On a theorem of Castelnuovo and applications to moduli

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To Professor Xavier Gómez-Mont on the occasion of his 60th birthday

Abstract In this paper we prove a theorem stated by Castelnuovo which bounds the dimension of linear systems of plane curves in terms of two invariants, one of which is the genus of the curves in the system. This extends a previous result of Castelnuovo and Enriques. We classify linear systems whose dimension belongs to certain intervals which naturally arise from Castelnuovo's theorem. Then we make an application to the following moduli problem: what is the maximum number of moduli of curves of geometric genus g varying in a linear system on a surface? It turns out that, for $g \geq 22$, the answer is $2g+1$, and it is attained by trigonal canonical curves varying on a balanced rational normal scroll.

0. Introduction

This paper was inspired by the following problem (see Problem 2.1). Consider the set $\mathcal{X}_g$, with $g \geq 2$, of all linear systems $\mathcal{L}$ of curves on a surface X such that the general curve of $\mathcal{L}$ is irreducible, with geometric genus g. For such an $\mathcal{L}$, consider its image in $\mathcal{M}_g$ via the obvious (rational) moduli map. What is the maximum dimension of this image (called the *number of moduli* of $\mathcal{L}$) when $\mathcal{L}$ varies in $\mathcal{X}_g$?

A naive expectation is that the larger the dimension of $\mathcal{L}$, the larger its number of moduli. So a related question is, what is the maximum of $r = \dim(\mathcal{L})$ as $\mathcal{L}$ varies in $\mathcal{X}_g$? This has a classical answer which goes back to Castelnuovo [4] and Enriques [11]. They proved an important result (see Theorem 1.1) to the effect that $r \leq 3g+5$ with two exceptions which, up to birational equivalence, are the following: either $\mathcal{L}$ is the linear system of plane cubics or the rational map determined by $\mathcal{L}$ realizes X as a scroll. In the former case the number of

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moduli is 1, in the latter it is zero. Castelnuovo and Enriques also classified the cases in which the bound $r = 3g + 5$ is attained: X is then rational and $\mathcal{L}$ is either (up to birational equivalence) the linear system of plane curves of degrees 2 or 4 or a suitable system of hyperelliptic curves. The Castelnuovo-Enriques theorem has been rediscovered and reconsidered a few times in the course of the years (see [10] for classical and more recent references).

At about the same time, Castelnuovo stated in [5], with a rather sketchy proof, a more general and interesting theorem which classifies linear systems on rational surfaces with $r > g$ (see Theorem 1.3). Castelnuovo's argument is based on an ingenious application of adjunction and on a basic inequality (see Theorem 1.2) which improves the original Castelnuovo-Enriques theorem. Castelnuovo's version of Theorem 1.3 is a very interesting result in birational geometry of surfaces, and more recent developments (e.g., [14, Corollary 1.1]) are reminiscent of it. Section 1 is devoted to its proof, following and clarifying Castelnuovo's original idea.

Castelnuovo's theorem applies to our original moduli problem, which we take up in Section 2, where we answer it at least when g is large enough. We prove (see Theorem 2.1) that the maximum number of moduli of a linear system of curves of genus $g \geq 22$ is $2g+1$, and it is attained by the linear systems of trigonal canonical curves on a balanced rational normal scroll in $\mathbb{P}^{g-1}$. (The bound $g \geq 22$ could be improved, but we thought it useless to dwell on this here.) It is remarkable that this maximum is not achieved by linear systems of the largest dimension $3g + 5$ compatible with a nontrivial map to moduli. Indeed, as we said, they consist of hyperelliptic curves, and in fact, they dominate the hyperelliptic locus, which has dimension $2g - 1$ (see Theorem 2.3). The proof of Theorem 2.1 relies on Castelnuovo's theorem, on the concept of *Castelnuovo pairs*, and on their classification and related moduli computation (see Section 2.1).

In conclusion, it is worth mentioning, on the same lines as the problem considered here, another more fascinating and complicated one (attributed to F. O. Schreyer). What is, for large enough g, the maximum dimension of a rational [resp., unirational, uniruled, rationally connected] subvariety of $\mathcal{M}_g$?

Notation, conventions, and generalities. We use standard notation in algebraic geometry. In particular, the symbol $\equiv$ denotes linear equivalence of divisors. If D is a divisor on a smooth, projective variety X, $|D|$ is the complete linear system of D. If $\mathcal{L}$ is a linear system of divisors on X of dimension r, $\phi_{\mathcal{L}} : X \dashrightarrow \mathbb{P}^r$ is the rational map defined by $\mathcal{L}$. The system $\mathcal{L}$ is said to be *simple* if $\phi_{\mathcal{L}}$ maps X birationally to its image.

Let X be a smooth irreducible projective surface. As usual, we denote by $K := K_X$ a *canonical divisor*, by $q := q(X) := h^1(X, \mathcal{O}_X)$ the *irregularity*, and by $p_g := p_g(X) := h^0(X, \mathcal{O}_X(K))$ the *geometric genus* of X.

