Global solutions to quasi-linear hyperbolic systems of viscoelasticity

Priyanjana M. N. Dharmawardane, Tohru Nakamura, and Shuichi Kawashima

Abstract In the present paper, we study a large-time behavior of solutions to a quasi-linear second-order hyperbolic system which describes a motion of viscoelastic materials. The system has dissipative properties consisting of a memory term and a damping term. It is proved that the solution exists globally in time in the Sobolev space, provided that the initial data are sufficiently small. Moreover, we show that the solution converges to zero as time tends to infinity. The crucial point of the proof is to derive uniform a priori estimates of solutions by using an energy method.

1. Introduction

The present paper is concerned with an asymptotic behavior of solutions to the following nonlinear second-order hyperbolic system:

$$u_{tt} - \sum_{j=1}^{n} b^j(\partial_x u)_{x_j} + \sum_{j,k=1}^{n} K^{jk} * u_{x_j x_k} + L u_t = 0. \tag{1.1}$$

We prescribe an initial condition for (1.1) as

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x). \tag{1.2}$$

In the system (1.1), u is an unknown m-vector function of $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $t \geq 0$; $b^j(v)$ are smooth m-vector functions of $v = (v_1, \dots, v_n) \in \mathbb{R}^{mn}$, where $v_j \in \mathbb{R}^m$ corresponds to u_{x_j}; $K^{jk}(t)$ are smooth $m \times m$ real matrix functions of $t \geq 0$ satisfying $K^{jk}(t)^T = K^{kj}(t)$ for each j, k, and t; L is an $m \times m$ real symmetric constant matrix; and the symbol "$*$" denotes the convolution with respect to t, that is,

$$K^{jk} * u_{x_j x_k} = \int_0^t K^{jk}(t-\tau) u_{x_j x_k}(\tau)\, d\tau.$$

We assume that the system (1.1) has a free energy $\phi(v)$ which is a smooth scalar function of $v = (v_1, \dots, v_n)$ satisfying

$$b^j(v) = D_{v_j} \phi(v), \tag{1.3}$$

Kyoto Journal of Mathematics, Vol. 51, No. 2 (2011), 467–483
DOI 10.1215/21562261-1214411,
Received May 31, 2010. Revised October 25, 2010. Accepted October 26, 2010.
2010 Mathematics Subject Classification: 35B40, 35L51, 74D10.

where $D_{v_j}\phi(v)$ denotes the Fréchet derivative of $\phi(v)$ with respect to v_j. We define $m\times m$ real matrix functions by

$$B^{jk}(v) := D_{v_k}b^j(v) = D_{v_k}D_{v_j}\phi(v).$$

Notice that $B^{jk}(v)^T = B^{kj}(v)$ holds for each j,k, and $v\in\mathbb{R}^{mn}$. Moreover, without loss of generality, we assume that

$$\phi(0)=0, \qquad b^j(0)=0 \quad \text{for } j=1,\dots,n.$$

Then, by virtue of the Taylor theorem, we see that

$$\phi(v) = \frac{1}{2}\sum_{j,k}\langle B^{jk}(0)v_j, v_k\rangle + O(|v|^3) \tag{1.4}$$

for $v=(v_1,\dots,v_n)\in\mathbb{R}^{mn}$.

We employ the following symbols of the differential operators associated with (1.1):

$$B_\omega(v) := \sum_{j,k} B^{jk}(v)\omega_j\omega_k, \qquad K_\omega(t) := \sum_{j,k} K^{jk}(t)\omega_j\omega_k \tag{1.5}$$

for $\omega=(\omega_1,\dots,\omega_n)\in S^{n-1}$. We see that $B_\omega(v)$ and $K_\omega(t)$ are real symmetric matrices. Using these symbols, we impose the following structural conditions.

[A1] $B_\omega(0)$ is positive definite for each $\omega\in S^{n-1}$, while $K_\omega(t)$ is nonnegative definite for each $\omega\in S^{n-1}$ and $t\ge 0$, and L is real symmetric and nonnegative definite.

[A2] $B_\omega(0)-\mathcal{K}_\omega(t)$ is positive definite for each $\omega\in S^{n-1}$ uniformly in $t\ge 0$, where $\mathcal{K}_\omega(t) := \int_0^t K_\omega(s)\,ds$.

[A3] $K_\omega(0)+L$ is (real symmetric and) positive definite for each $\omega\in S^{n-1}$.

[A4] $K_\omega(t)$ is smooth in $t\ge 0$ and decays exponentially as $t\to\infty$. Precisely, there are positive constants C_0 and c_0 such that $-C_0K_\omega(t)\le \dot K_\omega(t)\le -c_0K_\omega(t)$ and $-C_0K_\omega(t)\le \ddot K_\omega(t)\le C_0K_\omega(t)$ for $\omega\in S^{n-1}$ and $t\ge 0$, where $\dot K_\omega(t) := \partial_t K_\omega(t)$ and $\ddot K_\omega(t) := \partial_t^2 K_\omega(t)$.

There are many mathematical results related to the viscoelastic body. It is well known that the solution to a single equation with a damping term behaves like that of a heat equation as $t\to\infty$. For the case when the equation has a memory term and does not have a damping term, it was shown in [2], [8], and [9] that the solution also verifies a standard decay estimate. In the paper [10], Rivera, Naso, and Vegni studied a decay property of the regularity-loss type and showed that the solution verifies a weak decay estimate. This kind of the decay property of the regularity-loss type was also studied in [3] for a hyperbolic-elliptic system of a radiating gas and in [5] for the dissipative Timoshenko system. Dharmawardane, Rivera, and Kawashima considered a linearized problem of (1.1) in [1] and obtained a decay property of the solution by using an energy method in the Fourier space. They also treated the regularity-loss type. Thus the paper [1] generalized the previous results in [2], [7]–[10] for a single equation of viscoelasticity. For a nonlinear problem, Kawashima and Shibata [6] proved the global existence and exponential stability of small solutions for viscoelasticity in a bounded domain. The main aim of the present paper is to show the global existence of a solution to the nonlinear system (1.1) of which the dissipative property is obtained by combining a memory term and a damping term.

The contents of the present paper are as follows. In Section 2 we give some preliminary estimates for convolution-type operators of matrices with vectors, which will be used to estimate the memory term. In Section 3 we give the main theorem of the present paper, which shows the global existence of solutions to the problem (1.1) and (1.2). The proof is mainly based on deriving a priori estimates of solutions by using an energy method in an L^2-framework. This is discussed in Section 4.

2. Preliminaries

In the present section we give some standard notation and important results for convolution-type operators of matrices with vectors.

We denote by $\partial_x^k u$ the totality of all kth-order derivatives of u with respect to x. We denote by $\hat{u} = \mathcal{F}[u]$ the Fourier transform of a function u:

$$\hat{u}(\xi) = \mathcal{F}[u](\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} u(x) e^{-i\xi x}\, dx.$$

For $1 \le p \le \infty$, we denote by $L^p = L^p(\mathbb{R}^n)$ the standard Lebesgue space on $\mathbb{R}^n$ with the norm $\|\cdot\|_{L^p}$. For a nonnegative integer s, $H^s = H^s(\mathbb{R}^n)$ denotes the sth-order Sobolev space on $\mathbb{R}^n$ in the L^2-sense, equipped with the norm $\|\cdot\|_{H^s}$. Throughout the present paper, c and C denote generic positive constants.

