X_m is a finite set of m elements; thus we write $X_m=\{1,2,\ldots,m\}$. Similarly we write $Y_n=\{1,2,\ldots,n\}$.

2. Preliminaries

In this section we list some known results which will be used in the rest of the paper.

THEOREM 2.1

Let G be a finite group acting on $L(x_1,\ldots,x_m)$, the rational function field of m variables over a field L. Assume that (i) for any $\sigma\in G$, $\sigma(L)\subset L$, and (ii) the restriction of the action of G to L is faithful.

(a) ([HK3, Theorem 1]) *Assume furthermore that, for any $\sigma\in G$,*

$$\begin{pmatrix}\sigma(x_1)\\ \sigma(x_2)\\ \vdots\\ \sigma(x_m)\end{pmatrix} = A(\sigma)\cdot\begin{pmatrix}x_1\\ x_2\\ \vdots\\ x_m\end{pmatrix} + B(\sigma),$$

where $A(\sigma)\in\mathrm{GL}_m(L)$ and $B(\sigma)$ is an $m\times 1$ matrix over L. Then there exist $z_1,\ldots,z_m\in L(x_1,\ldots,x_m)$ so that $L(x_1,\ldots,x_m)=L(z_1,\ldots,z_m)$ with $\sigma(z_i)=z_i$ for any $\sigma\in G$ and any $1\le i\le m$.

In fact, there are $(a_{ij})_{1\le i,j\le n}\in\mathrm{GL}_n(L)$ and $c_j\in L$ such that, for $1\le j\le n$, $z_j=\sum_{1\le i\le n} a_{ij}x_i+c_j$. Moreover, if $B(\sigma)=0$ for all $\sigma\in G$, then we may choose z_j simply by $z_j=\sum_{1\le i\le n} a_{ij}x_i$.

(b) ([HK3, Theorem 1′]) *Assume furthermore that, for any $\sigma\in G$,*

$$\begin{pmatrix}\sigma(x_1)\\ \sigma(x_2)\\ \vdots\\ \sigma(x_m)\end{pmatrix} = A(\sigma)\begin{pmatrix}x_1\\ x_2\\ \vdots\\ x_m\end{pmatrix},$$

where $A(\sigma)\in\mathrm{GL}_m(L)$ and G acts on $L(x_1/x_m,x_2/x_m,\ldots,x_{m-1}/x_m)$ naturally. Then there exist $z_1,\ldots,z_m\in L(x_1,\ldots,x_m)$ so that $L(x_1/x_m,\ldots,x_{m-1}/x_m)=L(z_1/z_m,z_2/z_m,\ldots,z_{m-1}/z_m)$ and $\sigma(z_i/z_m)=z_i/z_m$ for any $\sigma\in G$ and any $1\le i\le m-1$.

THEOREM 2.2 ([AHK, THEOREM 3.1])

Let L be a field, let $L(x)$ be the rational function field of one variable over L, and let G be a finite group acting on $L(x)$. Suppose that, for any $\sigma \in G$, $\sigma(L) \subset L$ and $\sigma(x) = a_\sigma x + b_\sigma$ where $a_\sigma, b_\sigma \in L$ and $a_\sigma \neq 0$. Then $L(x)^G = L^G(f)$ for some polynomial $f \in L[x]$. In fact, if $m = \min\{\deg g(x) : g(x) \in L[x]^G \backslash L\}$, then any polynomial $f \in L[x]^G$ with $\deg f = m$ satisfies the property $L(x)^G = L^G(f)$.

DEFINITION 2.3

Let σ be a k-automorphism on the rational function field $k(x_1, \ldots, x_n)$; σ is called a purely monomial automorphism if $\sigma(x_j) = \prod_{1 \le i \le n} x_i^{a_{ij}}$ for $1 \le j \le n$ where $(a_{ij})_{1 \le i,j \le n} \in \mathrm{GL}_n(\mathbb{Z})$. The action of a finite group G acting on $k(x_1, \ldots, x_n)$ is called a purely monomial action if, for all $\sigma \in G$, σ acts on $k(x_1, \ldots, x_n)$ by a purely monomial k-automorphism (see [HK1]).

THEOREM 2.4 ([HK1], [HK2], [HR])

Let k be any field, and let G be a finite group acting on the rational function field $k(x_1, x_2, x_3)$ by purely monomial k-automorphisms. Then the fixed field $k(x_1, x_2, x_3)^G$ is k-rational.

THEOREM 2.5 (MAEDA [Ma])

Let k be any field, and let A_5 be the alternating group of degree 5 acting on $k(x_1, \ldots, x_5)$. Let A_5 act on $k(x_1, \ldots, x_5)$ via k-automorphisms defined by $\sigma \cdot x_i = x_{\sigma(i)}$ for any $\sigma \in A_5$ and any $1 \le i \le 5$. Then both the fixed fields $k(x_1/x_5, x_2/x_5, x_3/x_5, x_4/x_5)^{A_5}$ and $k(x_1, x_2, x_3, x_4, x_5)^{A_5}$ are k-rational.

When $n \ge 6$, it is still unknown whether $\mathbb{C}(x_1, \ldots, x_n)^{A_n}$ are $\mathbb{C}$-rational.

Recall the definition of $k(G)$ in the second paragraph of Section 1. The following theorem is a special case of Noether's problem, which has been investigated by many people (see [Sw]). For a proof, see [Le, Corollary 7.3].

THEOREM 2.6

Let k be any field. If $n \le 46$ and $8 \nmid n$, then $k(C_n)$ is k-rational.

3. Wreath products

Recall the definition of wreath products $H \wr G$ (or more precisely $H \wr_X G$) in [Ro, pp. 32 and 313], [Is, p. 73], and [DM, pp. 45–50].

DEFINITION 3.1

Let G and H be groups, and let G act on a set X from the left such that $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$, $1 \cdot x = x$ for any $x \in X$, any $g_1, g_2 \in G$. Let A be the set of all functions from X to H; A is a group by defining $\alpha \cdot \beta(x) := \alpha(x) \cdot \beta(x)$ for any $\alpha, \beta \in A$, any $x \in X$.

In case X is a finite set and $|X| = m$, we will write $X = X_m = \{1, 2, \ldots, m\}$ and $A = \prod_{1 \le i \le m} H_i$. Elements in $\prod_{1 \le i \le m} H_i$ are of the form $\alpha = (\alpha_1, \ldots, \alpha_m)$

where each $\alpha_i \in H$ and $\alpha = (\alpha_1, \ldots, \alpha_m)$ corresponds to the element $\alpha \in A$ satisfying $\alpha(i) = \alpha_i$ for $1 \le i \le m$.

The group G acts on A by $({}^g\alpha)(x) = \alpha(g^{-1} \cdot x)$ for any $g \in G$, $\alpha \in A$, $x \in X$. It is easy to verify that ${}^{g_1 g_2}\alpha = {}^{g_1}({}^{g_2}\alpha)$ for any $g_1, g_2 \in G$.

In case $X = X_m$ and $\alpha = (\alpha_1, \ldots, \alpha_m)$, ${}^g\alpha = (\alpha_{g^{-1}(1)}, \alpha_{g^{-1}(2)}, \ldots, \alpha_{g^{-1}(m)})$ where we write $g(i) = g \cdot i$ for any $g \in G$, any $i \in X_m$.

The wreath product $H \wr_X G$ is the semi-direct product $A \rtimes G$ defined by $(\alpha; g_1) \cdot (\beta; g_2) = (\alpha \cdot {}^{g_1}\beta; g_1 g_2)$ for any $\alpha, \beta \in A$, any $g_1, g_2 \in G$.

Sometimes we will write $H \wr G$ for $H \wr_X G$ if the set X is understood from the context, in particular, if $X = G$ and G acts on X by the left regular representation.

Since G and A may be identified as subgroups of $A \rtimes G = H \wr_X G$, we will identify $g \in G$ and $\alpha \in A$ as elements in $H \wr_X G$.

DEFINITION 3.2

Let G and H be groups acting on the sets X and Y from the left, respectively. Then the wreath product $H \wr_X G$ acts on the set $Y \times X$ by defining

$$(\alpha; g) \cdot (y, x) = \big(\big(\alpha\big(g(x)\big)\big)(y), g(x)\big)$$

for any $x \in X$, $y \in Y$, $g \in G$, $\alpha \in A$. It is routine to verify that $((\alpha; g_1) \cdot (\beta; g_2)) \cdot (y, x) = (\alpha; g_1) \cdot ((\beta; g_2) \cdot (y, x))$ for any $x \in X$, $y \in Y$, $\alpha, \beta \in A$, $g_1, g_2 \in G$.

In case $G \subset S_m$, $H \subset S_n$, we may regard $H \wr_{X_m} G$ as a subgroup of S_{mn} because $H \wr_{X_m} G$ acts faithfully on the set $Y_n \times X_m = \{(j, i) : 1 \le i \le m, 1 \le j \le n\}$.

If Y is the polynomial ring $k[y_1, \ldots, y_n]$ over a field k, then we require that the action of H on Y satisfies an extra condition, namely, for any $h \in H$, the map $\phi_h : f \mapsto h \cdot f$ is a k-algebra morphism where $f \in k[y_1, \ldots, y_n]$.

EXAMPLE 3.3

Let $G \subset S_m$, and let $H \subset S_n$. Then G acts on $X_m = \{1, 2, \ldots, m\}$ and H acts on $Y_n = \{1, 2, \ldots, n\}$. Thus $H \wr_{X_m} G$ acts faithfully on $Y_n \times X_m$ by

$$(\alpha_1, \alpha_2, \ldots, \alpha_m; g) \cdot (j, i) = \big(\alpha_{g(i)}(j), g(i)\big),$$

where $\alpha = (\alpha_1, \ldots, \alpha_m) \in A$, $g \in G$.