Let D be a divisor on X. We say that D is a *curve* on X if it is effective. If D is a reduced curve on X, the *geometric genus* g of D is the arithmetic genus of the normalization of D. Often we simply call g the *genus* of D. We use the notation $d = D^2$ and $r = \dim(|D|)$. Moreover, $D' \equiv K + D$ is an *adjoint divisor*, and $|D'|$ is the *adjoint linear system* to D. The system $|D|$ is called *nonspecial* if it is either empty or $h^1(X, \mathcal{O}_X(D)) = 0$.

Suppose that there is a morphism $f: X \to Y$, contracting a curve C of X to a smooth point p of a surface Y and inducing an isomorphism between $X - C$ and $Y - \{p\}$. The divisor E, supported on C, which is the scheme-theoretical fiber of f over p, is called a (-1)-*cycle*, or a (-1)-*curve* if $E = C$ is irreducible.

We consider pairs (X, D), with X a smooth irreducible projective surface and D a curve on it. We extend attributes of D (like being *nef, big, ample*, etc.) or of $|D|$ (like being *simple, special, very ample*, etc.) to the pair (X, D). We say that (X, D) is

- *minimal* if there is no (-1)-curve C on X such that $D \cdot C = 0$;
- an *h-scroll* if there is a smooth rational curve F on X such that $F^2 = 0$ and $D \cdot F = h$. A 1-scroll is simply called a *scroll*.

There are obvious notions of morphism, isomorphism, and rational and birational maps between pairs (see [3]). We are mainly interested in birational invariants of the linear system $|D|$ on X. If $|D|$ has no fixed curves and its general curve is irreducible, then by blowing up the base locus of $|D|$, we may assume that $|D|$ is base point free and the general curve of D is smooth. So we often assume that this is the case. In addition, we may assume that (X, D) is minimal by successively contracting all (-1)-curves E with $D \cdot E = 0$.

If $X \cong \mathbb{P}^2$ and ℓ is a line, the pair (X, D) with $D \equiv m\ell$ is called an *m-Veronese pair.*

As usual, we denote by $\mathbb{F}_a$ the *Hirzebruch surface* $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-a))$. The Picard group of $\mathbb{F}_a$ is freely generated by the classes of the divisors: E, a curve with $E^2 = -a$ (unique if $a > 0$), and F, a *ruling*, that is, a fiber of the structure morphism $f: \mathbb{F}_a \to \mathbb{P}^1$. One has $F^2 = 0, F \cdot E = 1$. A divisor $D \equiv \alpha E + \beta F$ is nef as soon as $D \cdot E = \beta - a\alpha \geq 0$. If $\alpha = 1$ and $\beta \geq a$, then $\phi_{|D|}$ birationally maps $\mathbb{F}_a$ to a *rational normal scroll* of degree $s - 1$ in $\mathbb{P}^s$ with $s = 2\beta - a + 1$. A pair (X, D) with $X \cong \mathbb{F}_a$ and $D \equiv 2E + (a + g - 1)F$ is nef, the general curve in $|D|$ is smooth of genus g, and $r = 3g + 5$. Such a pair is called an (a, g)-*Castelnuovo pair* (see [10]).

1. Castelnuovo's theorem

1.1. Castelnuovo-Enriques theorem

We recall the following theorem which extends results of Castelnuovo [4] and Enriques [11] (see also [10, Theorem 7.3] and [10] for classical and recent references).

THEOREM 1.1 (CASTELNUOVO-ENRIQUES THEOREM)

Let (X, D) be a pair with D an irreducible curve. Assume that $d > 0$, and assume *that (X, D) is not a scroll. Then*

$$d \leq 4g + 4 + \epsilon, \tag{1}$$

where $\epsilon = 1$ if $g = 1$ and $\epsilon = 0$ if $g \neq 1$. Consequently,

$$r \leq 3g + 5 + \epsilon, \tag{2}$$

and the equality holds in (1) if and only if it holds in (2).

If, in addition, the pair (X, D) is minimal, then the equality holds in (2) if and only if one of the following happens:

(i) *$g = 0$, $r = 5$, and (X, D) is a 2-Veronese pair;*
(ii) *$g = 1$, $r = 9$, and (X, D) is a 3-Veronese pair;*
(iii) *$g = 3$, $r = 14$, and (X, D) is a 4-Veronese pair;*
(iv) *(X, D) is a $(2, n + g + 1)$-Castelnuovo pair on $X \cong \mathbb{F}_n$, $n \geq 0$.*

1.2. Castelnuovo's inequality

In this section we prove a result of Castelnuovo [5], which specifies (1).

