Generalized eigenvalue-counting estimates for some random acoustic operators

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Abstract For some discrete random acoustic operators, we prove Wegner estimates. These estimates are applied to show some regularity of the integrated density of states. Moreover, we prove the generalized eigenvalue-counting estimates by using Combes, Germinet, and Klein's method. As an application, the multiplicity of the eigenvalues in some interval where the Anderson localization occurs is proven to be finite. For certain models, Poisson statistics for eigenvalues and Lifshitz tails are also studied.

1. Introduction

In this paper, we consider the following random acoustic operator in the discrete setting:

$$A_\omega := \nabla^*(\rho_0 + \rho_\omega)\nabla = A_0 + \sum_{\gamma \in \mathbf{Z}^d} \omega_\gamma \nabla^* \Pi_\gamma \nabla, \tag{1.1}$$

where ∇ is defined by the discrete analogue of the partial derivative, ∇^* is its adjoint, ρ_0 is a periodic term, and $\rho_\omega := \sum_{\gamma \in \mathbf{Z}^d} \omega_\gamma \Pi_\gamma$ is an Anderson-type random perturbation. The operator A_ω is defined on $\ell^2(\mathbf{Z}^d)$ (for details, see Section 2). In that section, we prove the Wegner-type estimate and the generalized eigenvalue-counting estimate for this operator. We apply the Wegner estimate to obtain some regularity for the integrated density of states (IDS). Moreover, we use the generalized eigenvalue-counting estimate to show that the multiplicity of the eigenvalues of (1.1) in some interval where the Anderson localization occurs is finite.

Recently, Combes, Germinet, and Klein [9] have given a nice proof of the generalized eigenvalue-counting estimate for discrete random Schrödinger operators, including the Anderson [4] model. This operator is defined by $H_\omega = -\Delta + V_\omega$, where $-\Delta$ is the discrete Laplacian and $V_\omega := \sum_{\gamma \in \mathbf{Z}^d} \omega_\gamma \Pi_\gamma$ is the random potential. Their eigenvalue-counting estimates give an upper bound of the probability that the restricted random operator H_ω^Λ has n eigenvalues in an interval I. This is an extension of the Wegner and Minami estimates. In this article, we use their ideas to study random acoustic operators under the discrete setting.

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The disorder associated with H_ω is a type of diagonal disorder, while our acoustic operator A_ω has a type of off-diagonal disorder. The random potential V_ω in H_ω is the sum of independent random rank-one operators; that is, the rank of Π_γ is just one. However, one difficulty in A_ω is that the rank of $\nabla^*\Pi_\gamma\nabla$ in (1.1) is more than two if $d \geq 2$. Therefore, we cannot use the method in [9] directly. To treat our random operator A_ω, by applying the general spectral averaging results in [11], [12], and [31], we generalize the method for the rank-one case to the case when the rank is more than two.

It is well known that there are many works on the Anderson model H_ω. Wegner [43] estimated the expectation of the number of eigenvalues of H_ω^Λ in I. This gives an upper bound of the probability that H_ω^Λ has at least one eigenvalue in I. First, this was used to show Lipschitz continuity of the IDS for H_ω or to give a bound of the density of states, the derivative of the IDS. Next, this was also used as a key estimate to prove the Anderson localization rigorously. This method is called the multiscale analysis (see, e.g., [8], [24], [28], [40]).

Figotin and Klein [21] proved the Wegner estimate for A_ω and showed the localization via the multiscale analysis when the probability distribution μ_γ of ω_γ has a bounded density. Our Wegner estimate is an extension to the case when μ_γ is general.

Minami [34] estimated the probability that H_ω^Λ has at least two eigenvalues in I and proved that the properly rescaled eigenvalues of H_ω^Λ behave according to a Poisson point process. This was first shown by Molchanov [33] for 1-dimensional continuous random Schrödinger operators. It is natural that we consider the application of our generalized eigenvalue-counting estimates. However, it is uncertain whether our estimate for A_ω is useful to study some behavior for eigenvalues of A_ω since our result is a weaker estimate if $d \geq 2$ (see Section 2 for details).

Klein and Molchanov [29] used the Minami estimate as an important probabilistic estimate to study the multiplicity of eigenvalues of H_ω. Moreover, the Minami estimate was generalized by [5], [9], and [25] to the probability that H_ω^Λ has at least n eigenvalues in I for all $n \in \mathbf{N}$.