For $1 \le l \le m$, $h \in H$, define $\alpha^{(l)}(h) \in A$ by $(\alpha^{(l)}(h))(l) = h \in H$ and $(\alpha^{(l)}(h))(i) = 1 \in H$ if $i \ne l$. It is clear that $H \wr_{X_m} G = \langle \alpha^{(l)}(h), g : 1 \le l \le m, g \in G, h \in H \rangle$.

In case G is a transitive subgroup of S_m, it is not difficult to verify that $H \wr_{X_m} G = \langle \alpha^{(1)}(h), g : g \in G, h \in H \rangle$. Note that

$$\alpha^{(1)}(h) : (j, i) \mapsto \begin{cases} (j, i) & \text{if } i \ne 1, \\ (h(j), 1) & \text{if } i = 1, \end{cases}$$

$$g : (j, i) \mapsto \big(j, g(i)\big).$$

EXAMPLE 3.4

Let p be a prime number, and let $G, H \subset S_p$. We denote $\lambda = (1, 2, \ldots, p) \in S_p$ and identify the set $Y_p \times X_p$ with the set X_{p^2} by the function

$$\varphi : Y_p \times X_p \to X_{p^2},$$
$$(j, i) \mapsto i + jp,$$

where the elements in X_{p^2} are taken modulo p^2.

Let $G = \langle\lambda\rangle$, $H = \langle\lambda\rangle$ act naturally on X_p and Y_p, respectively. Then $H \wr_{X_p} G$ is a group of order p^{1+p} acting on X_{p^2} by identifying $\sigma = \alpha^{(1)}(\lambda)$ and $\tau = \lambda$ with

$$\big(1, 1+p, 1+2p, \ldots, 1+(p-1)p\big) \qquad \text{and}$$
$$(1, 2, \ldots, p)(p+1, p+2, \ldots, 2p) \cdots \big((p-1)p+1, (p-1)p+2, \ldots, (p-1)p+p\big)$$

in S_{p^2}. Note that $H \wr_{X_p} G$ is a p-Sylow subgroup of S_{p^2}.

Inductively, let P_r be a p-Sylow subgroup of S_{p^r} constructed above. Let $H = \langle\lambda\rangle \subset S_p$, $G = P_r \subset S_{p^r}$. Then $H \wr_{X_{p^r}} G$ is a group of order $p^{1+p+p^2+\cdots+p^r}$ acting on $X_{p^{r+1}}$ (by the function $\varphi : Y_p \times X_{p^r} \to X_{p^{r+1}}$ defined by $\varphi(j, i) = i + j \cdot p^r$). Thus $H \wr_{X_{p^r}} G$ is a p-Sylow subgroup of $S_{p^{r+1}}$ (see [DM, p. 49]).

If n is a positive integer and we write $n = n_0 + n_1 p + n_2 p^2 + \cdots + n_t p^t$, where $0 \le n_i \le p-1$ and $n < p^{t+1}$, then a p-Sylow subgroup of S_n is isomorphic to

$$(P_1)^{n_1} \times \cdots \times (P_t)^{n_t},$$

where each P_i is isomorphic to a p-Sylow subgroup of S_{p^i} for $1 \le i \le t$.

We reformulate Theorem 1.3 as the following theorem.

THEOREM 3.5

Let k be any field, $G \subset S_m$, and let $H \subset S_n$. Let G and H act on the rational function fields $k(x_1, \ldots, x_m)$ and $k(y_1, \ldots, y_n)$, respectively, via k-automorphisms defined by $g \cdot x_i = x_{g(i)}$, $h \cdot y_j = y_{h(j)}$ for any $g \in G$, $h \in H$, $1 \le i \le m$, $1 \le j \le n$. Then $\widetilde{G} := H \wr_{X_m} G$ may be regarded as a subgroup of S_{mn} acting on the rational function field $k(x_{ij} : 1 \le i \le m, 1 \le j \le n)$ by Definition 3.2. Assume that both $k(x_1, \ldots, x_m)^G$ and $k(y_1, \ldots, y_n)^H$ are k-rational. Then $k(x_{ij} : 1 \le i \le m, 1 \le j \le n)^{\widetilde{G}}$ is also k-rational.

Proof

Adopt the notations in Example 3.3. For any $1 \le l \le m$, any $h \in H$, define $\alpha^{(l)}(h) \in \widetilde{G} = H \wr_{X_m} G$. Note that $A = \langle \alpha^{(l)}(h) : 1 \le l \le m, h \in H\rangle$. Then we find that, for any $g \in G$, any $\alpha^{(l)}(h)$, the actions are given by

$$g : x_{ij} \mapsto x_{g(i),j},$$
$$\alpha^{(l)}(h) : x_{ij} \mapsto \begin{cases} x_{ij} & \text{if } i \ne l, \\ x_{l,h(j)} & \text{if } i = l, \end{cases}$$

where $1 \le i \le m$, $1 \le j \le n$.

Since $k(y_1,\dots,y_n)^H$ is k-rational, we may find $F_1(y),\dots,F_n(y)\in k(y_1,\dots,y_n)$ such that $F_j(y)=F_j(y_1,\dots,y_n)\in k(y_1,\dots,y_n)$ for $1\le j\le n$, and $k(y_1,\dots,y_n)^H=k(F_1(y),\dots,F_n(y))$. It follows that $k(x_{ij}:1\le j\le n)^{\langle\alpha^{(i)}(h):h\in H\rangle}=k(F_1(x_{i1},\dots,x_{in}),F_2(x_{i1},\dots,x_{in}),\dots,F_n(x_{i1},\dots,x_{in}))$. Hence $k(x_{ij}:1\le i\le m,1\le j\le n)^A=k(F_1(x_{i1},\dots,x_{in}),\dots,F_n(x_{i1},\dots,x_{in}):1\le i\le m)$.

Note that $F_1(x_{i1},\dots,x_{in}),\dots,F_n(x_{i1},\dots,x_{in})$ (where $1\le i\le m$) are algebraically independent over k and $g\cdot F_j(x_{i1},\dots,x_{in})=F_j(x_{g(i),1},\dots,x_{g(i),n})$ for any $g\in G$. Denote $E_{ij}=F_j(x_{i1},\dots,x_{in})$ for $1\le i\le m$, $1\le j\le n$. We find that $k(E_{ij}:1\le i\le m,1\le j\le n)$ is a rational function field over k with G-actions given by $g\cdot E_{ij}=E_{g(i),j}$ for any $g\in G$.

It follows that $k(x_{ij}:1\le i\le m,1\le j\le n)^{\widetilde{G}}=\{k(x_{ij}:1\le i\le m,1\le j\le n)^A\}^G=k(E_{ij}:1\le i\le m,1\le j\le n)^G=k(E_{11},E_{21},\dots,E_{m,1})^G(t_{ij}:2\le i\le m,1\le j\le n)$ for some t_{ij} satisfying that $g(t_{ij})=t_{ij}$ for any $g\in G$ by applying Theorem 2.1.

Since $k(x_1,\dots,x_m)^G$ is k-rational, it follows that $k(E_{11},E_{21},\dots,E_{m,1})^G$ is also k-rational—hence the result. □

Proof of Theorem 1.4

Let P be a p-Sylow subgroup of S_n. We will show that if $k(C_p)$ is k-rational, then $k(x_1,\dots,x_n)^P$ is also k-rational.

Without loss of generality, we may assume that P is the p-Sylow subgroup constructed in Example 3.4.

Step 1. Consider the case $n=p^t$ first. Then the p-Sylow subgroup P_t is of the form $P_t=H\wr_{X_{p^{t-1}}}G$ where $H=\langle\lambda\rangle\simeq C_p$ (with $\lambda=(1,2,\dots,p)\in S_p$), and $G=P_{t-1}\subset S_{p^{t-1}}$. Note that $k(y_1,\dots,y_p)^H=k(y_1,\dots,y_p)^{\langle\lambda\rangle}\simeq k(C_p)$. By induction, $k(x_1,\dots,x_{p^{t-1}})^G$ is also k-rational. Applying Theorem 3.5, it follows that $k(z_1,\dots,z_{p^t})^{P_t}$ is k-rational.

Step 2. Suppose that $G_1\subset S_m$ and $G_2\subset S_n$. Thus G_1 acts on $k(x_1,\dots,x_m)$ and G_2 acts on $k(y_1,\dots,y_n)$. If $k(x_1,\dots,x_m)^{G_1}$ and $k(y_1,\dots,y_n)^{G_2}$ are k-rational, then $k(x_1,\dots,x_m)^{G_1}=k(F_1,\dots,F_m)$ and $k(y_1,\dots,y_n)^{G_2}=k(F_{m+1},\dots,F_{m+n})$ where $F_i=F_i(x_1,\dots,x_m)$ for $1\le i\le m$, and $F_{m+j}=F_{m+j}(y_1,\dots,y_n)$ for $1\le j\le n$. It follows that $k(x_1,\dots,x_m,\ y_1,\dots,y_n)^{G_1\times G_2}=k(F_1,F_2,\dots,F_{m+n})$, because $[k(x_1,\dots,x_m,y_1,\dots,y_n):k(F_1,\dots,F_{m+n})]=[k(x_1,\dots,x_m,y_1,\dots,y_n):k(F_1,\dots,F_m,y_1,\dots,y_n)]\cdot[k(F_1,\dots,F_m,y_1,\dots,y_n):k(F_1,\dots,F_{m+n})]=|G_1|\cdot|G_2|$. Thus $k(x_1,\dots,x_m,y_1,\dots,y_n)^{G_1\times G_2}$ is k-rational.