We consider here minimal pairs (X, D) with $p_g = q = 0$, where D is an irreducible, smooth curve of genus $g \geq 2$ with $d \geq 1$ and $r \geq 1$; hence D is nef. By [10, Proposition 7.1], an adjoint curve $D' \equiv K + D$ is nef and

$$d \leq 4(g - 1) + K^2 \leq 4g + 5, \tag{3}$$

which is basically the proof of (1). (In the last inequality we used the Miyaoka-Yau inequality.) Moreover, $\dim(|D'|) = g - 1$. Set $|D'| = P + |M|$, where P is the fixed divisor and $|M|$ is the movable part, called the *pure adjoint system* of D. We set $g' := p_a(M)$ and $d' = M^2$. One has $M \cdot D = 2g - 2$ and $P \cdot D = 0$, and for all curves $E \leq P$, one has $E^2 < 0$.

LEMMA 1.1

In the above setting, if $d \geq 5$ and $|D|$ is nonspecial, then $P = 0$.

Proof

Reider's theorem (see [2], [15]) implies that if x is a base point of $|D'|$, then there is an irreducible curve A containing x then such that either $A \cdot D = 1, A^2 = 0$ or $A \cdot D = 0, A^2 = -1$.

Let E be an irreducible curve contained in P. For all $x \in E$, we have a curve A_x as above. If $A_x \cdot D = 1$, then $A_x \neq E$. Moreover, $A_x^2 = 0$, so A_x moves in a base point–free pencil $|A|$. Since $A \cdot D = 1$, we would have $g = 0$, a contradiction. Hence $A_x \cdot D = 0$ and $E = A_x$. This shows that $E^2 = -1$.

Since $D \cdot E = 0$ and $r \geq 1$, then $D - E$ is effective. We have the exact sequence $0 \to \mathcal{O}_X(D - E) \to \mathcal{O}_X(D) \to \mathcal{O}_E(D) \cong \mathcal{O}_E \to 0$, which yields the exact sequence $H^1(X, \mathcal{O}_X(D)) \to H^1(E, \mathcal{O}_E) \to H^2(X, \mathcal{O}_X(D - E))$. Since $p_g = 0$, the last space is zero, and the first is zero by assumption. Hence $h^1(E, \mathcal{O}_E) = 0$, and E is rational.

In conclusion, E is a (-1)-curve such that $D\cdot E=0$, contradicting the minimality assumption. □

If $|M|$ is composed with a pencil $|L|$, then $|M|=|(g-1)L|$, $\dim(|L|)=1$, and $|L|$ has no base points on D. Then $D\cdot L=2$, D is hyperelliptic, and

(1) either $|D|$ cuts out a base point–free g^1_2 on the general curve L of $|L|$, hence there is a birational involution $\iota: X \dashrightarrow X$ that fixes all curves in $|D|$, which is not simple (in this case we say that $|D|$ is *composed* with the involution ι);

(2) or $|D|$ cuts out a g^2_2 on L and $|L|$ is a pencil of curves of genus zero.

If $d\geq 5$, then the index theorem implies that $L^2=0$.

THEOREM 1.2 (CASTELNUOVO'S INEQUALITY)

Let (X,D) be minimal with D smooth and irreducible, with $g\geq 2$, $d\geq 1$, and $r\geq 1$. Assume that either D is not hyperelliptic or that $|D|$ is not composed with an involution of X. Then

$$d\leq 3g+7-g', \tag{4}$$

and equality holds if and only if $X=\mathbb{P}^2$.

Proof

Suppose that first $|M|$ is composed with a pencil $|L|$. Then D is hyperelliptic with $D\cdot L=2$ and $|D|$ is not composed with an involution of X. Thus the curves in $|L|$ have genus zero, so X is rational, $L^2=0$, and $g'=2-g$. By (3), we have $d\leq 4g+5=3g+7-g'$. If equality holds, then $K^2=9$; hence $X=\mathbb{P}^2$.

Suppose next that $|M|$ is not composed with a pencil; hence $d'>0$. We have

$$h^0(M,\mathcal{O}_M(M))=h^0(X,\mathcal{O}_X(M))-1=h^0(X,\mathcal{O}_X(K+D))-1=g-1;$$

then $|\mathcal{O}_M(M)|=g^{g-2}_{d'}$. We have two cases:

(a) $K\cdot M\geq 0$;

(b) $K\cdot M<0$.

In case (a), since D' is nef, one has

$$(K+D)^2\geq (K+D)\cdot M\geq D\cdot M\geq 2g-2, \tag{5}$$

and equality implies that $K\cdot M=0$. Let $R\equiv D-M\equiv P-K$. We have two subcases:

(a.1) $|R|=\emptyset$;

(a.2) R is effective.

In case (a.1), one has $1>\chi(\mathcal{O}_X(R))$, which reads $d<3g-g'-2$, and (4) holds.