Let $\mathcal{X}^m$ be the totality of $m \times m$ real matrices, and let $\langle \cdot, \cdot \rangle$ be the standard inner product in $\mathbb{C}^m$. We introduce the operator norm of $A \in \mathcal{X}^m$ by

$$|A| := \sup_{\substack{\psi \in \mathbb{C}^m \\ \psi \neq 0}} \frac{|A\psi|}{|\psi|}.$$

If A is real symmetric and nonnegative definite, then

$$|A\psi|^2 \le |A| \langle A\psi, \psi \rangle \tag{2.1}$$

for $\psi \in \mathbb{C}^m$. Let $A(t) \in \mathcal{X}^m$ and $\psi(t) \in \mathbb{C}^m$. We define the convolution $A * \psi$ by

$$(A * \psi)(t) := \int_0^t A(t-\tau)\psi(\tau)\, d\tau.$$

We also introduce the related operator and the corresponding quadratic form, as in [9] and [10], defined by

$$(A \diamond \psi)(t) := \int_0^t A(t-\tau)\big(\psi(t) - \psi(\tau)\big)\, d\tau,$$

$$A[\psi, \zeta](t) := \int_0^t \big\langle A(t-\tau)\big(\psi(t) - \psi(\tau)\big), \zeta(t) - \zeta(\tau)\big\rangle\, d\tau$$

for $\psi(t), \zeta(t) \in \mathbb{C}^m$. Using the above definitions, we obtain the following relations. The convolution $A * \psi$ is related to $A \diamond \psi$ by

$$A * \psi = \mathcal{A}\psi - A \diamond \psi, \tag{2.2}$$

where $\mathcal{A}(t) := \int_0^t A(s)\,ds$. Direct computations show that

$$(A * \psi)_t = A(0)\psi + \dot{A} * \psi = A\psi - \dot{A} \diamond \psi, \tag{2.3}$$

where $\dot{A}(t) := dA(t)/dt$. In deriving the second equality in (2.3), we use (2.2) and the relation $\dot{A} * \psi = (A - A(0))\psi - \dot{A} \diamond \psi$. Moreover, differentiating (2.3) with respect to t, we have

$$(A * \psi)_{tt} = A(0)\psi_t + (\dot{A} * \psi)_t. \tag{2.4}$$

The following inequalities, which can be proved in a similar way to [1, Lemma 2.2], play important roles to control the memory term in deriving a priori estimates.

LEMMA 2.1
Let $\psi(t) \in \mathbb{C}^m$. Then we have

$$|K_\omega \diamond \psi|^2 \le C K_\omega[\psi, \psi], \tag{2.5}$$

$$|(K_\omega * \psi)_t|^2 \le C(K_\omega[\psi, \psi] + \langle K_\omega \psi, \psi \rangle). \tag{2.6}$$

Proof
The inequality (2.5) immediately follows from [1, Lemmas 2.2, 2.3]. Indeed, using (2.1) and the Hölder inequality as well as the exponentially decaying property of K_ω in condition [A4], we obtain

$$\begin{aligned} |(K_\omega \diamond \psi)(t)| &\le \int_0^t \big|K_\omega(t-\tau)\big(\psi(t) - \psi(\tau)\big)\big|\,d\tau \\ &\le \Big(\int_0^t |K_\omega(s)|\,ds\Big)^{1/2} K_\omega[\psi,\psi](t)^{1/2} \le C K_\omega[\psi,\psi](t)^{1/2}. \end{aligned}$$

In a similar computation, we also obtain

$$|\dot{K}_\omega \diamond \psi|^2 \le C K_\omega[\psi, \psi]. \tag{2.7}$$

To show (2.6), we use (2.1), the second equality in (2.3) and the inequalities (2.5) and (2.7). Namely we have

$$|(K_\omega * \psi)_t|^2 \le C(|K_\omega \psi|^2 + |\dot{K}_\omega \diamond \psi|^2) \le C(K_\omega[\psi, \psi] + \langle K_\omega \psi, \psi \rangle).$$

Therefore the proof of the lemma is complete. □

3. Global existence

In the present section, we give the main result on the global existence and asymptotic behavior of solutions to the problem (1.1) and (1.2). To this end, it is convenient to introduce the following quantities Q_K, $Q_K^\sharp$, and $Q_K^\flat$ defined by

$$Q_K[\partial_x u] := Q_K^\sharp[\partial_x u] + Q_K^\flat[\partial_x u], \tag{3.1}$$

$$Q_K^\sharp[\partial_x u] := \sum_{j,k} \int_{\mathbb{R}^n} K^{jk}[u_{x_j}, u_{x_k}]\, dx, \qquad Q_K^\flat[\partial_x u] := \sum_{j,k} \int_{\mathbb{R}^n} \langle K^{jk} u_{x_j}, u_{x_k} \rangle\, dx.$$

The Plancherel theorem yields

$$Q_K^\sharp[\partial_x u] = \int_{\mathbb{R}^n_\xi} |\xi|^2 K_\omega[\hat{u}, \hat{u}]\, d\xi, \qquad Q_K^\flat[\partial_x u] = \int_{\mathbb{R}^n_\xi} |\xi|^2 \langle K_\omega \hat{u}, \hat{u} \rangle\, d\xi, \tag{3.2}$$

where $\omega := \xi/|\xi|$. Notice that the nonnegativity of (3.2) follows from that of $K_\omega(t)$. We define the energy norm $E(t)$ and the corresponding dissipation norm $D(t)$ by

$$E(t)^2 := \sup_{0 \le \tau \le t} \|(u_t, \partial_x u)(\tau)\|_{H^s}^2 + \sum_{l=0}^{s} \sup_{0 \le \tau \le t} Q_K[\partial_x^{l+1} u](\tau),$$

$$D(t)^2 := \tilde{D}(t)^2 + \int_0^t \|(\partial_x u_t, \partial_x^2 u)(\tau)\|_{H^{s-1}}^2\, d\tau,$$

$$\tilde{D}(t)^2 := \int_0^t \|(I-P)u_t(\tau)\|_{H^s}^2\, d\tau + \sum_{l=0}^{s} \int_0^t Q_K[\partial_x^{l+1} u](\tau)\, d\tau,$$

where I and P are the identity matrix and the orthogonal projection matrix on $\ker(L)$, respectively. Under the structural conditions [A1]–[A4] and a smallness assumption on the initial data, the global existence of solutions to the problem (1.1) and (1.2) in the Sobolev space $H^s(\mathbb{R}^n)$ is shown in Theorem 3.1, where s is an integer satisfying $s \ge [n/2] + 2$.