Step 3. Consider the general case. As in Example 3.4, write $n=n_0+n_1p+n_2p^2+\cdots+n_tp^t$ where $0\le n_i\le p-1$ and $n<p^{t+1}$.

By Step 1, $k(x_1,\dots,x_{p^i})^{P_i}$ is k-rational for any $1\le i\le t$. Thus $k(x_1,\dots,x_{p^i},x_{p^i+1},\dots,x_{2p^i},\dots,x_{n_i\cdot p^i})^{P_i^{n_i}}$ is also k-rational by Step 2.

It follows that $k(x_1,\dots,x_{n-n_0})^P$ is k-rational when $P=(P_1)^{n_1}\times(P_2)^{n_2}\times\cdots\times(P_t)^{n_t}$. Thus $k(x_1,\dots,x_n)^P$ is also k-rational. □

REMARK

In [KP, Theorem 1.7], it was proved that if $k(G)$ and $k(H)$ are k-rational, then so is $k(H \wr G)$ where the group $H \wr G$ is actually $H \wr_X G$ with $X = G$ and G acting on X by the left regular representation. We remark that this result follows from Theorem 3.5 and Lemma 1.5 if G and H are transitive subgroups of S_m and S_n respectively.

According to [Kuy2, p. 38], it was proved in Kuyk [Kuy1] that if the exponent of $H \wr G$ is e and k is a field containing a primitive eth root of unity, then the k-rationality of $k(H)$ and $k(G)$ implies that of $k(H \wr G)$ and $k(H \times G)$.

Note that it was proved that if $k(G_1)$ and $k(G_2)$ are k-rational, then $k(G_1 \times G_2)$ is also k-rational (see [KP, Theorem 1.3]) without any assumption on the roots of unity. This result may be generalized to representations other than the regular representation as follows.

THEOREM 3.6

Let G_1, G_2 be finite groups, $G_1 \subset S_m$, $G_2 \subset S_n$, and $G := G_1 \times G_2$. Let G act naturally on the rational function field $k(z_{ij} : 1 \le i \le m, 1 \le j \le n)$ by $g_1 \cdot z_{ij} = z_{g_1(i),j}$, $g_2 \cdot z_{ij} = z_{i,g_2(j)}$ for any $g_1 \in G_1$, any $g_2 \in G_2$. If both $k(x_1, \dots, x_m)^{G_1}$ and $k(y_1, \dots, y_n)^{G_2}$ are k-rational, then $k(z_{ij} : 1 \le i \le m, 1 \le j \le n)^G$ is also k-rational.

Proof

Define an action of G on the rational function field $k(x_i, y_j : 1 \le i \le m, 1 \le j \le n)$ by $g_1 \cdot x_i = x_{g_1(i)}$, $g_1 \cdot y_j = y_j$, $g_2 \cdot x_i = x_i$, $g_2 \cdot y_j = y_{g_2(j)}$ for any $g_1 \in G_1$, any $g_2 \in G_2$.

The k-linear map $\Phi : (\bigoplus_{1 \le i \le m} k \cdot x_i) \oplus (\bigoplus_{1 \le j \le n} k \cdot y_j) \longrightarrow \bigoplus_{1 \le i \le m, 1 \le j \le n} k \cdot z_{ij}$ defined by $\Phi(x_i) = \sum_{1 \le j \le n} z_{ij}$ and $\Phi(y_j) = \sum_{1 \le i \le m} z_{ij}$ is G-equivariant.

By Theorem 2.1(a), $k(z_{ij} : 1 \le i \le m, 1 \le j \le n)^G$ is rational over $k(x_i, y_j : 1 \le i \le m, 1 \le j \le n)^G$. It is easy to see that $k(x_i, y_j : 1 \le i \le m, 1 \le j \le n)^G$ is k-rational. So is $k(z_{ij} : 1 \le i \le m, 1 \le j \le n)^G$. □

It is easy to adapt the proof of the above theorem to the following theorem.

THEOREM 3.7

Let G_1, G_2 be finite groups, $G := G_1 \times G_2$. Suppose that $\rho_1 : G_1 \to \mathrm{GL}(V)$, $\rho_2 : G_2 \to \mathrm{GL}(W)$ are faithful representations over a field k. Let G act on $V \otimes_k W$ by $g_1 \cdot (v \otimes w) = (g_1 \cdot v) \otimes w$, $g_2 \cdot (v \otimes w) = v \otimes (g_2 \cdot w)$ for any $g_1 \in G_1$, $g_2 \in G_2$, $v \in V, w \in W$. Assume that (i) V and W contain a trivial representation, and (ii) $k(V)^{G_1}$ and $k(W)^{G_2}$ are k-rational. Then $k(V \otimes_k W)^G$ is also k-rational.

Proof

Define a suitable action of G on $V \oplus W$ as in the proof of Theorem 3.6.

Let V^* and W^* be the dual spaces of V and W, respectively. Since V contains a trivial representation, it is possible to find a nonzero element $v_0 \in V^*$ such that $g_1 \cdot v_0 = v_0$ for any $g_1 \in G_1$. Similarly, find a nonzero element $w_0 \in W^*$ such that $g_2 \cdot w_0 = w_0$ for any $g_2 \in G_2$.

Define the embedding $\Phi : V^* \oplus W^* \longrightarrow V^* \otimes_k W^*$ given by $\Phi(x) = x \otimes w_0$ and $\Phi(y) = v_0 \otimes y$. Note that Φ is G-equivariant. The remaining proof is omitted. $\square$

Now let us turn to the polynomial invariants of wreath products.

Suppose that a group H acts on Y, which is a finitely generated commutative algebra over a field k. In this case, we require that, for any $h \in H$, the map $\varphi_h : Y \to Y$, defined by $\varphi_h(y) = h \cdot y$ for any $y \in Y$, is a k-algebra morphism. In Theorem 3.8, we take $Y = k[y_j : 1 \leq j \leq n]$ a polynomial ring; in that situation we require furthermore that $\varphi_h(y_j) = \sum_{1 \leq l \leq n} b_{lj}(h) y_l$, where $(b_{lj}(h))_{1 \leq l, j \leq n} \in \mathrm{GL}_n(k)$.

The method presented in Theorem 3.8 is valid for a more general setting; that is, H is a reductive group over a field k and Y is a finitely generated commutative k-algebra. To highlight the crucial idea of our method, we choose to formulate the results for some special cases only.

From here to the end of this section, $G \subset S_m$ and H is a finite group such that G acts on $X_m = \{1, 2, \ldots, m\}$, H acts on the polynomial ring $k[y_j : 1 \leq j \leq n]$ over a field k, and $\varphi_h(y_j) = \sum_{1 \leq l \leq n} b_{lj}(h) y_l$ for any $h \in H$, any $1 \leq j \leq n$ with $(b_{lj}(h))_{1 \leq l, j \leq n} \in \mathrm{GL}_n(k)$. Define $\widetilde{G} := H \wr_{X_m} G$, which acts on the polynomial ring $k[x_{ij} : 1 \leq i \leq m, 1 \leq j \leq n]$ defined by

$$g : x_{ij} \mapsto x_{g(i),j},$$

$$\alpha^{(l)}(h) : x_{ij} \mapsto \begin{cases} x_{ij} & \text{if } i \neq l, \\ \sum_{1 \leq t \leq n} b_{tj}(h) x_{lt} & \text{if } i = l, \end{cases}$$

where $g \in G$, $\alpha^{(l)}(h) \in A$.

The goal is to find the ring of invariants $k[x_{ij} : 1 \leq i \leq m, 1 \leq j \leq n]^{\widetilde{G}} := \{f \in k[x_{ij}] : \lambda(f) = f$ for any $\lambda \in \widetilde{G}\}$.

THEOREM 3.8

Let k be a field, and let $\widetilde{G} := H \wr_{X_m} G$ act on the polynomial ring $k[x_{ij} : 1 \leq i \leq m, 1 \leq j \leq n]$ as above. Assume that $\gcd\{|G|, \operatorname{char} k\} = 1$. Suppose that $k[y_1, \ldots, y_n]^H = k[F_1(y), \ldots, F_N(y)]$ where $F_t(y) = F_t(y_1, \ldots, y_n) \in k[y_1, \ldots, y_n]$ for $1 \leq t \leq N$ (where N is some integer greater than or equal to n). Define an action of G on the polynomial ring $k[X_{it} : 1 \leq i \leq m, 1 \leq t \leq N]$ by $g(X_{it}) = X_{g(i),t}$ for any $g \in G$, $1 \leq i \leq m$, $1 \leq t \leq N$. Define a k-algebra morphism

$$\begin{aligned} \Phi : & \, k[X_{it} : 1 \leq i \leq m, 1 \leq t \leq N] \\ & \to k\big[F_1(x_{i1}, \ldots, x_{in}), \ldots, F_N(x_{i1}, \ldots, x_{in}) : 1 \leq i \leq m\big] \end{aligned}$$

by $\Phi(X_{it}) = F_t(x_{i1}, \ldots, x_{in})$.