In case (a.2), one has $2g-2=(K+D)\cdot D=(K+D)\cdot(M+R)\geq (K+D)\cdot M=(K+M+R)\cdot M\geq 2g'-2$; then $g\geq g'$ and, by (5), $g+g'-2\leq 2g-2\leq (K+D)^2=4g-4+K^2-d$. Therefore (4) holds. If equality holds, then $K^2=9$

and $K \cdot M = 0$ (because equality holds in (5)). This cannot happen on a surface of general type because $d' > 0$; thus $X \cong \mathbb{P}^2$.

In case (b), one has $d' > 2g' - 2$. Then $g_{d'}^{g-2}$ is not special; hence $g - 1 = h^0(M, \mathcal{O}_M(M)) = d' - g' + 1$. Since $K + D \equiv P + M$ is nef, then $g + g' - 2 = d' = M^2 \le (K+D)^2 = 4g - 4 + K^2 - d$, so $d \le 3g - g' + (K^2 - 2) \le 3g + 7 - g'$, and if equality holds, then $K^2 = 9$ and, as above, $X \cong \mathbb{P}^2$.

Finally, if $X = \mathbb{P}^2$, then (4) holds with equality. □

1.3. Castelnuovo's theorem

In this section we prove a theorem of Castelnuovo stated in [5] (see also [7]) which classifies linear systems on rational surfaces with $r > g$. A remark is in order.

REMARK 1.1

Consider a pair (X, D) with D smooth and irreducible such that $r > g$. Then $h^0(D, \mathcal{O}_D(D)) > g$; hence $h^1(D, \mathcal{O}_D(D)) = 0$. Thus $d - g + 1 = h^0(D, \mathcal{O}_D(D)) > g$, and therefore $d \ge 2g$, so that $K \cdot D < 0$, which implies that X has negative Kodaira dimension. This shows that the rationality assumption on X in Theorem 1.3 below is no restriction. In this case, one has $h^1(X, \mathcal{O}_X(D)) = 0$; that is, $|D|$ is nonspecial.

THEOREM 1.3 (CASTELNUOVO'S THEOREM)

Let (X, D) be minimal with D smooth, irreducible, and of genus $g \ge 2$, X rational, $|D|$ not composed with an involution of X, and

$$r \ge \tau(\mu, g) := \frac{(\mu+2)g + \epsilon_\mu}{\mu} + 2\mu + 3, \tag{6}$$

where

$$\epsilon_\mu = \begin{cases} 1 & \text{for } \mu \text{ odd,} \\ 2 & \text{for } \mu \text{ even.} \end{cases}$$

Then

(i) *either there is a birational morphism $\phi : X \to \mathbb{P}^2$ such that $|D|$ is the proper transform of a linear system of plane curves of degree $m \le 2\mu + 1$ with base points of multiplicity $k \le \ulcorner \mu/2 \urcorner - 1$,*

(ii) *or (X, D) is an m-scroll with $m \le \mu$, and precisely there is a birational map $\phi : X \dashrightarrow \mathbb{F}_a$, for some $a \ge 0$, such that $|D|$ is the proper transform of a linear system of m-secant curves to the ruling of $\mathbb{F}_a$, with base points of multiplicity $k \le \ulcorner \mu/2 \urcorner - 1$.*

Proof

Since $\tau(\mu, g) > g$, Remark 1.1 applies.

For $\mu = 1$, one has $\tau(1, g) = 3g + 6$, and the assertion follows by the Castelnuovo-Enriques theorem (see Theorem 1.1). So we may assume that $\mu \ge 2$.

If the general curve in $|M|$ is reducible, then $|M|$ is composed with a pencil $|L|$ of curves of genus zero such that $L\cdot D=2$ (see the proof of Castelnuovo's inequality Theorem 1.2). Then there is a birational morphism $\psi: X\to\mathbb{F}_n$ such that $|L|$ is the proper transform of the ruling of $\mathbb{F}_n$, and $|D|$ is the proper transform of a linear system of 2-secant curves to the ruling of $\mathbb{F}_n$ with at most double base points. By performing elementary transformations based at these double base points, we find case (ii) with $\mu=2$. So from now on we may assume the general curve in $|M|$ to be irreducible.

Let $\mu=2$. We have $r=d-g+1\geq\tau(2,g)=2g+8$, which is equivalent to $d\geq 3g+7$. By Castelnuovo's inequality, if $3g+7\leq d\leq 3g+7-g'$, then $g'\leq 0$. Since M is irreducible, we have $g'\geq 0$; thus $g'=0$ and equality holds in (5); hence $X=\mathbb{P}^2$ and we are in case (i).