As relating works, Faris [18]–[20] studied a simpler random acoustic model:

$$\Gamma_\omega := \frac{1}{\sqrt{\rho_\omega}}(-\Delta)\frac{1}{\sqrt{\rho_\omega}}, \tag{1.2}$$

To show the Anderson localization, Faris gave the Wegner estimate for this operator. In this article, this random operator is also discussed (see Section 3 for details).

Our random operator A_ω in this article has the continuous version, which is formally defined by $-\nabla(\rho_0 + \rho_\omega)\nabla = A_0 + \sum_{\gamma\in\mathbf{Z}^d}\omega_\gamma(-\nabla u_\gamma\nabla)$ on $L^2(\mathbf{R}^d)$, where $\rho_0(x)$ is a $\mathbf{Z}^d$-periodic function and $\rho_\omega(x) := \sum_{\gamma\in\mathbf{Z}^d}\omega_\gamma u_\gamma(x)$ is an Anderson-type random perturbation. In [22] Figotin and Klein gave the Wegner estimate and applied this to prove the localization of acoustic waves. Unfortunately, the techniques in this article do not work for this continuous model because of the following. Although the operator $-\nabla u_\gamma\nabla$ is nonnegative as in the discrete model A_ω, this is not a bounded operator. Thus, we cannot use the general spectral averaging results in [11], [12], and [31] (see Proposition 2.1 in Section 2).

In [21], the method for the acoustic model was also applied to random Maxwell operators M_ω, formally defined by $\nabla^* \times (1/\varepsilon_\omega)\nabla\times$ on $\ell^2(\mathbf{Z}^3; \mathbf{C}^3)$, where ε_ω is the position-dependent random dielectric constant. The Wegner estimate for M_ω was given to prove the localization of electromagnetic waves (see [21] for the discrete case and [13], [23] for the continuous case). In this article, we only treat random acoustic models; however, the method in this article works also for M_ω.

This article is organized as follows. In Section 2, we first prove the Wegner estimate for A_ω by using the general spectral averaging result. Next, we extend the method of the article by Combes, Germinet, and Klein [9] for the operator A_ω and prove the generalized eigenvalue-counting estimate. In Section 3, we study the random operator Γ_ω, including (1.2) as another acoustic model. To consider the same problem for Γ_ω, we introduce an auxiliary random Schrödinger operator. In Section 3, we also state Poisson statistics for Γ_ω. In Appendix A, we present a simple proof of Lifshitz tails for Γ_ω. In Appendix B, we discuss some relations between A_ω and Γ_ω, which are especially studied in the one-dimensional case. In Appendix C, we verify a condition used in Section 3.

2. Main results

2.1. Generalized eigenvalue-counting estimate for A_ω

In general, acoustic operators are defined by

$$A_{ab} := \frac{1}{\sqrt{a}}\nabla^* \frac{1}{b}\nabla \frac{1}{\sqrt{a}} = \sum_{j=1}^{d} \frac{1}{\sqrt{a}}\partial_j^* \frac{1}{b}\partial_j \frac{1}{\sqrt{a}} \tag{2.1}$$

acting on the Hilbert space $\ell^2(\mathbf{Z}^d)$. The multiplicative operators $1/\sqrt{a}$ and $1/b$ are defined by

$$\frac{1}{\sqrt{a}} := \sum_{\gamma\in\mathbf{Z}^d} \frac{1}{\sqrt{a_\gamma}}\Pi_\gamma \qquad \text{and} \qquad \frac{1}{b} := \sum_{\gamma\in\mathbf{Z}^d} \frac{1}{b_\gamma}\Pi_\gamma \tag{2.2}$$

for some positive real numbers $\{a_\gamma\}_{\gamma\in\mathbf{Z}^d}$ and $\{b_\gamma\}_{\gamma\in\mathbf{Z}^d}$, where Π_γ is an orthogonal projector to the Kronecker delta function $\delta_\gamma(x) \in \ell^2(\mathbf{Z}^d)$, that is, $\Pi_\gamma := |\delta_\gamma\rangle\langle\delta_\gamma| = \langle\delta_\gamma, \cdot\rangle\delta_\gamma$. The discrete analogue of the partial derivative ∂_j is defined by $(\partial_j f)(x) := f(x) - f(x - e_j)$ for $f(x) \in \ell^2(\mathbf{Z}^d)$ and $j = 1, \dots, d$, where $\{e_j\}_{j=1}^d$ are the standard basis vectors in the lattice $\mathbf{Z}^d$. The adjoint of ∂_j is given by $(\partial_j^* f)(x) = f(x) - f(x + e_j)$. The acoustic operator A_{ab} appears in the discrete wave equation