If $k[X_{it} : 1 \leq i \leq m, 1 \leq t \leq N]^G = k[H_1(X), \ldots, H_M(X)]$ where $H_s(X) = H_s(X_{11}, \ldots, X_{it}, \ldots, X_{m,N}) \in k[X_{it} : 1 \leq i \leq m, 1 \leq t \leq N]$ for $1 \leq s \leq M$, then $k[x_{ij} : 1 \leq i \leq m, 1 \leq j \leq n]^{\widetilde{G}} = k[\Phi(H_1(X)), \ldots, \Phi(H_M(X))]$.

REMARK

Even without the assumption that $\gcd\{|G|, \operatorname{char} k\} = 1$, it is still known that $k[x_{ij} : 1 \le i \le m, 1 \le j \le n]^{\widetilde{G}}$ is finitely generated over k. With the assumption that $\gcd\{|G|, \operatorname{char} k\} = 1$, the ring of invariants $k[x_{ij} : 1 \le i \le m, 1 \le j \le n]^{\widetilde{G}}$ can be computed effectively (see Example 3.9).

Proof

Step 1. For $1 \le l \le m$, define $H^{(l)} = \langle \alpha^{(l)}(h) : h \in H \rangle$. Then $A = \langle H^{(l)} : 1 \le l \le m \rangle$ and $k[x_{ij} : 1 \le i \le m, 1 \le j \le n]^A = \bigcap_{1 \le l \le m} k[x_{ij} : 1 \le i \le m, 1 \le j \le n]^{H^{(l)}}$.

On the other hand, from the definition of $F_1(y), \ldots, F_N(y)$, it is clear that $k[x_{ij} : 1 \le i \le m, 1 \le j \le n]^{H^{(l)}} = k[F_1(x_{l1}, \ldots, x_{ln}), \ldots, F_N(x_{l1}, \ldots, x_{ln})][x_{ij} : i \ne l, 1 \le i \le m, 1 \le j \le n]$.

It follows that $k[x_{ij} : 1 \le i \le m, 1 \le j \le n]^A = k[F_1(x_{i1}, \ldots, x_{in}), \ldots, F_N(x_{i1}, \ldots, x_{in}) : 1 \le i \le m]$ and G acts on it by

$$g : F_t(x_{i1}, \ldots, x_{in}) \mapsto F_t(x_{g(i),1}, \ldots, x_{g(i),n}),$$

where $g \in G$, $1 \le t \le N$.

Step 2. It is clear that Φ is an equivariant G-map.

We claim that $k[F_1(x_{i1}, \ldots, x_{in}), \ldots, F_N(x_{i1}, \ldots, x_{in}) : i \le m]^G = \Phi(k[X_{it} : 1 \le i \le m, 1 \le t \le N]^G)$.

If $h \in k[F_1(x_{i1}, \ldots, x_{in}), \ldots, F_N(x_{i1}, \ldots, x_{in}) : 1 \le i \le m]^G$, then choose a preimage $\tilde{h}$ of h, that is, $\Phi(\tilde{h}) = h$. Since $h = (\sum_{g \in G} g(h))/|G|$ because $g(h) = h$ for any $g \in G$, it follows that $h = \Phi(\tilde{h}) = \Phi(\sum_{g \in G} g(\tilde{h}))/|G|$. Since $\sum_{g \in G} g(\tilde{h}) \in k[X_{it} : 1 \le i \le m, 1 \le t \le N]^G$, it follows that h belongs to the image of $k[X_{it} : 1 \le i \le m, 1 \le t \le N]^G$. $\square$

EXAMPLE 3.9

Let $G = S_m$ act on the polynomial ring $k[X_{it} : 1 \le i \le m, 1 \le t \le N]$ by $g(X_{it}) = X_{g(i),t}$ for any $g \in G$, any $1 \le i \le m$, $1 \le t \le N$.

Let $f_1, \ldots, f_m$ be the elementary symmetric functions of $X_1, \ldots, X_m$; that is, $f_1 = \sum_{1 \le i \le m} X_i$, $f_2 = \sum_{1 \le i < j \le m} X_i X_j, \ldots, f_m = X_1 X_2 \cdots X_m$.

The polarized polynomials of f_2 with respect to the variables X_{i1} and X_{i2} where $1 \le i \le m$ are

$$\sum_{1 \le i < j \le m} X_{i1} X_{j1}, \qquad \sum_{\substack{1 \le i,j \le m \\ i \ne j}} X_{i1} X_{j2}, \qquad \sum_{1 \le i < j \le m} X_{i2} X_{j2}.$$

Similarly we may define the polarized polynomials of f_2 with respect to the variables $X_{i1}, X_{i2}, X_{i3}, \ldots, X_{im}$ where $1 \le i \le N$. See [Sm, pp. 60–61] for details.

Assume that $1/|G|! \in k$. Then $k[X_{it} : 1 \le i \le m, 1 \le t \le N]^{S_m}$ is generated over k by all the polarized polynomials of $f_1, \ldots, f_m$ (see [Sm, p. 68, Theorem 3.4.1]).

If we assume only that $1/|G| \in k$ and $G \subset \mathrm{GL}_m(k)$ is a finite group, then it is still possible to compute $k[X_{it} : 1 \le i \le m, 1 \le t \le N]^G$ effectively. See [Fl] and [Fo] for details.

EXAMPLE 3.10

Let $\sigma=(1,2)$, and let $G=\langle\sigma\rangle$ act on $X_2=\{1,2\}$. Also let $H=\langle\tau\rangle\simeq C_3$ act on $\mathbb{C}[y_1,y_2,y_3]$ by $\tau: y_1\mapsto y_1,\ y_2\mapsto \omega y_2,\ y_3\mapsto\omega^2 y_3$, where $\omega=e^{2\pi\sqrt{-1}/3}$. Define $\widetilde{G}=H\wr_{X_2}G$ and let it act on $\mathbb{C}[x_{ij}:1\le i\le 2, 1\le j\le 3]$ by

$$\sigma: x_{ij}\mapsto x_{\sigma(i),j},$$

$$\tau_1: x_{11}\mapsto x_{11}, x_{12}\mapsto \omega x_{12}, x_{13}\mapsto\omega^2 x_{13}, x_{2j}\mapsto x_{2j},$$

$$\tau_2: x_{21}\mapsto x_{21}, x_{22}\mapsto \omega x_{22}, x_{23}\mapsto\omega^2 x_{23}, x_{1j}\mapsto x_{1j}.$$

Then $\widetilde{G}=\langle\sigma,\tau_1,\tau_2\rangle$ and $\mathbb{C}[x_{ij}:1\le i\le 2,1\le j\le 3]^{\langle\tau_1,\tau_2\rangle}=\mathbb{C}[x_{11},f_1,f_2,f_3, x_{21},f_1',f_2',f_3']$ where $f_1=x_{12}^3,\ f_2=x_{12}x_{13},\ f_3=x_{13}^3,\ f_1'=x_{22}^3,\ f_2'=x_{22}x_{23}$, $f_3'=x_{23}^3$. Moreover, $\sigma: x_{11}\leftrightarrow x_{21},\ f_1\leftrightarrow f_1',\ f_2\leftrightarrow f_2',\ f_3\leftrightarrow f_3'$.

Define X_{it} (where $1\le i\le 2,\ 1\le t\le 4$) as in Theorem 3.8 with $\sigma: X_{ij}\leftrightarrow X_{\sigma(i),j}$. It is easy to verify that $\mathbb{C}[x_{ij}:1\le i\le 2,1\le j\le 3]^{\widetilde{G}}=\mathbb{C}[x_{11}+x_{21}, f_1+f_1', f_2+f_2', f_3+f_3', x_{11}x_{21}, f_1f_1', f_2f_2', f_3f_3', x_{11}f_1'+x_{21}f_1, x_{11}f_2'+x_{21}f_2, x_{11}f_3'+x_{21}f_3, f_1f_2'+f_1'f_2, f_1f_3'+f_1'f_3, f_2f_3'+f_2'f_3]$.

4. Proof of Theorem 1.2

Let G be a subgroup of S_6 acting naturally on $k(x_1,\dots,x_6)$. We will study the rationality of $k(x_1,\dots,x_6)^G$.

If G is not transitive, say, G leaves invariant $k(x_1,\dots,x_4)$ and $k(x_5,x_6)$, then we may apply Theorem 2.2 and work on $k(x_1,\dots,x_4,x_5/x_6)^G$. The proof is similar to the first paragraph in the proof of [KW, Theorem 3.4]. In case where G leaves invariant $k(x_1,x_2,x_3)$ and $k(x_4,x_5,x_6)$, the restrictions of G to $k(x_1,x_2,x_3)$ and $k(x_4,x_5,x_6)$ are isomorphic to S_3, C_3, or C_2; thus we may either apply Theorem 2.1 or solve the rationality problem separately for $k(x_1,x_2,x_3)$ and $k(x_4,x_5,x_6)$. The details are omitted.

From now on, we will assume that G is a transitive subgroup of S_6.