THEOREM 3.1 (GLOBAL EXISTENCE AND ASYMPTOTIC DECAY)

Suppose that all the conditions [A1]–[A4] are satisfied. Assume that $u_0 \in H^{s+1}$ *and* $u_1 \in H^s$ *for* $s \ge [n/2] + 2$, *and put*

$$E_0 := \|(u_1, \partial_x u_0)\|_{H^s}.$$

Then there is a positive constant δ_0 *such that if* $E_0 \le \delta_0$, *then the problem (1.1) and (1.2) has a unique global solution* $u(x,t)$ *verifying*

$$u_t, \partial_x u \in C([0,\infty); H^s).$$

The solution satisfies the uniform energy estimate

$$\sup_{t \in [0,\infty)} \|(u_t, \partial_x u)(t)\|_{H^s}^2 + \int_0^\infty \left(\|(\partial_x u_t, \partial_x^2 u)(t)\|_{H^{s-1}}^2 + \|(I-P)u_t(t)\|_{H^s}^2 \right) dt \le C E_0^2, \tag{3.3}$$

where C *is a positive constant. Moreover, the solution converges to zero as* $t \to \infty$ *in the following sense:*

$$\lim_{t \to \infty} \|(\partial_x^j u_t, \partial_x^{j+1} u)(t)\|_{L^\infty} = 0, \tag{3.4}$$

where $0 \le j \le s - [n/2] - 2$.

The key to the proof of this global existence result is to combine a local existence result summarized in Lemma 3.2 and the corresponding a priori estimates of solutions in Proposition 3.3. The following lemma can be proved by applying the theory in [4] for the solvability of quasi-linear second-order hyperbolic systems since the memory term $\sum_{j,k} K^{jk} * u_{x_j x_k}$ in (1.1) is regarded as a lower-order term. Thus we omit the details of the proof.

LEMMA 3.2 (LOCAL EXISTENCE)

Suppose that the matrix $B_\omega(v)$ in (1.5) is real symmetric and positive definite for each $v \in \mathbb{R}^{mn}$ and $\omega \in S^{n-1}$, and suppose that $K_\omega(t)$ is smooth in $t \geq 0$. Assume that $u_0 \in H^{s+1}$ and $u_1 \in H^s$ for $s \geq [n/2]+2$. Then there is a positive constant T_0 depending only on E_0 such that the problem (1.1) and (1.2) admits a unique solution $u(x,t)$ with

$$u_t, \partial_x u \in C([0,T_0]; H^s),$$

which satisfies the estimate

$$\|(u_t, \partial_x u)(t)\|_{H^s} \leq C E_0$$

for $t \in [0,T_0]$, where C is a positive constant depending on E_0.

The a priori estimates of solution u to the problem (1.1) and (1.2) are given as follows.

PROPOSITION 3.3 (A PRIORI ESTIMATE)

Assume the same conditions as in Theorem 3.1. Let $u(x,t)$ be a solution to the problem (1.1) and (1.2) satisfying

$$u_t, \partial_x u \in C([0,T]; H^s)$$

for a certain $T > 0$. Then there is a positive constant δ_1 independent of T such that if $E(T) \leq \delta_1$, then the solution verifies the following uniform estimate:

$$E(t)^2 + D(t)^2 \leq C E_0^2 \tag{3.5}$$

for an arbitrary $t \in [0,T]$, where C is a positive constant independent of T.

The proofs of Theorem 3.1 and Proposition 3.3 are given in Section 4. In deriving the a priori estimate (3.5), we utilize the following inequalities to control nonlinear terms.

LEMMA 3.4

Let $1 \leq p,q,r \leq \infty$ and $1/p = 1/q + 1/r$. Then we have

$$\|\partial_x^k (uv)\|_{L^p} \leq C(\|u\|_{L^q} \|\partial_x^k v\|_{L^r} + \|v\|_{L^q} \|\partial_x^k u\|_{L^r}) \quad \text{for } k \geq 0, \tag{3.6}$$

$$\|[\partial_x^k, u]\partial_x v\|_{L^p} \leq C(\|\partial_x u\|_{L^q} \|\partial_x^k v\|_{L^r} + \|\partial_x v\|_{L^q} \|\partial_x^k u\|_{L^r}) \quad \text{for } k \geq 1, \tag{3.7}$$

where $[\partial_x^k, u]v := \partial_x^k (uv) - u \partial_x^k v$ is a commutator.

LEMMA 3.5

Let $f(u)$ be a smooth function of u. Assume that $\|u\|_{L^\infty} \le M$ for a certain constant $M>0$. Then we have

$$\|\partial_x^k f(u)\|_{L^p} \le C(M)(1+\|u\|_{L^\infty})^{k-1}\|\partial_x^k u\|_{L^p} \tag{3.8}$$

for $1\le p\le\infty$ and $k\ge 1$, where $C(M)$ is a constant depending on M.

These inequalities for nonlinear terms are well known and proved in [3], for instance, so that we omit the proof. We also use the following inequalities to estimate linear terms arising from the memory term.

LEMMA 3.6

Let u be a function satisfying $\partial_x u \in C([0,T];H^s)$, and let $f\in H^1$. Then we have

$$\Big|\sum_{j,k}\int_{\mathbb{R}^n}\langle K^{jk}\diamond u_{x_j}, f_{x_k}\rangle\,dx\Big| \le C\|\partial_x f\|_{L^2} Q_K^\sharp[\partial_x u]^{1/2}, \tag{3.9}$$

$$\Big|\sum_{j,k}\int_{\mathbb{R}^n}\langle (K^{jk}* u_{x_j})_t, f_{x_k}\rangle\,dx\Big| \le C\|\partial_x f\|_{L^2} Q_K[\partial_x u]^{1/2}, \tag{3.10}$$

$$\Big|\sum_{j,k}\int_{\mathbb{R}^n}\langle (K^{jk}* u_{x_jx_k})_t, f\rangle\,dx\Big| \le C\|f\|_{L^2} Q_K[\partial_x^2 u]^{1/2}. \tag{3.11}$$

Proof

Using the Plancherel theorem and the Schwarz inequality, we have

$$\begin{aligned}\Big|\sum_{j,k}\int_{\mathbb{R}^n}\langle K^{jk}\diamond u_{x_j}, f_{x_k}\rangle\,dx\Big| &= \Big|\int_{\mathbb{R}^n_\xi}|\xi|^2\langle K_\omega\diamond\hat u,\hat f\rangle\,d\xi\Big| \\ &\le \|\partial_x f\|_{L^2}\Big(\int_{\mathbb{R}^n_\xi}|\xi|^2|K_\omega\diamond\hat u|^2\,d\xi\Big)^{1/2}.\end{aligned} \tag{3.12}$$

Moreover, the inequality (2.5) and the property (3.2) of $Q_K^\sharp$ yield

$$\int_{\mathbb{R}^n_\xi}|\xi|^2|K_\omega\diamond\hat u|^2\,d\xi \le C\int_{\mathbb{R}^n_\xi}|\xi|^2 K_\omega[\hat u,\hat u]\,d\xi = CQ_K^\sharp[\partial_x u]. \tag{3.13}$$

Substituting (3.13) in (3.12) gives the inequality (3.9).