Next, we assume that $\mu\geq 3$, and we make induction on μ. By Castelnuovo's inequality, we have $r=d-g+1\leq 2g-g'+8$. If equality holds, then $X\cong\mathbb{P}^2$ and (X,D) is an m-Veronese pair; that is, $D\in|\mathcal{O}_{\mathbb{P}^2}(m)|$. We claim that $m\leq 2\mu+1$, that is, that we are in case (i). Indeed, $g=(m-1)(m-2)/2$, $r=m(m+3)/2$, and (6) reads $2m^2-6m(\mu+1)+4\mu^2+8\mu+2(\epsilon_\mu+2)\leq 0$. The polynomial $h(x)=2x^2-6x(\mu+1)+4\mu^2+8\mu+2(\epsilon_\mu+2)$ has its critical value at $x_0=3(\mu+1)/2<2\mu$, so that h is strictly increasing in $[x_0,+\infty)$. If $m\geq 2\mu+2$, we would have $0\geq h(m)>h(2\mu+2)=2\epsilon_\mu$, a contradiction.

Now we analyze the case $r\leq 2g-g'+7$. Then $2g-g'+7\geq r\geq\tau(\mu,g)$ implies that $g\geq(\mu g'+\epsilon_\mu)/(\mu-2)+2\mu$. Thus

$$\text{(7)}\qquad \dim(|D'|)=\dim(|M|)=g-1\geq\frac{\mu g'+\epsilon_{\mu-2}}{\mu-2}+2\mu-1=\tau(\mu-2,g').$$

By induction, we may assume that $r<\tau(\mu-1,g)$; hence $\tau(\mu-1,g)>\tau(\mu,g)$, which yields $g>\mu(\mu-1)+1/2((\mu-1)\epsilon_\mu-\mu\epsilon_{\mu-1})$. Thus

$$\text{(8)}\qquad g>\begin{cases}\mu(\mu-1)+\frac{\mu-2}{2} & \text{for } \mu \text{ an even number,}\\ \mu(\mu-1)-\frac{\mu+1}{2} & \text{for } \mu \text{ an odd number.}\end{cases}$$

In particular, if $\mu\geq 3$, then $g\geq 4$ and $d\geq 2g-2\geq 6$. Then, by Lemma 1.1, $P=0$ and $|D'|=|M|$.

In view of (7), we would like to apply induction on $|M|$, which we can do only if M verifies the hypotheses of the theorem.

First, we dispose of the case $g'=1$, in which (7) implies that $\dim(|M|)\geq 9$ and equality holds only for $\mu=3$. Then, by the Castelnuovo-Enriques theorem (see Theorem 1.1), $\mu=3$, (X,M) is a 3-Veronese pair, and D is a smooth plane sextic; that is, we are in case (i). Hence from now on we may assume that $g'\geq 2$.

CLAIM 1.1

The system $|M|$ is not composed with an involution of X.

Proof
Suppose that M is composed with an involution ι of X, defined in the Zariski open subset U. Consider the incidence variety $\mathcal{V}$ which is the Zariski closure in $X \times X \times |D|$ of the set $\{(p,q,D) : p,q \in D, D \in |D|, \iota(p) = q\} \subset U \times U \times |D|$, with the projections $\pi_1 : \mathcal{V} \to X \times X$, $\pi_2 : \mathcal{V} \to |D|$ to the factors. The image of π_1 is the graph Γ of ι. Since $|D|$ is not composed with ι, the general fiber of π_1 has dimension $r-2$. Hence $\mathcal{V}$ has an irreducible component $\mathcal{W}$ which dominates Γ via π_1 and has dimension r.

If $\pi_{2|\mathcal{W}}$ is surjective, then the general curve in $|D|$ is hyperelliptic. Since $\mu \geq 3$ and $g \geq 4$, (7) yields $r \geq 14$; hence $d \geq 17$. By Reider's theorem (see again [15], [2]), there is a curve A such that $0 \leq A \cdot D - 2 \leq A^2 < (A \cdot D)/2 < 2$. The index theorem implies that $A^2 = 0$. Then $A \cdot D = 2$, and there is a base point–free pencil $|A|$ which cuts the g^1_2 on the general curve of $|D|$. Since $|D|$ is not composed with an involution, the curves in $|A|$ have genus zero (see the discussion before Theorem 1.2). Then $0 = A \cdot (K + D) = A \cdot M$. Since the general curve in $|M|$ is irreducible and $\dim(|M|) = g - 1 \geq 3$, this is a contradiction.

If $\dim(\pi_2(\mathcal{W})) < r$, let $D \in \pi_2(\mathcal{W})$ be a general element. If the general fiber of π_2 has dimension at most 1, then it has dimension one; hence $\dim(\pi_2(\mathcal{W})) = r-1$. Now repeat the same argument as above. $\square$

The pair (X, M) could be not minimal. If E is a (-1)-curve such that $E \cdot M = 0$, then $E \cdot D = 1$. By contracting these (-1)-curves, we have a birational morphism $f : X \to X'$, and there are irreducible curves D' and M' on X' whose proper transform on X are D and M. The linear system $|D|$ is the proper transform of the sublinear system of $|D'|$ formed by the curves passing through the points which are blown up in $f : X \to X'$.