According to [DM, p. 60], such a group is conjugate to one of the following 16 groups:

$$G_1=\big\langle(1,2,3,4,5,6)\big\rangle\simeq C_6,$$

$$G_2=\big\langle(1,2)(3,4)(5,6),(1,3,5)(2,6,4)\big\rangle\simeq S_3,$$

$$G_3=\big\langle(1,2,3,4,5,6),(1,6)(2,5)(3,4)\big\rangle\simeq D_6,$$

$$G_4=\big\langle(1,2,3)(4,5,6),(1,2)(4,5),(1,4)\big\rangle\simeq S_2\wr_{X_3}S_3,$$

$$G_5=\big\langle(1,2,3)(4,5,6),(1,2)(4,5),(1,4)(2,5)\big\rangle=G_4\cap A_6,$$

$$G_6=\big\langle(1,2,3)(4,5,6),(1,5,4,2)\big\rangle\simeq S_4,$$

$$G_7=\big\langle(1,2,3)(4,5,6),(1,4)(2,5)\big\rangle=G_6\cap A_6,$$

$$G_8=\big\langle(1,2,3)(4,5,6),(1,4)\big\rangle\simeq C_2\wr_{X_3}C_3,$$

$$G_9=\big\langle(1,2,3),(1,2),(1,4)(2,5)(3,6)\big\rangle\simeq S_3\wr_{X_2}C_2,$$

$$
\begin{aligned}
G_{10} &= \big\langle (1,2,3), (1,4,2,5)(3,6), (1,2)(4,5) \big\rangle = G_9 \cap A_6, \\
G_{11} &= \big\langle (1,2,3), (1,2)(4,5), (1,4)(2,5)(3,6) \big\rangle \simeq C_3^2 \rtimes C_2^2, \\
G_{12} &= \big\langle (1,2,3), (1,4)(2,5)(3,6) \big\rangle \simeq C_3 \wr_{X_2} C_2, \\
G_{13} &= \big\langle (0,1,2,3,4), (0,\infty)(1,4)(1,2,4,3) \big\rangle \simeq \mathrm{PGL}_2(\mathbb{F}_5), \\
G_{14} &= \big\langle (0,1,2,3,4), (0,\infty)(1,4) \big\rangle \simeq \mathrm{PSL}_2(\mathbb{F}_5), \\
G_{15} &= A_6, \\
G_{16} &= S_6.
\end{aligned}
$$

Be aware that the above descriptions of the groups G_2, G_4, and G_{10} are different from those in [DM, p. 60], because the presentation there contains some minor mistakes.

Note that the rationality of $k(x_1,\ldots,x_6)^{G_{16}}$ is easy. On the other hand, the rationality of $k(x_1,\ldots,x_6)^{G_{15}}$ is still an open problem. When $G = G_9$, G_{10}, G_{11}, or G_{12}, the rationality of $k(x_1,\ldots,x_6)^G$ was proved in [Zh, Section 3].

When $G = G_4$, G_8, G_9, or G_{12}, the group is a wreath product. We may apply Theorem 3.5, because $k(x_1,x_2,x_3)^{S_3}$, $k(C_2)$, $k(C_3)$ are k-rational by Theorem 1.1. For example, consider the case $G = G_4$. Note that S_3 acts transitively on $X_3 = \{1,2,3\}$. Define $\widetilde{G} = S_2 \wr_{X_3} S_3$, define $G = S_3$, and let $H = S_2 = \langle \tau \rangle$ act on $Y_2 = \{1,2\}$. In the notation of Section 2, we have $A = \prod_{1\le i\le 3} H_i$ where each $H_i = H$. It follows that $\widetilde{G} = \langle \sigma_1, \sigma_2, \alpha^{(1)}(\tau)\rangle$ where $\sigma_1 = (1,2,3)$, $\sigma_2 = (1,2) \in G$ by Example 3.3. It is not difficult to show that $\widetilde{G} \simeq G_4$.

Thus it remains to study the rationality of $k(x_1,\ldots,x_6)^G$ when $G = G_1$, G_2, G_3, G_5, G_6, G_7, G_{13}, and G_{14}. We study the case G_{13} and G_{14} first.

THEOREM 4.1

If $G = G_{13}$ or G_{14}, then $\mathbb{C}(x_1,\ldots,x_6)^G$ is $\mathbb{C}$-rational, and $k(x_1,\ldots,x_6)^G$ is stably k-rational where k is any field.

Proof

We will prove that G_{13} is isomorphic to S_5 as abstract groups. Then it will be shown that the permutation representation of G_{13} as a subgroup of S_6 is equivalent to the direct sum of the trivial representation and a 5-dimensional irreducible representation of S_5 over $\mathbb{Q}$. Then we will apply the results of [Sh].

Step 1. Since $G_{13} = \mathrm{PGL}_2(\mathbb{F}_5)$ is the automorphism group of the projective line over $\mathbb{F}_5$, it acts naturally on $\mathbb{F}_5 \cup \{\infty\}$. For example, the fractional linear transformations $x \mapsto x+1$, $x \mapsto 2/x$, and $x \mapsto 4/x$ correspond to the permutations $(0,1,2,3,4)$, $(0,\infty)(1,4)(1,2,4,3)$ $(= (0,\infty)(1,2)(3,4))$, and $(0,\infty)(1,4)$, respectively.

We rewrite the points 0, 1, 2, 3, 4, ∞ as 1, 2, 3, 4, 5, 6. Thus G_{13} and G_{14} are defined by $G_{13} = \langle (1,2,3,4,5), (1,6)(2,3)(4,5)\rangle \subset S_6$, $G_{14} = \langle (1,2,3,4,5), (1,6)(2,5)\rangle \subset S_6$.

Define a group homomorphism $\rho : S_5 \to S_6$ by $\rho : (1,2) \mapsto (1,6)(2,3)(4,5)$, $(2,3) \mapsto (1,5)(2,6)(3,4)$, $(3,4) \mapsto (1,2)(3,6)(4,5)$, $(4,5) \mapsto (1,5)(2,3)(4,6)$.

Note that the group S_5 is defined by generators $\{(i,i+1) : 1 \le i \le 4\}$ with relations $(i,i+1)^2 = 1$ (for $1 \le i \le 4$), $((i,i+1)(i+1,i+2))^3 = 1$ (for $1 \le i \le 3$), $((i,i+1)(j,j+1))^2 = 1$ if $|j-i| \ge 2$. These relations are preserved by $\{\rho((i,i+1)) : 1 \le i \le 4\}$. Hence ρ is a well-defined group homomorphism.

We will show that $\rho(S_5) = G_{13}$ and $\mathrm{Ker}(\rho) = \{1\}$, that is, $S_5 \simeq G_{13}$ as abstract groups. By the definition of ρ, it is easy to verify that $\rho((1,2,3,4,5)) = (1,2,3,4,5) \in S_6$. Since $S_5 = \langle (1,2,3,4,5), (1,2) \rangle$ and $\rho((1,2,3,4,5))$, $\rho((12)) \in G_{13}$, it follows that $\rho(S_5) \subset G_{13}$. Since $A_5 \not\subset \mathrm{Ker}(\rho)$, it follows that ρ is injective and $\rho(S_5) = G_{13}$ because $|S_5| = 120 = |G_{13}|$.

It is possible to construct an embedding of S_5 as a transitive subgroup of S_6 by other methods (see, for example, [Di, Section 4]).

Since $G_{14} = \mathrm{PSL}_2(\mathbb{F}_5)$ is a subgroup of $G_{13} = \mathrm{PGL}_2(\mathbb{F}_5)$ of index 2, it follows that the restriction of ρ to A_5 gives an isomorphism of A_5 to G_{14}.

Step 2. Let $\rho' : S_6 \to \mathrm{GL}_6(k)$ be the natural representation of S_6 where k is any field. Then $\rho' \circ \rho : S_5 \to \mathrm{GL}_6(k)$ provides the permutation representation of S_5 when it is embedded in S_6 via ρ. It follows that S_5 acts on $k(x_1, \ldots, x_6)$ via $\rho' \circ \rho$.

When $\mathrm{char}\, k = 0$, by checking the character table, we find that the representation $\rho' \circ \rho$ decomposes into $\mathbb{1} \oplus \rho_0$ where $\mathbb{1}$ is the trivial representation of S_5, and ρ_0 is the 5-dimensional irreducible representation of S_5 which is equivalent to the representation W' in [FH, p. 28].

Step 3. For any field k, let S_5 act on the rational function field $k(y_1, \ldots, y_5)$ by $\sigma(y_i) = y_{\sigma(i)}$ for any $\sigma \in S_5$. Since $G_{13} \simeq S_5$ by Step 1, we may consider the action of G_{13} (resp., G_{14}) on $k(x_1, \ldots, x_6)$ also. Thus G_{13} and G_{14} act on $k(x_1, \ldots, x_6, y_1, \ldots, y_5)$.

Apply Theorem 2.1(a) to $k(x_1, \ldots, x_6, y_1, \ldots, y_5)^G$ where $G = G_{13}$ or G_{14}. We find that $k(x_1, \ldots, x_6, y_1, \ldots, y_5)^G = k(x_1, \ldots, x_6)^G(t_1, \ldots, t_5)$, where $g(t_i) = t_i$ for all $g \in G$, all $1 \le i \le 5$.

On the other hand, apply Theorem 2.1(a) to $k(x_1, \ldots, x_6, y_1, \ldots, y_5)^G$ again with $L = k(y_1, \ldots, y_5)$. We get $k(x_1, \ldots, x_6, y_1, \ldots, y_5)^G = k(y_1, \ldots, y_5)^G(s_1, \ldots, s_6)$, where $g(s_i) = s_i$ for all $g \in G$, all $1 \le i \le 6$.

Since $k(y_1, \ldots, y_5)^G$ is k-rational when $G = G_{13} \simeq S_5$, and when $G = G_{14} \simeq A_5$ by Maeda's [Ma] theorem (i.e., Theorem 2.5), we find that $k(x_1, \ldots, x_6)^G$ is stably k-rational.

Step 4. We will show that $\mathbb{C}(x_1, \ldots, x_6)^{G_{13}}$ is $\mathbb{C}$-rational. Recall a result of Shepherd-Barron [Sh] that if $S_5 \to \mathrm{GL}(V)$ is any irreducible representation over $\mathbb{C}$, then $\mathbb{C}(V)^G$ is $\mathbb{C}$-rational.