Next we show (3.10). From the inequality (2.6) and properties (3.1) and (3.2) of Q_K, we have

$$\int_{\mathbb{R}^n_\xi}|\xi|^2|(K_\omega*\hat u)_t|^2\,d\xi \le C\int_{\mathbb{R}^n_\xi}|\xi|^2\big(K_\omega[\hat u,\hat u]+\langle K_\omega\hat u,\hat u\rangle\big)\,d\xi = CQ_K[\partial_x u],$$

which immediately yields the inequality (3.10). To show (3.11), we use (2.6) and the Plancherel theorem to obtain

$$\begin{aligned}\Big|\sum_{j,k}\int_{\mathbb{R}^n}\langle (K^{jk}* u_{x_jx_k})_t, f\rangle\,dx\Big| &= \Big|\int_{\mathbb{R}^n_\xi}|\xi|^2\langle (K_\omega*\hat u)_t,\hat f\rangle\,d\xi\Big| \\ &\le C\|f\|_{L^2}(I+J)^{1/2},\end{aligned} \tag{3.14}$$

where

$$I := \int_{\mathbb{R}^n_\xi} |\xi|^4 \langle K_\omega \hat{u}, \hat{u}\rangle \, d\xi, \qquad J := \int_{\mathbb{R}^n_\xi} |\xi|^4 K_\omega[\hat{u}, \hat{u}] \, d\xi.$$

We compute I as

$$I = \sum_{j,k,l} \int_{\mathbb{R}^n_\xi} \langle K^{jk} \xi_j \xi_l \hat{u}, \xi_k \xi_l \hat{u}\rangle \, d\xi = \sum_{j,k} \int_{\mathbb{R}^n} \langle K^{jk} \partial_x u_{x_j}, \partial_x u_{x_k}\rangle \, dx = Q_K^\flat[\partial_x^2 u].$$

In a similar computation, we see that $J = Q_K^\sharp[\partial_x^2 u]$. Substituting these equalities of I and J in (3.14) yields the inequality (3.11). Therefore we complete the proof. □

REMARK 3.7

By similar computations as in Lemma 3.6, we have

$$\Big|\sum_{j,k} \int_{\mathbb{R}^n} \langle (\dot{K}^{jk} * u_{x_j})_t, f_{x_k}\rangle \, dx\Big| \le C\|\partial_x f\|_{L^2} Q_K[\partial_x u]^{1/2}, \tag{3.15}$$

which is obtained by replacing K^{jk} by $\dot{K}^{jk}$ in (3.10) and using assumption [A4].

4. Proof of a priori estimates

This section is devoted to showing the proof of Proposition 3.3. Namely, we obtain the a priori estimate (3.5), provided that $E(t)$ is sufficiently small. To do this, we employ $N(t)$ defined by

$$N(t) := \sup_{0 \le \tau \le t} \|(\partial_x u, \partial_x u_t, \partial_x^2 u)(\tau)\|_{L^\infty}$$

and obtain the energy estimates under the smallness assumption on $N(t)$. The proof of Proposition 3.3 is divided into several steps.

LEMMA 4.1

Assume the same conditions as in Proposition 3.3. Then we have

$$\begin{aligned} &\|(u_t, \partial_x u)(t)\|_{L^2}^2 + Q_K[\partial_x u](t) + \int_0^t \big(\|(I-P)u_t(\tau)\|_{L^2}^2 + Q_K[\partial_x u](\tau)\big)\, d\tau \\ &\quad \le C\|(u_1, \partial_x u_0)\|_{L^2}^2. \end{aligned} \tag{4.1}$$

Proof

Take the inner product of (1.1) with u_t to get

$$\langle u_{tt}, u_t\rangle - \Big\langle \sum_j b^j(\partial_x u)_{x_j}, u_t\Big\rangle + \Big\langle \sum_{j,k} K^{jk} * u_{x_j x_k}, u_t\Big\rangle + \langle Lu_t, u_t\rangle = 0. \tag{4.2}$$

By using the property (1.3) of the free energy ϕ, we rewrite the second term on the left-hand side of (4.2) as

$$-\Big\langle \sum_j b^j(\partial_x u)_{x_j}, u_t\Big\rangle = -\sum_j \{\langle b^j(\partial_x u), u_t\rangle\}_{x_j} + \phi(\partial_x u)_t. \tag{4.3}$$

By straightforward computations using (2.2), we see that the third term on the left-hand side of (4.2) is rewritten to be

$$
\begin{aligned}
\Big\langle \sum_{j,k} K^{jk} * u_{x_j x_k}, u_t \Big\rangle &= \frac{1}{2}\Big\{ -\sum_{j,k} \langle \mathcal{K}^{jk} u_{x_j}, u_{x_k} \rangle + \sum_{j,k} K^{jk}[u_{x_j}, u_{x_k}] \Big\}_t \\
&\quad + \sum_{j,k} \{ \langle K^{jk} * u_{x_j}, u_t \rangle \}_{x_k} - \frac{1}{2} \sum_{j,k} \dot{K}^{jk}[u_{x_j}, u_{x_k}] + \frac{1}{2} \sum_{j,k} \langle K^{jk} u_{x_j}, u_{x_k} \rangle.
\end{aligned} \tag{4.4}
$$

Substituting (4.3) and (4.4) in (4.2) yields

$$
\partial_t \mathcal{E}_1^{(0)} + \sum_k \partial_{x_k} \mathcal{F}_{1k}^{(0)} + \mathcal{D}_1^{(0)} = 0, \tag{4.5}
$$

where

$$
\begin{aligned}
\mathcal{E}_1^{(0)} &:= \frac{1}{2}\Big\{ |u_t|^2 + 2\phi(\partial_x u) - \sum_{j,k} \langle \mathcal{K}^{jk} u_{x_j}, u_{x_k} \rangle + \sum_{j,k} K^{jk}[u_{x_j}, u_{x_k}] \Big\}, \\
\mathcal{F}_{1k}^{(0)} &:= -\langle b^k(\partial_x u), u_t \rangle + \sum_j \langle K^{jk} * u_{x_j}, u_t \rangle, \\
\mathcal{D}_1^{(0)} &:= \langle L u_t, u_t \rangle + \frac{1}{2}\Big\{ -\sum_{j,k} \dot{K}^{jk}[u_{x_j}, u_{x_k}] + \sum_{j,k} \langle K^{jk} u_{x_j}, u_{x_k} \rangle \Big\}.
\end{aligned}
$$

We next show that $\mathcal{E}_1^{(0)}$ satisfies

$$
\int_{\mathbb{R}^n} \mathcal{E}_1^{(0)}\, dx \geq c(\|(u_t, \partial_x u)\|_{L^2}^2 + Q_K^\sharp[\partial_x u]), \tag{4.6}
$$

provided that $\|\partial_x u\|_{L^\infty}$ is sufficiently small. To show (4.6), we use (1.4) and the assumption [A2] to obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \Big(2\phi(\partial_x u) - \sum_{j,k} \langle \mathcal{K}^{jk} u_{x_j}, u_{x_k} \rangle \Big)\, dx \\
&\quad = \int_{\mathbb{R}^n} \sum_{j,k} \langle (B^{jk}(0) - \mathcal{K}^{jk}) u_{x_j}, u_{x_k} \rangle\, dx \\
&\qquad + \int_{\mathbb{R}^n} \Big(2\phi(\partial_x u) - \sum_{j,k} \langle B^{jk}(0) u_{x_j}, u_{x_k} \rangle \Big)\, dx \\
&\quad \geq (c - C\|\partial_x u\|_{L^\infty}) \|\partial_x u\|_{L^2}^2,
\end{aligned}
$$

which yields the lower estimate (4.6) using the definition of $Q_K^\sharp$. The assumptions [A1] and [A4] give the lower estimate of the dissipative term as

$$
\int_{\mathbb{R}^n} \mathcal{D}_1^{(0)}\, dx \geq c(\|(I-P)u_t\|_{L^2}^2 + Q_K[\partial_x u]). \tag{4.7}
$$