$$\frac{\partial^2 f(x,t)}{\partial t^2} + A_{ab} f(x,t) = 0.$$

We note that the coefficient $a = \{a_\gamma\}_{\gamma\in\mathbf{Z}^d}$ corresponds to the local propagation speed and $b = \{b_\gamma\}_{\gamma\in\mathbf{Z}^d}$ corresponds to the mass density. Refer to [13], [18]–[22] for the physical interpretation.

In this section, we consider the operator defined by

$$A_\omega := \nabla^*(\rho_0+\rho_\omega)\nabla = \sum_{j=1}^{d} \partial_j^*(\rho_0+\rho_\omega)\partial_j \tag{2.3}$$

on $\ell^2(\mathbf{Z}^d)$. We set ρ_0 is a periodic term defined by $\rho_0 := \sum_{\gamma\in\mathbf{Z}^d} q_\gamma \Pi_\gamma$, where $\gamma \mapsto q_\gamma$ is a nonnegative bounded periodic function on $\mathbf{Z}^d$. More precisely, assume that there exists $p=(p_1,\dots,p_d)\in\mathbf{Z}^d$ such that $q_\gamma = q_{\gamma+np}$ for any $\gamma, n=(n_1,\dots,n_d)\in\mathbf{Z}^d$, where $np := (n_1p_1,\dots,n_dp_d)$. Let M_0 be the upper bound of $\{q_\gamma\}_{\gamma\in\mathbf{Z}^d}$, and set $A_0 := \nabla^*\rho_0\nabla$. We next set ρ_ω as a positive Anderson-type random perturbation defined by $\rho_\omega := \sum_{\gamma\in\mathbf{Z}^d}\omega_\gamma\Pi_\gamma$, where $\omega=\{\omega_\gamma\}_{\gamma\in\mathbf{Z}^d}$ are nontrivial random variables taking values in the finite interval $[m,M]$ for some $0<m<M<\infty$. For $\gamma\in\mathbf{Z}^d$, let μ_γ be the probability distribution of the random variable ω_γ. We always assume that $\{\omega_\gamma\}_{\gamma\in\mathbf{Z}^d}$ are independent. A_ω is a nonnegative, bounded self-adjoint operator. If we assume that

$$\{\omega_\gamma\}_{\gamma\in\mathbf{Z}^d} \text{ are identically distributed} \tag{2.4}$$

with the common probability distribution μ, then A_ω is a $\mathbf{Z}^d$-ergodic operator. It is known that there exist closed and nonrandom sets Σ, Σ_{pp}, Σ_{ac}, and Σ_{sc} of $\mathbf{R}$ such that $\Sigma=\sigma(A_\omega)$, $\Sigma_{pp}=\sigma_{pp}(A_\omega)$, $\Sigma_{ac}=\sigma_{ac}(A_\omega)$, and $\Sigma_{sc}=\sigma_{sc}(A_\omega)$ almost surely (see [21, Theorem 1], [8], [28], [37]).

To state the main theorems for A_ω, we define $S_\nu(t)$ by

$$S_\nu(t) := \sup_{c\in\mathbf{R}} \nu([c,c+t])$$

for the probability measure ν on $\mathbf{R}$ and $t>0$. For a Borel set $J\subset\mathbf{R}$ and a self-adjoint operator T, we denote the associated spectral projection by $\chi_J(T)$. For $L\in\mathbf{N}$, set a cube $\Lambda=\Lambda_L(0)$ centered at zero with side length L, that is, $\Lambda := \{k\in\mathbf{Z}^d \mid \max_{i=1,2,\dots,d}|k_i|\le L/2\}\subset\mathbf{Z}^d$. We denote by A_ω^Λ some self-adjoint restriction of A_ω to the cube Λ. We realize A_ω^Λ simply by taking the matrix elements $A_\omega^\Lambda(n,m)$ as $\langle\delta_n, A_\omega\delta_m\rangle$ whenever both n and m belong to Λ. If we let χ_Λ be the characteristic function on Λ, that is, $\chi_\Lambda=\sum_{\gamma\in\Lambda}\Pi_\gamma$, then we can also write $A_\omega^\Lambda=\chi_\Lambda A_\omega\chi_\Lambda$. In Appendix B, we also use the periodic boundary condition. We write $\mathbf{E}=\mathbf{E}_\omega$ for the expectation with respect to $\omega=\{\omega_j\}_{j\in\mathbf{Z}^d}$. We first state the following estimate.