By Step 2, since the representation $\rho' \circ \rho$ decomposes, we may write $\mathbb{C}(x_1, \ldots, x_6) = \mathbb{C}(t_1, \ldots, t_6)$ where $g(t_6) = t_6$ for any $g \in G_{13}$, and $G_{13} \simeq S_5$ acts on $\bigoplus_{1 \le i \le 5} \mathbb{C} \cdot t_i$ irreducibly.

Apply Shepherd-Barron's [Sh] theorem. We get that $\mathbb{C}(t_1,\dots,t_5)^{G_{13}}$ is $\mathbb{C}$-rational. Hence $\mathbb{C}(x_1,\dots,x_6)^{G_{13}} = \mathbb{C}(t_1,\dots,t_6)^{G_{13}} = \mathbb{C}(t_1,\dots,t_5)^{G_{13}}(t_6)$ is also $\mathbb{C}$-rational.

Step 5. We will show that $\mathbb{C}(x_1,\dots,x_6)^{G_{14}}$ is $\mathbb{C}$-rational.

By Step 1, $G_{14} \simeq A_5$ as abstract groups.

By [FH, p. 29], A_5 has a faithful complex irreducible representation $A_5 \to \mathrm{GL}(V)$ where $\dim_{\mathbb{C}} V = 3$. Let z_1, z_2, z_3 be a dual basis of V. Then $\mathbb{C}(V)^{G_{14}} = \mathbb{C}(z_1,z_2,z_3)^{G_{14}} = \mathbb{C}(z_1/z_3, z_2/z_3, z_3)^{G_{14}}$.

Consider $\mathbb{C}(x_1,\dots,x_6)^{G_{14}}$. Define $y_0 = \sum_{1\le i\le 6} x_i$, $y_i = x_i - (y_0/6)$. Sine G_{14} permutes $x_1,\dots,x_6$, it follows that G_{14} permutes $y_1,\dots,y_6$ where $\sum_{1\le i\le 6} y_i = 0$. Thus $\mathbb{C}(x_1,\dots,x_6)^{G_{14}} = \mathbb{C}(y_1,\dots,y_5)^{G_{14}}(y_0)$.

Since $\mathbb{C}(y_1,\dots,y_5)^{G_{14}} = \mathbb{C}(y_1/y_5, y_2/y_5, y_3/y_5, y_4/y_5, y_5)^{G_{14}}$ and $g(y_5) = a_g \cdot y_5 + b_g$ for some $a_g, b_g \in \mathbb{C}(y_1/y_5,\dots,y_4/y_5)$, we may apply Theorem 2.2. We find that $\mathbb{C}(y_1/y_5,\dots,y_4/y_5,y_5)^{G_{14}} = \mathbb{C}(y_1/y_5,\dots,y_4/y_5)^{G_{14}}(t)$ for some t with $g(t) = t$ for any $g \in G_{14}$. In conclusion, $\mathbb{C}(x_1,\dots,x_6)^{G_{14}} = \mathbb{C}(y_1/y_5, y_2/y_5, y_3/y_5, y_4/y_5)^{G_{14}}(t, y_0)$.

On the other hand, apply Theorem 2.1(b) to $\mathbb{C}(y_1/y_5, y_2/y_5, y_3/y_5, y_4/y_5, z_1/z_3, z_2/z_3)^{G_{14}}$. It follows that $\mathbb{C}(y_1/y_5,\dots,y_4/y_5, z_1/z_3, z_2/z_3)^{G_{14}} = \mathbb{C}(y_1/y_5, \dots, y_4/y_5)^{G_{14}}(t_1,t_2)$ where $g(t_1) = t_1$, $g(t_2) = t_2$ for any $g \in G$. Thus $\mathbb{C}(x_1,\dots, x_6)^{G_{14}} = \mathbb{C}(y_1/y_5, y_2/y_5, y_3/y_5, y_4/y_5)^{G_{14}}(t, y_0) \simeq \mathbb{C}(y_1/y_5,\dots,y_4/y_5)^{G_{14}}(t_1,t_2) = \mathbb{C}(y_1/y_5,\dots,y_4/y_5, z_1/z_3, z_2/z_3)^{G_{14}}$.

But G_{14} acts faithfully also on $\mathbb{C}(z_1/z_3, z_2/z_3)$ because $G_{14} \simeq A_5$ is a simple group. Apply Theorem 2.1(b) to $\mathbb{C}(y_1/y_5,\dots,y_4/y_5, z_1/z_3, z_2/z_3)^{G_{14}}$ again with $L = \mathbb{C}(z_1/z_3, z_2/z_3)^{G_{14}}$. We get $\mathbb{C}(y_1/y_5,\dots,y_4/y_5, z_1/z_3, z_2/z_3)^{G_{14}} = \mathbb{C}(z_1/z_3, z_2/z_3)^{G_{14}}(s_1,s_2,s_3,s_4)$ with $g(s_i) = s_i$ for all $g \in G_{14}$, for all $1 \le i \le 4$.

We conclude that $\mathbb{C}(x_1,\dots,x_6)^{G_{14}} \simeq \mathbb{C}(z_1/z_3, z_2/z_3)^{G_{14}}(s_1,s_2,s_3,s_4)$.

By Castelnuovo's theorem (see [Za]), $\mathbb{C}(z_1/z_3, z_2/z_3)^{G_{14}}$ is $\mathbb{C}$-rational. Hence $\mathbb{C}(x_1,\dots,x_6)^{G_{14}}$ is $\mathbb{C}$-rational. □

REMARK

In the last paragraph of Step 5, if we use Zariski–Castelnuovo's theorem instead of Castelnuovo's original theorem, then we find a slightly general result as follows. If k is an algebraically closed field with $\operatorname{char} k \neq 2, 5$, then $k(x_1,\dots,x_6)^{G_{14}}$ is k-rational. Note that the assumption that $\operatorname{char} k \neq 2,5$ is added in order to guarantee the existence of the 3-dimensional irreducible representation in [FH, p. 29].

Proof of Theorem 1.2

It remains to prove that, for any field k, $k(x_1,\dots,x_6)^G$ is k-rational where $G = G_i$ with $1 \le i \le 3$ or $5 \le i \le 7$.

Case 1. $G = G_1$. Since $G_1 = \langle (1,2,3,4,5,6) \rangle$, $k(x_1,\dots,x_6)^{G_1} = k(G_1)$ is k-rational by Theorem 2.6.

Case 2. $G = G_2 = \langle (1,2)(3,4)(5,6), (1,3,5)(2,6,4) \rangle$. Write $\sigma = (1,3,5)(2,6,4)$, $\tau = (1,2)(3,4)(5,6)$. Then the actions are given by

$$\sigma : x_1 \mapsto x_3 \mapsto x_5 \mapsto x_1, x_2 \mapsto x_6 \mapsto x_4 \mapsto x_2,$$

$$\tau : x_1 \leftrightarrow x_2, x_3 \leftrightarrow x_4, x_5 \leftrightarrow x_6.$$

Define $y_1 = x_1/x_2$, $y_2 = x_3/x_6$, $y_3 = x_5/x_4$. Then we get

$$\sigma : y_1 \mapsto y_2 \mapsto y_3 \mapsto y_1,$$

$$\tau : y_1 \mapsto 1/y_1, y_2 \mapsto 1/y_3, y_3 \mapsto 1/y_2.$$

It follows that $k(x_1,\ldots,x_6)^{G_2} = k(y_1,y_2,y_3,x_2,x_4,x_6)^{G_2}$. Applying Theorem 2.1(a) with $L = k(y_1,y_2,y_3)$, we find that $k(y_1,y_2,y_3,x_2,x_4,x_6)^{G_2} = k(y_1,y_2,y_3)^{G_2}(t_1,t_2,t_3)$ where $g(t_i) = t_i$ for all $g \in G_2$, for all $1 \le i \le 3$.

Since G_2 acts on $k(y_1,y_2,y_3)$ by purely monomial k-automorphisms, we may apply Theorem 2.4. Hence $k(y_1,y_2,y_3)^{G_2}$ is k-rational.

Case 3. $G = G_3 = \langle (1,2,3,4,5,6), (1,6)(2,5)(3,4) \rangle$. Write $\sigma = (1,2,3,4,5,6)$, $\tau = (1,6)(2,5)(3,4)$. Then σ and τ act on $k(x_1,\ldots,x_6)$ by

$$\sigma : x_1 \mapsto x_2 \mapsto x_3 \mapsto x_4 \mapsto x_5 \mapsto x_6 \mapsto x_1,$$

$$\tau : x_1 \leftrightarrow x_6, x_2 \leftrightarrow x_5, x_3 \leftrightarrow x_4.$$

Subcase 3.1. $\operatorname{char} k \neq 2$. Define $y_1 = x_1 - x_4$, $y_2 = x_2 - x_5$, $y_3 = x_3 - x_6$, $y_4 = x_1 + x_4$, $y_5 = x_2 + x_5$, $y_6 = x_3 + x_6$.

Then $k(x_1,\ldots,x_6) = k(y_1,\ldots,y_6)$ and

$$\sigma : y_1 \mapsto y_2 \mapsto y_3 \mapsto -y_1, y_4 \mapsto y_5 \mapsto y_6 \mapsto y_4,$$

$$\tau : y_1 \mapsto -y_3, y_2 \mapsto -y_2, y_3 \mapsto -y_1, y_4 \leftrightarrow y_6, y_5 \mapsto y_5.$$

Apply Theorem 2.1(a). We get that $k(y_1,\ldots,y_6)^{G_3} = k(y_1,y_2,y_3)^{G_3}(t_1,t_2,t_3)$ where $\sigma(t_i) = \tau(t_i) = t_i$ for $1 \le i \le 3$.