Therefore, integrating (4.5) over $\mathbb{R}^n \times (0,t)$, substituting (4.6) and (4.7) in the resultant inequality, and using a simple inequality

$$
Q_K[\partial_x u] \leq C(\|\partial_x u\|_{L^2}^2 + Q_K^\sharp[\partial_x u]),
$$

which follows from the boundedness of K_ω, we obtain the desired estimate (4.1). $\square$

We next derive the energy inequality for derivatives $\partial_x^l u$ for $1 \le l \le s$.

LEMMA 4.2

Assume the same conditions as in Proposition 3.3. Then we have

$$\begin{aligned} &\|(\partial_x^l u_t, \partial_x^{l+1} u)(t)\|_{L^2}^2 + Q_K[\partial_x^{l+1} u](t) \\ &\quad + \int_0^t \big(\|(I-P)\partial_x^l u_t(\tau)\|_{L^2}^2 + Q_K[\partial_x^{l+1} u](\tau)\big)\, d\tau \\ &\quad \le C E_0^2 + C N(t) D(t)^2 \end{aligned} \tag{4.8}$$

for an arbitrary integer l satisfying $1 \le l \le s$.

Proof

Applying ∂_x^l to (1.1) and using the fact $b^j(\partial_x u)_{x_j} = \sum_k B^{jk}(\partial_x u) u_{x_j x_k}$, we have

$$\partial_x^l u_{tt} - \sum_{j,k} B^{jk}(\partial_x u)\partial_x^l u_{x_j x_k} + \sum_{j,k} K^{jk} * \partial_x^l u_{x_j x_k} + L\partial_x^l u_t = f^{(l)}, \tag{4.9}$$

$$f^{(l)} := \sum_{j,k} [\partial_x^l, B^{jk}(\partial_x u)] u_{x_j x_k}.$$

Taking the inner product of (4.9) with $\partial_x^l u_t$ yields

$$\begin{aligned} &\langle \partial_x^l u_{tt}, \partial_x^l u_t\rangle - \Big\langle \sum_{j,k} B^{jk}(\partial_x u)\partial_x^l u_{x_j x_k}, \partial_x^l u_t \Big\rangle + \Big\langle \sum_{j,k} K^{jk} * \partial_x^l u_{x_j x_k}, \partial_x^l u_t \Big\rangle \\ &\quad + \langle L\partial_x^l u_t, \partial_x^l u_t\rangle = \langle f^{(l)}, \partial_x^l u_t\rangle. \end{aligned} \tag{4.10}$$

The second term on the left-hand side of (4.10) is computed using the following equality:

$$\begin{aligned} -\Big\langle \sum_{j,k} B^{jk}(\partial_x u) u_{x_j x_k}, u_t \Big\rangle &= \frac{1}{2}\sum_{j,k} \{\langle B^{jk}(\partial_x u) u_{x_j}, u_{x_k}\rangle\}_t \\ &\quad - \sum_{j,k} \{\langle B^{jk}(\partial_x u) u_{x_j}, u_t\rangle\}_{x_k} \\ &\quad - \frac{1}{2}\sum_{j,k} \langle B^{jk}(\partial_x u)_t u_{x_j}, u_{x_k}\rangle \\ &\quad + \sum_{j,k} \langle B^{jk}(\partial_x u)_{x_k} u_{x_j}, u_t\rangle. \end{aligned}$$

The third term on the left-hand side of (4.10) is handled in the same way as in Lemma 4.1 by using the equality (4.4). Thus the equality (4.9) is reduced to

$$\partial_t \mathcal{E}_1^{(l)} + \sum_k \partial_{x_k} \mathcal{F}_{1k}^{(l)} + \mathcal{D}_1^{(l)} = \mathcal{R}_1^{(l)}, \tag{4.11}$$

where we put

$$\begin{aligned}
\mathcal{E}_1^{(l)} &:= \frac{1}{2}\Big\{|\partial_x^l u_t|^2 + \sum_{j,k}\big\langle \big(B^{jk}(\partial_x u) - \mathcal{K}^{jk}\big)\partial_x^l u_{x_j}, \partial_x^l u_{x_k}\big\rangle \\
&\qquad + \sum_{j,k} K^{jk}[\partial_x^l u_{x_j}, \partial_x^l u_{x_k}]\Big\}, \\
\mathcal{F}_{1k}^{(l)} &:= -\sum_j \langle B^{jk}(\partial_x u)\partial_x^l u_{x_j}, \partial_x^l u_{x_k}\rangle + \sum_j \langle K^{jk} * \partial_x^l u_{x_j}, \partial_x^l u_t\rangle, \\
\mathcal{D}_1^{(l)} &:= \langle L\partial_x^l u_t, \partial_x^l u_t\rangle + \frac{1}{2}\Big\{-\sum_{j,k}\dot{K}^{jk}[\partial_x^l u_{x_j}, \partial_x^l u_{x_k}] + \sum_{j,k}\langle K^{jk}\partial_x^l u_{x_j}, \partial_x^l u_{x_k}\rangle\Big\}, \\
\mathcal{R}_1^{(l)} &:= \langle f^{(l)}, \partial_x^l u_t\rangle + \frac{1}{2}\sum_{j,k}\langle B^{jk}(\partial_x u)_t \partial_x^l u_{x_j}, \partial_x^l u_{x_k}\rangle \\
&\qquad - \sum_{j,k}\langle B^{jk}(\partial_x u)_{x_k}\partial_x^l u_{x_j}, \partial_x^l u_t\rangle.
\end{aligned}$$

By computations similar to Lemma 4.1, we have the estimates from below for $\mathcal{E}_1^{(l)}$ and $\mathcal{D}_1^{(l)}$ as

$$\int_{\mathbb{R}^n} \mathcal{E}_1^{(l)}\, dx \geq c\big(\|(\partial_x^l u_t, \partial_x^{l+1} u)\|_{L^2}^2 + Q_K^\sharp[\partial_x^{l+1} u]\big), \tag{4.12}$$

$$\int_{\mathbb{R}^n} \mathcal{D}_1^{(l)}\, dx \geq c\big(\|(I-P)\partial_x^l u_t\|_{L^2}^2 + Q_K[\partial_x^{l+1} u]\big). \tag{4.13}$$