Finally, we may apply induction to the pair (X', M'), and

(i') either there is a birational morphism $\phi' : X' \to \mathbb{P}^2$ and $|M'|$ is the proper transform of a linear system $|C|$ of curves of degree $d \leq 2\mu - 3$ with base points of multiplicity $k \leq \ulcorner \mu/2 \urcorner - 2$,

(ii') or there is a birational map $\phi' : X' \dashrightarrow \mathbb{F}_a$ and $|M'|$ is the proper transform of a linear system $|C|$ of m-secant curves to the ruling of $\mathbb{F}_a$ with $m \leq \mu - 2$ with base points of multiplicity $k \leq \ulcorner \mu/2 \urcorner - 2$.

In case (i'), consider $\phi = \phi' \circ f : X \to \mathbb{P}^2$. If ℓ is a line in $\mathbb{P}^2$, set $H = \phi^*(\ell)$. We have $M \equiv dH - \sum_i k_i E_i$, where E_i are (-1)-cycles contracted by ϕ and $k_i \leq \ulcorner (\mu-2)/2 \urcorner - 1$. We also have $K_X \equiv -3H + \sum_i E_i$. Then $D \equiv (d+3) - \sum_i (k_i + 1) E_i$, and we are in case (i). The analysis of (i") is similar and leads to case (ii). $\square$

2. Castelnuovo pairs and their moduli

The Castelnuovo-Enriques theorem (see Theorem 1.1) classifies minimal pairs (X, D) for which (7) holds with $\mu = 1$. For higher μ's we define the concept of *μ-Castelnuovo pairs.*

2.1. Castelnuovo pairs

We call a pair (X,D) as in Castelnuovo's theorem (see Theorem 1.3) a *μ-Castelnuovo pair* with $\mu \geq 2$ if

$$\tau(\mu-1,g) > r \geq \tau(\mu,g),$$

which implies that (8) holds (see the proof of Theorem 1.3). If for such a pair case (i) (resp., case (ii)) occurs, we say that it presents the *planar case* (resp., the *scroll case*). Here we list μ-Castelnuovo's pairs for $2 \leq \mu \leq 4$. The reader may check the details.

PROPOSITION 2.1

If (X,D) is a μ-Castelnuovo pair with $2 \leq \mu \leq 4$ with D smooth of genus $g \geq 2$, then

(i) *(X,D) is either an m-Veronese pair with*

$$\begin{aligned} 4 \leq m \leq 5 &\quad \textit{if } \mu = 2 \textit{ and } r = \ulcorner\tau(2,g)\urcorner, \\ 6 \leq m \leq 7 &\quad \textit{if } \mu = 3 \textit{ and } r = \ulcorner\tau(3,g)\urcorner + \eta, \\ 8 \leq m \leq 9 &\quad \textit{if } \mu = 4 \textit{ and } r = \ulcorner\tau(4,g)\urcorner + \eta, \end{aligned}$$

where, in the last two cases, $\eta = 0$ if m is odd and $\eta = 1$ if m is even, or $X \cong \mathbb{F}_1$, and $D = (m-1)E + mF$ with

$$\begin{aligned} m = 6 &\quad \textit{if } \mu = 3, \\ m = 8 &\quad \textit{if } \mu = 4, \end{aligned}$$

and $r = \ulcorner\tau(\mu,g)\urcorner$ in both cases;

(ii) *$X \cong \mathbb{F}_a$ and $D \equiv \mu E + \alpha F$ with*

$$\alpha \geq \begin{cases} 4 & \textit{if } a = 0, \\ 5 & \textit{if } a = 1, \\ a\mu & \textit{if } a \geq 2, \end{cases}$$

$$g = \begin{cases} \alpha - a - 1 & \textit{if } \mu = 2, \\ 2\alpha - 3a - 2 & \textit{if } \mu = 3, \\ 3\alpha - 6a - 3 & \textit{if } \mu = 4, \end{cases}$$

and $r = \ulcorner\tau(\mu-1,g)\urcorner - 1$. In particular, for $\mu = 2$, (X,D) is an (a,g)-Castelnuovo pair.

2.2. Number of moduli, I

Consider a pair (X,D) with D irreducible and smooth of genus $g > 0$. We denote by $\mathcal{X}_g$ the set of all these pairs. Given $(X,D) \in \mathcal{X}_g$, we have the rational *moduli map* $\nu_D : |D| \dashrightarrow \mathcal{M}_g$. The dimension of the image of ν_D is called the *number of moduli* of (X,D), denoted by $\nu(X,D)$.

PROBLEM 2.1

Given g, what is the maximum of $\nu(X,D)$ as (X,D) varies in $\mathcal{X}_g$?

One might expect that the larger the dimension of $|D|$, the larger $\nu(X,D)$. This is not exactly the case, as we will see by looking at the μ-Castelnuovo pair with $2 \le \mu \le 4$.