Write $k(y_1,y_2,y_3) = k(y_1/y_3, y_2/y_3, y_3)$. Apply Theorem 2.1. We get that $k(y_1/y_3,y_2/y_3,y_3)^{G_3} = k(y_1/y_3,y_2/y_3)^{G_3}(t)$ where $\sigma(t) = \tau(t) = t$.

Note that G_3 acts on y_1/y_3 and y_2/y_3 by monomial k-automorphisms. By Hajja's [Ha] theorem, $k(y_1/y_3,y_2/y_3)^{G_3}$ is k-rational.

Subcase 3.2. $\operatorname{char} k = 2$. Define $y_1 = x_1/(x_1+x_4)$, $y_2 = x_2/(x_2+x_5)$, $y_3 = x_3/(x_3+x_6)$, $y_4 = x_1+x_4$, $y_5 = x_2+x_5$, $y_6 = x_3+x_6$. Then $k(x_1,\ldots,x_6) = k(y_1,\ldots,y_6)$ and

$$\sigma : y_1 \mapsto y_2 \mapsto y_3 \mapsto y_1 + 1, y_4 \mapsto y_5 \mapsto y_6 \mapsto y_4,$$

$$\tau : y_1 \mapsto y_3 + 1, y_2 \mapsto y_2 + 1, y_3 \mapsto y_1 + 1, y_4 \leftrightarrow y_6, y_5 \mapsto y_5.$$

Apply Theorem 2.1(a). We get that $k(y_1,\ldots,y_6)^{G_3} = k(y_1,y_2,y_3)^{G_3}(t_1,t_2,t_3)$ where $\sigma(t_i) = \tau(t_i) = t_i$ for $1 \le i \le 3$.

Define $z_1 = y_1(y_1+1)$, $z_2 = y_1 + y_2$, $z_3 = y_2 + y_3$.

Then $k(y_1,y_2,y_3)^{\langle \sigma^3 \rangle} = k(z_1,z_2,z_3)$ and

$$\sigma : z_1 \mapsto z_1 + z_2^2 + z_2, z_2 \mapsto z_3 \mapsto z_2 + z_3 + 1 \mapsto z_2,$$

$$\tau : z_1 \mapsto z_1 + z_2^2 + z_3^2 + z_2 + z_3, z_2 \leftrightarrow z_3.$$

Apply Theorem 2.2 to $k(z_1, z_2, z_3)^{\langle \sigma, \tau \rangle}$ with $L = k(z_2, z_3)$. We get that $k(z_1, z_2, z_3)^{G_3} = k(z_2, z_3)^{G_3}(t)$ where $\sigma(t) = \tau(t) = t$.

Define $z_4 = z_2 + z_3 + 1$. Then $z_2 + z_3 + z_4 = 1$ and $\sigma : z_2 \mapsto z_3 \mapsto z_4 \mapsto z_2$, $\tau : z_2 \leftrightarrow z_3$, $z_4 \mapsto z_4$. Thus $\langle \sigma, \tau \rangle \simeq S_3$ on $k(z_2, z_3, z_4)$ with $z_2 + z_3 + z_4 = 1$.

Define $u = z_2z_3 + z_2z_4 + z_3z_4 = z_2^2 + z_2z_3 + z_3^2 + z_2 + z_3$, $v = z_2z_3z_4 = z_2^2z_3 + z_2z_3^2 + z_2z_3$.

Since $k(u, v) \subset k(z_2, z_3)^{G_3}$ and $[k(z_2, z_3) : k(u, v)] \le 6 = [k(z_2, z_3) : k(z_2, z_3)^{G_3}]$, it follows that $k(z_2, z_3)^{G_3} = k(u, v)$. Hence $k(y_1, y_2, y_3)^{G_3}$ is k-rational.

Case 4. $G = G_5 = \langle (1, 2, 3)(4, 5, 6), (1, 2)(4, 5), (1, 4)(2, 5) \rangle$. Write $\sigma = (1, 2, 3)(4, 5, 6)$, $\tau = (1, 2)(4, 5)$, $\lambda_1 = (1, 4)(2, 5)$, $\lambda_2 = \sigma\lambda_1\sigma^{-1} = (2, 5)(3, 6)$. Note that $\langle \lambda_1, \lambda_2 \rangle \simeq C_2 \times C_2$. The action of G_5 is given by

$$\lambda_1 : x_1 \leftrightarrow x_4, x_2 \leftrightarrow x_5, x_3 \mapsto x_3, x_6 \mapsto x_6,$$

$$\lambda_2 : x_1 \mapsto x_1, x_4 \mapsto x_4, x_2 \leftrightarrow x_5, x_3 \leftrightarrow x_6,$$

$$\sigma : x_1 \mapsto x_2 \mapsto x_3 \mapsto x_1, x_4 \mapsto x_5 \mapsto x_6 \mapsto x_4,$$

$$\tau : x_1 \leftrightarrow x_2, x_3 \mapsto x_3, x_4 \leftrightarrow x_5, x_6 \mapsto x_6.$$

Subcase 4.1. $\operatorname{char} k \neq 2$. Define $y_1 = x_1 - x_4$, $y_2 = x_2 - x_5$, $y_3 = x_3 - x_6$, $y_4 = x_1 + x_4$, $y_5 = x_2 + x_5$, $y_6 = x_3 + x_6$.

Then $k(x_1, \dots, x_6) = k(y_1, \dots, y_6)$ and

$$\lambda_1 : y_1 \mapsto -y_1, y_2 \mapsto -y_2, y_3 \mapsto y_3, y_4 \mapsto y_4, y_5 \mapsto y_5, y_6 \mapsto y_6,$$

$$\lambda_2 : y_1 \mapsto y_1, y_2 \mapsto -y_2, y_3 \mapsto -y_3, y_4 \mapsto y_4, y_5 \mapsto y_5, y_6 \mapsto y_6,$$

$$\sigma : y_1 \mapsto y_2 \mapsto y_3 \mapsto y_1, y_4 \mapsto y_5 \mapsto y_6 \mapsto y_4,$$

$$\tau : y_1 \leftrightarrow y_2, y_3 \mapsto y_3, y_4 \leftrightarrow y_5, y_6 \mapsto y_6.$$

Apply Theorem 2.1(a). We get that $k(x_1, \dots, x_6)^{G_5} = k(y_1, \dots, y_6)^{G_5} = k(y_1, y_2, y_3)^{G_5}(t_1, t_2, t_3)$, where $g(t_i) = t_i$ for any $g \in G_5$, any $1 \le i \le 3$.

Define $z_1 = y_2y_3/y_1$, $z_2 = y_1y_3/y_2$, $z_3 = y_1y_2/y_3$. It is not difficult to show that $k(y_1, y_2, y_3)^{\langle \lambda_1, \lambda_2 \rangle} = k(z_1, z_2, z_3)$ and the actions of σ and τ are given by

$$\begin{aligned} &\sigma : z_1 \mapsto z_2 \mapsto z_3 \mapsto z_1, \\ &\tau : z_1 \leftrightarrow z_2, z_3 \mapsto z_3. \end{aligned} \tag{4.1}$$

Hence $k(z_1, z_2, z_3)^{\langle \sigma, \tau \rangle} = k(s_1, s_2, s_3)$ is k-rational where s_1, s_2, s_3 are the elementary symmetric functions in z_1, z_2, z_3.

Subcase 4.2. $\operatorname{char} k = 2$. Define $y_1 = x_1/(x_1 + x_4)$, $y_2 = x_2/(x_2 + x_5)$, $y_3 = x_3/(x_3 + x_6)$, $y_4 = x_1 + x_4$, $y_5 = x_2 + x_5$, $y_6 = x_3 + x_6$. Then $k(x_1, \dots, x_6) = k(y_1, \dots, y_6)$ and

$$\lambda_1 : y_1 \mapsto y_1 + 1, y_2 \mapsto y_2 + 1, y_3 \mapsto y_3, y_4 \mapsto y_4, y_5 \mapsto y_5, y_6 \mapsto y_6,$$

$$\lambda_2 : y_1 \mapsto y_1, y_2 \mapsto y_2 + 1, y_3 \mapsto y_3 + 1, y_4 \mapsto y_4, y_5 \mapsto y_5, y_6 \mapsto y_6,$$

$$\sigma : y_1 \mapsto y_2 \mapsto y_3 \mapsto y_1, y_4 \mapsto y_5 \mapsto y_6 \mapsto y_4,$$

$$\tau : y_1 \leftrightarrow y_2, y_3 \mapsto y_3, y_4 \leftrightarrow y_5, y_6 \mapsto y_6.$$

Apply Theorem 2.1(a). We get that $k(y_1,\dots,y_6)^{G_5} = k(y_1,y_2,y_3)^{G_5}(t_1,t_2,t_3)$ where $g(t_i)=t_i$ for any $g \in G_5$, any $1 \le i \le 3$.