THEOREM 2.1

Fix $E_0\in(0,\infty)$. Let I be any bounded interval in $[E_0,\infty)$. Then we have the bound

$$\mathbf{E}\,\mathrm{Tr}\,\chi_I(A_\omega^\Lambda) \le C_W Q_A(|I|)|\Lambda|, \tag{2.5}$$

where $Q_A(|I|)=\sup_\gamma Q_{A,\gamma}(|I|)$,

$$Q_{A,\gamma}(|I|) = \begin{cases} 2d|g_\gamma|_\infty |I| & \textit{if } \mu_\gamma \textit{ has a bounded density } g_\gamma,\\ 8d(1+2d)S_{\mu_\gamma}(|I|) & \textit{otherwise,}\end{cases} \tag{2.6}$$

and C_W is a constant which depends only on d, M_0, M, and E_0.

REMARK 1
The above constant C_W is given by $(d+1)^2(M_0+M)^2/E_0^2$.

We next state the generalized eigenvalue-counting estimate for A_ω.

THEOREM 2.2
Fix $E_0 \in (0,\infty)$. Let I be any bounded interval in $[E_0,\infty)$. For $n=0,1,2,\ldots,$ we have the bound

$$\mathbf{P}\big(\operatorname{Tr}\chi_I(A_\omega^\Lambda) \geq nd+1\big) \leq C_G\big(Q_A(|I|)|\Lambda|\big)^{n+1}, \tag{2.7}$$

where C_G is a constant which depends only on $n, d, M_0, M, m,$ and E_0.

REMARK 2
The above constant C_G is given by

$$\frac{(d+1)^{2(n+1)}\prod_{j=0}^{n}\left(M_0+2^jM-2^jm+m\right)^2}{E_0^{2(n+1)}\prod_{j=0}^{n}(jd+1)}.$$

Let us explain that (2.7) is weaker than the corresponding estimate given in [9]. The generalized eigenvalue-counting estimate for H_ω^Λ in [9] is the following: for any $n \in \mathbf{N}$,

$$\mathbf{P}\big(\operatorname{Tr}\chi_I(H_\omega^\Lambda) \geq n\big) \leq \frac{1}{n!}\big(Q(|I|)|\Lambda|\big)^n, \tag{2.8}$$

where $Q(|I|) = \sup_\gamma Q_\gamma(|I|)$ and

$$Q_\gamma(|I|) = \begin{cases} |g_\gamma|_\infty |I| & \text{if } \mu_\gamma \text{ has a bounded density } g_\gamma, \\ 8S_{\mu_\gamma}(|I|) & \text{otherwise.} \end{cases}$$

Since we have $\mathbf{P}(\operatorname{Tr}\chi_I(A_\omega^\Lambda) \geq n+1) \geq \mathbf{P}(\operatorname{Tr}\chi_I(A_\omega^\Lambda) \geq nd+1)$, (2.7) is a weaker version of (2.8). However, (2.7) is the same type of estimate as (2.8) if $d=1$. Moreover, (2.7) always gives the Wegner estimate by taking $n=0$.

To state a corollary of Theorem 2.1, we introduce the IDS for A_ω. This function is defined as follows. Since the spectrum of A_ω^Λ is discrete, we can define the eigenvalue-counting function by $N_\omega^\Lambda(E) := \operatorname{Tr}\chi_{(-\infty,E]}(A_\omega^\Lambda)$. If we assume (2.4), that is, that $\{\omega_\gamma\}_{\gamma\in\mathbf{Z}^d}$ are independent and identically distributed (i.i.d.), we can show the existence of a nonrandom, right continuous, and increasing function N_A such that

$$N_A(E) = \lim_{|\Lambda|\to\infty}\frac{N_\omega^\Lambda(E)}{|\Lambda|}$$

for any point of continuity of $N_A(E)$ and for almost all ω. We call $N_A(E)$ the IDS for A_ω (see, e.g., [8], [21], [28], [37]). We say that the common probability measure μ of $\{\omega_\gamma\}_{\gamma\in\mathbf{Z}^d}$ is α-Hölder continuous if there are $c>0$ and $\alpha \in (0,1]$ such that