2.2.1. Veronese pairs

Consider an m-Veronese pair (X,D) so that $g = \binom{m-1}{2}$. A classical theorem of Noether (see [6, Sezione 3]) asserts that two smooth plane curves of degree m are isomorphic if and only if they are projectively equivalent. As a consequence, we have the following.

PROPOSITION 2.2

If (X,D) is an m-Veronese pair with $m \ge 3$, then $\nu(X,D) = g + 3m - 9$.

The moduli map is dominant if and only if $3 \le m \le 4$, whereas for $m \gg 0$, $\nu(X,D) = o(g)$.

2.2.2. Castelnuovo pairs

Consider an (a,g)-Castelnuovo pair (X,D) with $g \ge 2$, which is then a 2-Castelnuovo pair. The curve D is hyperelliptic; hence the image of ν_D is contained in the hyperelliptic locus $\mathcal{H}_g$ in $\mathcal{M}_g$, and therefore $\nu(X,D) \le 2g-1$.

PROPOSITION 2.3

If (X,D) is an (a,g)-Castelnuovo pair, then $\nu(X,D) = 2g-1$, that is, $\mathrm{Im}(\nu_D) = \mathcal{H}_g$.

Proof

We have $X = \mathbb{F}_a$ and $D = 2E + (a+g+1)F$. Consider the exact sequence $0 \to T_D \to T_X|_D \to N_{D,S} \to 0$. To prove the assertion it suffices to prove that

$$\dim(\mathrm{Im}(H^0(D,N_{D,X}) \to H^1(D,T_D))) = 2g-1. \tag{9}$$

We have $h^0(D,N_{D,X}) = r = 3g+5$ and $h^0(D,T_D) = 0$. So (9) is equivalent to $h^0(T_X|_D) = g+6$. Consider the structure morphism $f : X \to \mathbb{P}^1$, and let T_f be the relative tangent sheaf. We have the exact sequence $0 \to T_f|_D \to T_X|_D \to \mathcal{O}_X(2F)|_D \to 0$, from which $\deg(T_f|_D) = 2g+2$; hence $h^1(D,T_f|_D) = 0$, and therefore $h^0(D,T_X|_D) = h^0(D,T_f|_D) + h^0(D,\mathcal{O}_D(2F)) = g+6$, as needed. $\square$

Next, we consider μ-Castelnuovo pairs as in Proposition 2.1(ii) with $3 \le \mu \le 4$. The analysis of the moduli maps in these cases could be done, as in the proof of Proposition 2.3, by studying the coboundary map in (9). There is, however, a quicker way, which parallels Noether's theorem for plane curves.

PROPOSITION 2.4

Let (X,D) be a μ-Castelnuovo pair as in Proposition 2.1(ii) with $3 \le \mu \le 4$ and $g \ge 4$. Then two smooth curves $C, C' \in |D|$ are isomorphic if and only if there is an automorphism ω of $X \cong \mathbb{F}_a$ such that $C' = \omega(C)$. Accordingly

$$\nu(X,D) = \begin{cases} 2g+1 & \text{if } \mu = 3 \text{ and } a = 0, \\ 2g+2-a & \text{if } \mu = 3 \text{ and } a > 0, \\ \frac{5g}{3}+3 & \text{if } \mu = 4 \text{ and } a = 0, \\ \frac{5g}{3}+4-a & \text{if } \mu = 4 \text{ and } a > 0. \end{cases} \tag{10}$$

In particular, for $\mu = 3$ and $0 \le a \le 1$, the image of ν_D is the whole trigonal locus.

Proof

Consider the case where $\mu = 3$. Then $D \equiv 3E + \alpha F$ with $\alpha \ge 3a$. Set $H \equiv D + K = E + (\alpha - a - 2)F$. Then $\phi_{|H|}$ is a morphism mapping X to a rational normal scroll S in $\mathbb{P}^{g-1}$, and the smooth curves in $|D|$ are mapped to canonical curves. Two such curves C, C' are isomorphic if and only if there is a projective transformation ω of $\mathbb{P}^{g-1}$ such that $C' = \omega(C)$. Since $\omega(S) = S$, the first assertion follows.

Note that $r = \tau(2,g) - 1 = 2g - 7$. The automorphism group of $\mathbb{F}_a$ has dimension $a+5$ if $a > 0$ and 6 if $a = 0$, which explains the first two lines of (10).