We use inequalities (3.6)–(3.8) to get the estimate for nonlinear terms

$$\begin{aligned}
&\int_{\mathbb{R}^n} \mathcal{R}_1^{(l)}\, dx \\
&\quad \leq C\|f^{(l)}\|_{L^2}\|\partial_x^l u_t\|_{L^2} + C\|\partial_x u_t\|_{L^\infty}\|\partial_x^{l+1} u\|_{L^2}^2 \\
&\qquad + C\|\partial_x^2 u\|_{L^\infty}\|\partial_x^{l+1} u\|_{L^2}\|\partial_x^l u_t\|_{L^2} \\
&\quad \leq C\|(\partial_x u_t, \partial_x^2 u)\|_{L^\infty}\|(\partial_x^l u_t, \partial_x^{l+1} u)\|_{L^2}^2,
\end{aligned} \tag{4.14}$$

where we have used the fact

$$\|f^{(l)}\|_{L^2} \leq C\|\partial_x^2 u\|_{L^\infty}\|\partial_x^{l+1} u\|_{L^2}.$$

Integrating (4.11) over $\mathbb{R}^n \times (0,t)$ and substituting the estimates (4.12)–(4.14) in the resultant equality, we obtain the desired inequality (4.8). □

Summing up (4.8) for $l = 1, \dots, s$ and adding (4.1) to the resultant inequality, we have the basic energy inequality (4.15) below.

COROLLARY 4.3

Assume the same conditions as in Proposition 3.3. Then we have

$$E(t)^2 + \tilde{D}(t)^2 \le CE_0^2 + CN(t)D(t)^2. \tag{4.15}$$

We next obtain the estimate for the dissipative term $\partial_x u_t$.

LEMMA 4.4

Assume the same conditions as in Proposition 3.3. For an arbitrary constant $\lambda > 0$, we have

$$\begin{aligned} \int_0^t \|\partial_x u_t(\tau)\|_{H^{s-1}}^2 \, d\tau &\le CE_0^2 + CN(t)D(t)^2 \\ &\quad + \lambda \int_0^t \|\partial_x^2 u(\tau)\|_{H^{s-1}}^2 \, d\tau + C_\lambda \tilde{D}(t)^2, \end{aligned} \tag{4.16}$$

where C_λ is a positive constant depending on λ.

Proof

Let l be an integer satisfying $0 \le l \le s-1$. Taking the inner product of (4.9) with $\sum_{j,k}(K^{jk} * \partial_x^l u_{x_j x_k})_t$ and using the equality

$$\begin{aligned} &\sum_{j,k} \langle u_{tt}, (K^{jk} * u_{x_j x_k})_t \rangle \\ &= \sum_{j,k} \big\{ \langle u_{tt}, (K^{jk} * u_{x_j})_t \rangle \big\}_{x_k} - \sum_{j,k} \big\{ \langle u_{x_k t}, (K^{jk} * u_{x_j})_t \rangle \big\}_t \\ &\qquad + \sum_{j,k} \langle u_{x_k t}, (K^{jk} * u_{x_j})_{tt} \rangle \\ &= \sum_{j,k} \big\{ \langle u_{tt}, (K^{jk} * u_{x_j})_t \rangle \big\}_{x_k} - \sum_{j,k} \big\{ \langle u_{x_k t}, (K^{jk} * u_{x_j})_t \rangle \big\}_t \\ &\qquad + \sum_{j,k} \langle u_{x_k t}, K^{jk}(0) u_{x_j t} + (\dot{K}^{jk} * u_{x_j})_t \rangle, \end{aligned}$$

which follows from (2.4), we have the second energy equality as

$$\partial_t \mathcal{E}_2^{(l)} + \sum_k \partial_{x_k} \mathcal{F}_{2k}^{(l)} + \mathcal{D}_2^{(l)} = \mathcal{M}_2^{(l)} + \mathcal{R}_2^{(l)}, \tag{4.17}$$

where

$$\begin{aligned} \mathcal{E}_2^{(l)} &:= \frac{1}{2} \Big| \sum_{j,k} K^{jk} * \partial_x^l u_{x_j x_k} \Big|^2 - \sum_{j,k} \langle (K^{jk} * \partial_x^l u_{x_j})_t, \partial_x^l u_{x_k t} \rangle, \\ \mathcal{F}_{2k}^{(l)} &:= \sum_j \langle (K^{jk} * \partial_x^l u_{x_j})_t, \partial_x^l u_{tt} + L \partial_x^l u_t - f^{(l)} \rangle, \\ \mathcal{D}_2^{(l)} &:= \sum_{j,k} \big\langle \big(K^{jk}(0) + L\big) \partial_x^l u_{x_j t}, \partial_x^l u_{x_k t} \big\rangle, \end{aligned}$$

$$\begin{aligned}\mathcal{M}_2^{(l)} &:= -\sum_{j,k}\langle(\dot K^{jk}*\partial_x^l u_{x_j})_t, \partial_x^l u_{x_k t}\rangle \\ &\quad + \Big\langle\sum_{j,k}(K^{jk}*\partial_x^l u_{x_j x_k})_t, \sum_{j,k}B^{jk}(\partial_x u)\partial_x^l u_{x_j x_k}\Big\rangle \\ &\quad + \sum_{j,k}\langle(K^{jk}*\partial_x^l u_{x_j})_t + \partial_x^l u_{x_j t}, L\partial_x^l u_{x_k t}\rangle, \\ \mathcal{R}_2^{(l)} &:= -\sum_{j,k}\langle(K^{jk}*\partial_x^l u_{x_j})_t, f_{x_k}^{(l)}\rangle.\end{aligned}$$

The second term in $\mathcal{E}_2^{(l)}$ is estimated using (3.10) as

$$\Big|\int_{\mathbb{R}^n}\sum_{j,k}\langle(K^{jk}*\partial_x^l u_{x_j})_t, \partial_x^l u_{x_k t}\rangle\,dx\Big| \le C(\|\partial_x^{l+1}u_t\|_{L^2}^2 + Q_K[\partial_x^{l+1}u]). \tag{4.18}$$

The assumption [A3] gives the estimate for $\mathcal{D}_2^{(l)}$ from below:

$$\int_{\mathbb{R}^n}\mathcal{D}_2^{(l)}\,dx \ge c\|\partial_x^{l+1}u_t\|_{L^2}^2. \tag{4.19}$$

We next estimate each term in $\mathcal{M}_2^{(l)}$. From (3.15), we have

$$\Big|\int_{\mathbb{R}^n}\sum_{j,k}\langle(\dot K^{jk}*\partial_x^l u_{x_j})_t, \partial_x^l u_{x_k t}\rangle\,dx\Big| \le \varepsilon\|\partial_x^{l+1}u_t\|_{L^2}^2 + C_\varepsilon Q_K[\partial_x^{l+1}u],$$

where $\varepsilon > 0$ is a constant and $C_\varepsilon > 0$ is a constant depending on ε. The inequality (3.11) yields the estimate of the second term in $\mathcal{M}_2^{(l)}$ as

$$\begin{aligned}&\Big|\int_{\mathbb{R}^n}\Big\langle\sum_{j,k}(K^{jk}*\partial_x^l u_{x_j x_k})_t, \sum_{j,k}B^{jk}(\partial_x u)\partial_x^l u_{x_j x_k}\Big\rangle\,dx\Big| \\ &\le \lambda\|\partial_x^{l+2}u\|_{L^2}^2 + C_\lambda Q_K[\partial_x^{l+2}u].\end{aligned}$$