Define $z_1 = y_1(y_1+1)$, $z_2 = y_2(y_2+1)$, $z_3 = y_1+y_2+y_3$. It is not difficult to verify that $k(y_1,y_2,y_3)^{\langle \lambda_1,\lambda_2\rangle} = k(z_1,z_2,z_3)$ and

$$\sigma : z_1 \mapsto z_2 \mapsto z_1 + z_2 + z_3^2 + z_3, z_3 \mapsto z_3,$$

$$\tau : z_1 \leftrightarrow z_2, z_3 \mapsto z_3.$$

Define $z_4 = z_1 + z_2 + z_3^2 + z_3$. It follows that $\sigma : z_1 \mapsto z_2 \mapsto z_4 \mapsto z_1$ and $z_1 + z_2 + z_4 = z_3^2 + z_3$. Define $u = z_1z_2 + z_1z_4 + z_2z_4 = z_1^2 + z_2^2 + z_1z_2 + z_1z_3 + z_2z_3 + z_1z_3^2 + z_2z_3^2$, $v = z_1z_2z_4 = z_1^2z_2 + z_1z_2^2 + z_1z_2z_3 + z_1z_2z_3^2$. It follows that $k(z_1,z_2,z_3)^{\langle\sigma,\tau\rangle} = k(z_3,u,v)$ is k-rational.

Case 5. $G = G_6$ or G_7, where $G_6 = \langle (1,2,3)(4,5,6),(1,5,4,2)\rangle$, and $G_7 = \langle (1,2,3)(4,5,6),(1,4)(2,5)\rangle$. Write $\sigma = (1,2,3)(4,5,6)$, $\tau = (1,5,4,2)$, $\lambda_1 = \tau^2 = (1,4)(2,5)$, $\lambda_2 = \sigma\lambda_1\sigma^{-1} = (2,5)(3,6)$. Note that $\langle \lambda_1,\lambda_2\rangle \simeq C_2 \times C_2$. Then $k(x_1,\dots,x_6)^{G_6} = k(x_1,\dots,x_6)^{\langle \lambda_1,\lambda_2,\sigma,\tau\rangle}$, $k(x_1,\dots,x_6)^{G_7} = k(x_1,\dots,x_6)^{\langle\lambda_1,\lambda_2,\sigma\rangle}$, and the actions are given by

$$\sigma : x_1 \mapsto x_2 \mapsto x_3 \mapsto x_1, x_4 \mapsto x_5 \mapsto x_6 \mapsto x_4,$$

$$\tau : x_1 \mapsto x_5 \mapsto x_4 \mapsto x_2 \mapsto x_1, x_3 \mapsto x_3, x_6 \mapsto x_6,$$

$$\lambda_1 : x_1 \leftrightarrow x_4, x_2 \leftrightarrow x_5, x_3 \mapsto x_3, x_6 \mapsto x_6,$$

$$\lambda_2 : x_1 \mapsto x_1, x_2 \leftrightarrow x_5, x_3 \leftrightarrow x_6, x_4 \mapsto x_4.$$

The proof is similar to the proof of Case 4.

Subcase 5.1. $\operatorname{char} k \neq 2$. Define $y_1 = x_1 - x_4$, $y_2 = x_2 - x_5$, $y_3 = x_3 - x_6$, $y_4 = x_1 + x_4$, $y_5 = x_2 + x_5$, $y_6 = x_3 + x_6$. Then we find that

$$\lambda_1 : y_1 \mapsto -y_1, y_2 \mapsto -y_2, y_3 \mapsto y_3, y_4 \mapsto y_4, y_5 \mapsto y_5, y_6 \mapsto y_6,$$

$$\lambda_2 : y_1 \mapsto y_1, y_2 \mapsto -y_2, y_3 \mapsto -y_3, y_4 \mapsto y_4, y_5 \mapsto y_5, y_6 \mapsto y_6,$$

$$\sigma : y_1 \mapsto y_2 \mapsto y_3 \mapsto y_1, y_4 \mapsto y_5 \mapsto y_6 \mapsto y_4,$$

$$\tau : y_1 \mapsto -y_2, y_2 \mapsto y_1, y_3 \mapsto y_3, y_4 \leftrightarrow y_5, y_6 \mapsto y_6.$$

Apply Theorem 2.1(a). It remains to prove that $k(y_1,y_2,y_3)^G$ is k-rational, where $G = G_6$ or G_7.

Define $z_1 = y_2y_3/y_1$, $z_2 = y_1y_3/y_2$, $z_3 = y_1y_2/y_3$. Then $k(y_1,y_2,y_3)^{\langle\lambda_1,\lambda_2\rangle} = k(z_1,z_2,z_3)$ and

$$\sigma : z_1 \mapsto z_2 \mapsto z_3 \mapsto z_1,$$

$$\tau : z_1 \mapsto -z_2, z_2 \mapsto -z_1, z_3 \mapsto -z_3.$$

It follows that $k(z_1,z_2,z_3)^{\langle\sigma\rangle} = k(C_3)$ is k-rational by Theorem 2.6. Hence $k(x_1,\dots,x_6)^{G_7}$ is k-rational.

For $k(x_1,\dots,x_6)^{G_6}$, note that $k(z_1,z_2,z_3)^{\langle\sigma,\tau\rangle} = k(z_1/z_3, z_2/z_3, z_3)^{\langle\sigma,\tau\rangle}$. Apply Theorem 2.2. We have $k(z_1/z_3,z_2/z_3,z_3)^{\langle\sigma,\tau\rangle} = k(z_1/z_3,z_2/z_3)^{\langle\sigma,\tau\rangle}(t)$ where $\sigma(t)=\tau(t)=t$.

On the other hand, in the last part of Subcase 4.1, we have $k(z_1,z_2,z_3)^{\langle\sigma,\tau\rangle}$ (see (4.1)). By the same method as above, we have that $k(z_1,z_2,z_3)^{\langle\sigma,\tau\rangle} = k(z_1/z_3,z_2/z_3)^{\langle\sigma,\tau\rangle}(s)$ where $\sigma(s)=\tau(s)=s$.

Note that the actions of σ, τ on z_1/z_3 and z_2/z_3 in (4.1) and in the present situation are the same. Since $k(z_1,z_2,z_3)^{\langle\sigma,\tau\rangle}$ is k-rational in Subcase 4.1, so is $k(z_1,z_2,z_3)^{\langle\sigma,\tau\rangle}$ in the present case.

Subcase 5.2. $\operatorname{char} k = 2$. Define $y_1 = x_1/(x_1+x_4)$, $y_2 = x_2/(x_2+x_5)$, $y_3 = x_3/(x_3+x_6)$, $y_4 = x_1+x_4$, $y_5 = x_2+x_5$, $y_6 = x_3+x_6$. Then we have

$$\lambda_1 : y_1 \mapsto y_1+1, y_2 \mapsto y_2+1, y_3 \mapsto y_3, y_4 \mapsto y_4, y_5 \mapsto y_5, y_6 \mapsto y_6,$$
$$\lambda_2 : y_1 \mapsto y_1, y_2 \mapsto y_2+1, y_3 \mapsto y_3+1, y_4 \mapsto y_4, y_5 \mapsto y_5, y_6 \mapsto y_6,$$
$$\sigma : y_1 \mapsto y_2 \mapsto y_3 \mapsto y_1, y_4 \mapsto y_5 \mapsto y_6 \mapsto y_4,$$
$$\tau : y_1 \mapsto y_2+1, y_2 \mapsto y_1, y_3 \mapsto y_3, y_4 \leftrightarrow y_5, y_6 \mapsto y_6.$$

Apply Theorem 2.1(a). It remains to prove that $k(y_1,y_2,y_3)^G$ is k-rational, where $G = G_6$ or G_7.

Define $z_1 = y_1(y_1+1)$, $z_2 = y_2(y_2+1)$, $z_3 = y_1+y_2+y_3$. Then $k(y_1,y_2,y_3)^{\langle\lambda_1,\lambda_2\rangle} = k(z_1,z_2,z_3)$ and

$$\sigma : z_1 \mapsto z_2 \mapsto z_1+z_2+z_3^2+z_3, z_3 \mapsto z_3,$$
$$\tau : z_1 \leftrightarrow z_2, z_3 \mapsto z_3+1.$$

Define $z_4 = z_1+z_3^2+z_3$, $z_5 = z_2+z_3^2+z_3$. Then $k(z_1,z_2,z_3) = k(z_3,z_4,z_5)$ and

$$\sigma : z_3 \mapsto z_3, z_4 \mapsto z_5 \mapsto z_4+z_5,$$
$$\tau : z_3 \mapsto z_3+1, z_4 \leftrightarrow z_5.$$

Apply Theorem 2.2. We get that $k(z_3,z_4,z_5) = k(z_4,z_5)(t)$ where $\sigma(t) = \tau(t) = t$. Thus it remains to consider $k(z_4,z_5)^{\langle\sigma\rangle}$ and $k(z_4,z_5)^{\langle\sigma,\tau\rangle}$.

Note that $\langle\sigma,\tau\rangle \simeq S_3$ on $k(z_4,z_5)$. Let t_1, t_2, t_3 be the elementary symmetric functions of z_4, z_5, and z_4+z_5. Be aware that $t_1 = z_4+z_5+(z_4+z_5) = 0$. It is easy to see that $k(z_4,z_5)^{\langle\sigma,\tau\rangle} = k(t_2,t_3)$ is k-rational. Hence $k(x_1,\dots,x_6)^{G_6}$ is k-rational.

Consider $k(z_4,z_5)^{\langle\sigma\rangle} = k(z_4/z_5,z_5)^{\langle\sigma\rangle}$. Apply Theorem 2.2. We get that $k(z_4/z_5,z_5)^{\langle\sigma\rangle} = k(z_4/z_5)^{\langle\sigma\rangle}(s)$ where $\sigma(s)=s$, since $k(z_4/z_5)^{\langle\sigma\rangle}$ is k-rational by Lüroth's theorem. Thus $k(x_1,\dots,x_6)^{G_7}$ is k-rational. □

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