Now look at the case where $\mu = 4$. Assume first that $a \ge 3$. Then $D \equiv 4E + \alpha F$ with $\alpha \ge 4a$. Set $H \equiv E + \beta F$ with β verifying

$$\alpha \le 4\beta \le 2\alpha - 2a - 2. \tag{11}$$

Since $\alpha \ge 4a$ and $a \ge 3$, such a β exists. In addition, $\beta \ge \alpha/4 \ge a$; hence $\phi_{|H|}$ maps X to a rational normal scroll in $\mathbb{P}^s$ with $s = 2\beta - a + 1$. The curves in $|D|$ map to curves of degree $n = 4\beta + \alpha - 4a$. Note that $n - 1 = 3(s-1) + \epsilon$ with $\epsilon = \alpha - a - 2\beta - 1$. By (11), one has $0 \le \epsilon < s - 1$. Then the maximal genus of curves of degree n in $\mathbb{P}^s$ is $3\alpha - 6n - 3 = g$ (see, e.g., [12, p. 527]). Hence the smooth curves in $|D|$ are Castelnuovo curves in $\mathbb{P}^s$. By a result of Accola (see [1] and also [6, Teorema 2.11]), two smooth curves $C, C' \in |D|$ are isomorphic if and only if they are projectively equivalent in $\mathbb{P}^s$. The conclusion is as for $\mu = 3$.

In the case where $0 \le a \le 2$, the same argument as above applies if $\alpha \ge 5 + 2a$, since in this case still there is an integer β verifying (11). So we are left to consider the cases where $0 \le a \le 2$ with $\alpha \le 4 + 2a$. Then $\alpha = 4 + 2a$ by Proposition 2.1(ii) for $a = 0, 2$. In the case where $a = 1$, also with $\alpha = 4 + 2a = 6$ because $\mu = 4, \alpha = 5$ does not correspond to a 4-Castelnuovo pair. The argument is similar to the above and therefore we will be brief. For $a = 0, 2$ we map $\mathbb{F}_a$ to a quadric S in $\mathbb{P}^3$. Then the curves in $|D|$ are complete intersections of S with a surface of degree 4. Again, two smooth curves $C, C' \in |D|$ are isomorphic if and only if they are projectively equivalent in $\mathbb{P}^3$ (see [9] or [6, Corollario 4.8]). If $a = 1$, then $\mathbb{F}_1$ birationally maps to the plane by contracting E, and $|D|$ is the proper transform of the linear system of curves of degree 6 with a base point of multiplicity 2. Two such curves with only one node are birational if and only if they are projectively equivalent in $\mathbb{P}^2$ (see [6, Osservazione 2.19]), and the conclusion is as above. $\square$

The last assertion in Proposition 2.4 is no news: indeed it goes back to Maroni [13].

2.3. Number of moduli, II

In this section we answer Problem 2.1.

THEOREM 2.1

Let (X,D) be a minimal pair with D a smooth curve of genus $g \geq 22$. Then $\nu(X,D) \leq 2g+1$, and equality holds if and only if (X,D) is a 3-Castelnuovo pair with $X \cong \mathbb{F}_a$ and $0 \leq a \leq 1$, in which case the image of $|D|$ via ν_D is the trigonal locus in $\mathcal{M}_g$.

Proof

By Remark 1.1, we may assume that X has negative Kodaira dimension; otherwise, $r \leq g$. If $q > 0$, there is a curve C of genus q and a surjective morphism $f : X \to C$; hence all curves in $|D|$ map to C, and therefore $\nu(X,D) \leq 2g-2$ (see [8]). So we may assume that $q = 0$.

If $|D|$ is composed with an involution, then the general curve $D \in |D|$ has a nonconstant morphism $D \to C$ to a curve. If C is rational, then D is hyperelliptic and $\nu(X,D) \leq 2g-1$; if C is irrational, one has $\nu(X,D) \leq 2g-2$ as above. Thus if $\nu(X,D) \geq 2g$, we may assume that (X,D) verifies the hypotheses of Castelnuovo's theorem (see Theorem 1.3).

By the Castelnuovo-Enriques theorem (see Theorem 1.1), we may assume that $r \leq 3g+5 = \tau(1,g)-1$. If $r \geq 2g+8 = \tau(2,g)$, then (X,D) is a 2-Castelnuovo pair. By Propositions 2.1–2.3, the image of ν_D is $\mathcal{H}_g$ and $\nu(X,D) = 2g-1$. If $\tau(2,g)-1 \geq r \geq \tau(3,g) = (5g+1)/3+9$, then (X,D) is a 3-Castelnuovo pair. By Propositions 2.1, 2.2, and 2.4, the maximum of $\nu(X,D)$ is attained if (X,D) is as in Proposition 2.1(ii) with $0 \leq a \leq 1$. In this case $\nu(X,D) = 2g+1$, and the image of $|D|$ via ν_D is the trigonal locus. If $\tau(3,g) > g \geq \tau(4,g) = (3g+1)/2+11$, then the maximum of $\nu(X,D)$ is $(5g)/3+3$ (see Proposition 2.4), which is smaller than $2g+1$ if $g \geq 7$. Finally, if $r < (3g+1)/2+11$, then also $\nu(X,D) \leq (3g+1)/2+11$, and this is smaller than $2g+1$ if $g \geq 22$. $\square$

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