We estimate the third term in $\mathcal{M}_2^{(l)}$ by using (3.10) as

$$\begin{aligned}&\Big|\int_{\mathbb{R}^n}\sum_{j,k}\langle(K^{jk}*\partial_x^l u_{x_j})_t + \partial_x^l u_{x_j t}, L\partial_x^l u_{x_k t}\rangle\,dx\Big| \\ &\le C\big(\|(I-P)\partial_x^{l+1}u_t\|_{L^2}^2 + Q_K[\partial_x^{l+1}u]\big).\end{aligned}$$

Combining the above three inequalities, we get the estimate for $\mathcal{M}_2^{(l)}$ as

$$\begin{aligned}\Big|\int_{\mathbb{R}^n}\mathcal{M}_2^{(l)}\,dx\Big| &\le \varepsilon\|\partial_x^{l+1}u_t\|_{L^2}^2 + \lambda\|\partial_x^{l+2}u\|_{L^2}^2 \\ &\quad + C\|(I-P)\partial_x^{l+1}u_t\|_{L^2}^2 + C_\varepsilon Q_K[\partial_x^{l+1}u] + C_\lambda Q_K[\partial_x^{l+2}u].\end{aligned} \tag{4.20}$$

Finally, we consider the nonlinear term $\mathcal{R}_2^{(l)}$. Using the inequality

$$\|\partial_x f^{(l)}\|_{L^2} \le \|f^{(l+1)}\|_{L^2} + \sum_{j,k}\|\partial_x B^{jk}(\partial_x u)\partial_x^l u_{x_j x_k}\|_{L^2} \le C\|\partial_x^2 u\|_{L^\infty}\|\partial_x^{l+2}u\|_{L^2}$$

and (3.10), we have

$$\left|\int_{\mathbb{R}^n} \mathcal{R}_2^{(l)}\,dx\right| \le \|\partial_x f^{(l)}\|_{L^2} Q_K[\partial_x^{l+1}u]^{1/2} \le C\|\partial_x^2 u\|_{L^\infty}(\|\partial_x^{l+2}u\|_{L^2}^2 + Q_K[\partial_x^{l+1}u]). \tag{4.21}$$

Therefore, integrating (4.17) over $\mathbb{R}^n \times (0,t)$, substituting the estimates (4.18)–(4.21), and letting ε be suitably small, we have

$$\begin{aligned}\Big\|\sum_{j,k} K^{jk} * \partial_x^l u_{x_j x_k}\Big\|_{L^2}^2 + \int_0^t \|\partial_x^{l+1}u_t\|_{L^2}^2\,d\tau \le CE_0^2 + CN(t)D(t)^2 \\ + \lambda\int_0^t \|\partial_x^{l+2}u\|_{L^2}^2\,d\tau + C_\lambda \int_0^t Q_K[\partial_x^{l+2}u]\,d\tau \\ + C(\|\partial_x^{l+1}u_t\|_{L^2}^2 + Q_K[\partial_x^{l+1}u]) \\ + C\int_0^t \big(\|(I-P)\partial_x^{l+1}u_t\|_{L^2}^2 + Q_K[\partial_x^{l+1}u]\big)\,d\tau.\end{aligned} \tag{4.22}$$

Substituting the basic estimate (4.15) in the fifth and sixth terms on the right-hand side of (4.22) and summing up the resultant inequality for $l = 0,\dots,s-1$, we obtain the desired estimate (4.16). □

We next show the estimate for the dissipative term $\partial_x^2 u$.

LEMMA 4.5

Assume the same conditions as in Proposition 3.3. Then we have

$$\int_0^t \|\partial_x^2 u(\tau)\|_{H^{s-1}}^2\,d\tau \le CE_0^2 + CN(t)D(t)^2 + C\int_0^t \|\partial_x u_t(\tau)\|_{H^{s-1}}^2\,d\tau. \tag{4.23}$$

Proof

Let l be an integer satisfying $0 \le l \le s-1$. Apply ∂_x^{l+1} to (1.1) to get

$$\begin{aligned}\partial_x^{l+1}u_{tt} - \sum_{j,k} B^{jk}(0)\partial_x^{l+1}u_{x_j x_k} + \sum_{j,k} K^{jk} * \partial_x^{l+1}u_{x_j x_k} + L\partial_x^{l+1}u_t \\ = \sum_j \partial_x^{l+1} g^j(\partial_x u)_{x_j},\end{aligned} \tag{4.24}$$

$$g^j(\partial_x u) := b^j(\partial_x u) - \sum_k B^{jk}(0)u_{x_k}.$$

Notice that $g^j(\partial_x u) = O(|\partial_x u|^2)$. Taking the inner product of (4.24) with $\partial_x^{l+1}u$ and using (2.2), we have

$$\partial_t \mathcal{E}_3^{(l)} + \sum_k \partial_{x_k}\mathcal{F}_{3k}^{(l)} + \mathcal{D}_3^{(l)} = \mathcal{M}_3^{(l)} + \mathcal{R}_3^{(l)}, \tag{4.25}$$

where we put

$$\mathcal{E}_3^{(l)} := \langle \partial_x^{l+1}u_t, \partial_x^{l+1}u\rangle + \frac{1}{2}\langle L\partial_x^{l+1}u, \partial_x^{l+1}u\rangle,$$

$$\mathcal{F}_{3k}^{(l)} := -\langle \partial_x^{l+1} b^k(\partial_x u), \partial_x^{l+1} u\rangle - \sum_j \langle K^{jk} * \partial_x^{l+1} u_{x_j}, \partial_x^{l+1} u\rangle,$$

$$\mathcal{D}_3^{(l)} := \sum_{j,k} \big\langle (B^{jk}(0) - \mathcal{K}^{jk})\partial_x^{l+1} u_{x_j}, \partial_x^{l+1} u_{x_k}\big\rangle,$$

$$\mathcal{M}_3^{(l)} := |\partial_x^{l+1} u_t|^2 - \sum_{j,k} \langle K^{jk} \diamond \partial_x^{l+1} u_{x_j}, \partial_x^{l+1} u_{x_k}\rangle,$$

$$\mathcal{R}_3^{(l)} := -\sum_k \langle \partial_x^{l+1} g^k(\partial_x u), \partial_x^{l+1} u_{x_k}\rangle.$$

The estimates for $\mathcal{E}_3^{(l)}$ and $\mathcal{D}_3^{(l)}$ are given by

$$\Big|\int_{\mathbb{R}^n} \mathcal{E}_3^{(l)}\,dx\Big| \le C\|(\partial_x^{l+1} u_t, \partial_x^{l+1} u)\|_{L^2}^2, \qquad \int_{\mathbb{R}^n} \mathcal{D}_3^{(l)}\,dx \ge c\|\partial_x^{l+2} u\|_{L^2}^2. \tag{4.26}$$

In deriving the lower estimate for $\mathcal{D}_3^{(l)}$, we have used the assumption [A2]. For an arbitrary positive constant ε, by using (3.9), we have the estimate for $\mathcal{M}_3^{(l)}$ as

$$\Big|\int_{\mathbb{R}^n} \mathcal{M}_3^{(l)}\,dx\Big| \le \varepsilon\|\partial_x^{l+2} u\|_{L^2}^2 + C_\varepsilon(\|\partial_x^{l+1} u_t\|_{L^2}^2 + Q_K^\sharp[\partial_x^{l+2} u]). \tag{4.27}$$

The nonlinear term $\mathcal{R}_3^{(l)}$ satisfies

$$\Big|\int_{\mathbb{R}^n} \mathcal{R}_3^{(l)}\,dx\Big| \le C\|\partial_x u\|_{L^\infty}\|\partial_x^{l+2} u\|_{L^2}^2 \tag{4.28}$$

since we have the estimate for $g^k(\partial_x u)$ from Lemma 3.4 as

$$\|\partial_x^{l+1} g^k(\partial_x u)\|_{L^2} \le C\|\partial_x u\|_{L^\infty}\|\partial_x^{l+2} u\|_{L^2}.$$

Therefore, integrating (4.25) over $\mathbb{R}^n \times (0,t)$, substituting (4.26)–(4.28), and letting ε be suitably small, we get

$$\begin{aligned} \int_0^t \|\partial_x^{l+2} u\|_{L^2}^2\,d\tau &\le CE_0^2 + CN(t)D(t)^2 + C\int_0^t \|\partial_x^{l+1} u_t\|_{L^2}^2\,d\tau \\ &\quad + C\|(\partial_x^{l+1} u_t, \partial_x^{l+1} u)\|_{L^2}^2 + C\int_0^t Q_K^\sharp[\partial_x^{l+2} u]\,d\tau. \end{aligned} \tag{4.29}$$

We substitute (4.15) in the fourth and fifth terms on the right-hand side of (4.29) and sum up the resultant inequality for $l = 0, \dots, s-1$. These computations yield the desired inequality (4.23). □

Proof of Proposition 3.3

We give the proof of the a priori estimate (3.5) by combining Corollary 4.3 and Lemmas 4.4 and 4.5. Multiplying (4.23) by a positive constant α and adding the resultant inequality to (4.16), we get

$$\begin{aligned} &(1-\alpha C)\int_0^t \|\partial_x u_t\|_{H^{s-1}}^2\,d\tau + (\alpha - \lambda)\int_0^t \|\partial_x^2 u\|_{H^{s-1}}^2\,d\tau \\ &\quad \le CE_0^2 + CN(t)D(t)^2 + C_\lambda \tilde{D}(t)^2. \end{aligned} \tag{4.30}$$

We fix α to satisfy $1-\alpha C = 1/2$ and let $\lambda = \alpha/2$. Successively we substitute (4.15) in the last term on the right-hand side of (4.30). These computations yield

$$\int_0^t \|(\partial_x u_t, \partial_x^2 u)\|_{H^{s-1}}^2 \, d\tau \le CE_0^2 + CN(t)D(t)^2. \tag{4.31}$$

Combining (4.15) and (4.31), we have

$$E(t)^2 + D(t)^2 \le CE_0^2 + CN(t)D(t)^2.$$

Since $N(t) \le CE(t)$ by the Sobolev inequality, we deduce the desired estimate (3.5) by assuming a smallness condition on $E(t)$. Therefore we complete the proof of Proposition 3.3. □

Proof of Theorem 3.1

We finally show the proof of Theorem 3.1. The existence of the solution globally in time is proved by combining the local existence result in Lemma 3.2 and the a priori estimate (3.5) with a standard continuation argument. Moreover, we see that the solution satisfies the estimate (3.5) for any $t \ge 0$. Next we show the asymptotic behavior (3.4). Let $v := (u_t, \partial_x u)$, and let j be an integer satisfying $0 \le j \le s - s_0 - 1$, where $s_0 := [n/2] + 1$. From the Gagliardo-Nirenberg inequality and (3.5), we have

$$\begin{aligned} \|\partial_x^j v(t)\|_{L^\infty} &\le C\|\partial_x^j v(t)\|_{L^2}^{1-\theta} \|\partial_x^{j+s_0} v(t)\|_{L^2}^{\theta} \\ &\le C\|\partial_x^{j+s_0} v(t)\|_{L^2}^{\theta}, \quad \theta := \frac{n}{2s_0}. \end{aligned} \tag{4.32}$$

Thus, to show (3.4), it suffices to show that

$$I(t) := \|\partial_x^{j+s_0} v(t)\|_{L^2}^2 \to 0 \quad \text{as } t \to \infty. \tag{4.33}$$

By straightforward computations from (3.5), under the condition $j + s_0 + 1 \le s$, we see that

$$\int_0^\infty |I(t)| \, dt \le C, \qquad \int_0^\infty |I'(t)| \, dt \le C,$$

which yield the convergence (4.33). Consequently, we complete the proof of Theorem 3.1. □

References

[1] P. M. N. Dharmawardane, J. E. Muñoz Rivera, and S. Kawashima, *Decay property for second order hyperbolic systems of viscoelastic materials*, J. Math. Anal. Appl. **366** (2010), 621–635.

[2] M. Fabrizio and B. Lazzari, *On the existence and the asymptotic stability of solutions for linearly viscoelastic solids*, Arch. Rational Mech. Anal. **116** (1991), 139–152.

[3] T. Hosono and S. Kawashima, *Decay property of regularity-loss type and application to some nonlinear hyperbolic-elliptic system*, Math. Models Methods Appl. Sci. **16** (2006), 1839–1859.

[4] T. J. R. Hughes, T. Kato, and J. E. Marsden, *Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity*, Arch. Rational Mech. Anal. **63** (1976), 273–294.

[5] K. Ide, K. Haramoto, and S. Kawashima, *Decay property of regularity-loss type for dissipative Timoshenko system*, Math. Models Methods Appl. Sci. **18** (2008), 647–667.

[6] S. Kawashima and Y. Shibata, *Global existence and exponential stability of small solutions to nonlinear viscoelasticity*, Comm. Math. Phys. **148** (1992), 189–208.

[7] M. de Lima Santos, *Decay rates for solutions of a system of wave equations with memory*, Electron. J. Differential Equations (2002), no. 38.

[8] Z. Liu and S. Zheng, *On the exponential stability of linear viscoelasticity and thermoviscoelasticity*, Quart. Appl. Math. **54** (1996), 21–31.

[9] J. E. Muñoz Rivera, *Asymptotic behaviour in linear viscoelasticity*, Quart. Appl. Math. **52** (1994), 628–648.

[10] J. E. Muñoz Rivera, M. G. Naso, and F. M. Vegni, *Asymptotic behavior of the energy for a class of weakly dissipative second-order systems with memory*, J. Math. Anal. Appl. **286** (2003), 692–704.

Dharmawardane: Graduate School of Mathematics, Kyushu University, Fukuoka 819-0395, Japan; pdharmawardane@yahoo.com

Nakamura: Faculty of Mathematics, Kyushu University, Fukuoka 819-0395, Japan; tohru@math.kyushu-u.ac.jp

Kawashima: Faculty of Mathematics, Kyushu University, Fukuoka 819-0395, Japan; kawashim@math.kyushu-u.ac.jp