$$S_\mu(t) \leq ct^\alpha \tag{2.9}$$

for small $t>0$. From the definition of the IDS and Fatou's lemma, it follows that

$$\begin{aligned} N_A(E)-N_A(E') &= \mathbf{E}\Big[\lim_{|\Lambda|\to\infty}\frac{N_\omega^\Lambda(E)-N_\omega^\Lambda(E')}{|\Lambda|}\Big] \\ &\le \liminf_{|\Lambda|\to\infty}\frac{1}{|\Lambda|}\mathbf{E}[N_\omega^\Lambda(E)-N_\omega^\Lambda(E')] \\ &= \liminf_{|\Lambda|\to\infty}\frac{1}{|\Lambda|}\mathbf{E}[\mathrm{Tr}\,\chi_{(E',E]}(A_\omega^\Lambda)] \end{aligned}$$

for any point of continuity $E'<E\in\mathbf{R}$. By Theorem 2.1 and α-Hölder continuity of μ, we have the following.

COROLLARY 2.1

Assume (2.4) and (2.9). Then the IDS for A_ω is α-Hölder continuous.

We next state an application of Theorem 2.2. We say that

$$A_\omega \text{ exhibits the Anderson localization in some interval } I \tag{2.10}$$

if, with probability one, A_ω has only pure point spectrum in I and the corresponding eigenfunctions decay exponentially. For $x\ge 0$, we let $[x]$ be the integral part of x. We prove the following result.

COROLLARY 2.2

Assume (2.4), (2.9), and (2.10). Then, with probability one, every eigenvalue of A_ω in I has multiplicity less than or equal to $[\alpha^{-1}]d$. In particular, if $\alpha>1/2$ and $d=1$, with probability one, every eigenvalue of A_ω in I is simple.

REMARK 3

Figotin and Klein [21] showed the localization for A_ω near the band edge of the spectrum under some assumptions on the periodic operator A_0 and on the distribution of $\{\omega_\gamma\}_{\gamma\in\mathbf{Z}^d}$.

2.2. Proofs of Theorems 2.1 and 2.2

We quote an essential proposition to show the main theorems.

PROPOSITION 2.1

Let ν denote the probability distribution of the random variable ω. Let I be a finite interval in $\mathbf{R}$. Let H and B be two self-adjoint operators on a separable Hilbert space $\mathcal{H}$, and suppose that B is bounded and nonnegative. Then, for any $\varphi\in\mathcal{H}$ and any $k>0$, we have the bound

$$\begin{aligned} &\int\langle\varphi, B\chi_I(H+\omega B)B\varphi\rangle\nu(d\omega) \\ &\quad\le\begin{cases}\|B\|_{op}|g|_\infty|I|\|\varphi\|^2 & \text{if } \nu \text{ has a bounded density } g,\\ 4\|B\|_{op}(k+\|B\|_{op})S_\nu(|I|/k)\|\varphi\|^2 & \text{otherwise,}\end{cases} \end{aligned} \tag{2.11}$$

where $\|\cdot\|_{op}$ *is the operator norm and* $\|\cdot\|$ *is the norm in* $\mathcal{H}$.

This is often called the spectral averaging (see, e.g., [11, 12, 31, 42]). Combes and Hislop [11] and Kotani and Simon [31] studied the case when the probability measure ν has a bounded density. Combes and colleagues [12] proved the general case. Since [12] considered the case $k=1$, (2.11) is slightly generalized (see [9, Appendix] of for this extension). Now, we prepare the following lemma.

LEMMA 2.1

Let $I=[a,b]$ *be any finite interval in* $(0,\infty)$*; then we have*

$$\operatorname{Tr}\chi_I(A_\omega^\Lambda)\le C\sum_{x\in\Lambda}\sum_{\gamma\in K_x}\langle\delta_x,B_\gamma\chi_I(A_\omega^\Lambda)B_\gamma\delta_x\rangle, \tag{2.12}$$

where $C=C(a,d,M_0,M)=(d+1)(M_0+M)^2/a^2$, $K_x=\{x,x+e_1,\dots,x+e_d\}\subset\mathbf{Z}^d$, *and* B_γ *is the restriction of* $\nabla^*\Pi_\gamma\nabla$ *to the cube* Λ.

Proof

We first note that $A_\omega^\Lambda=\sum_{\gamma\in\tilde\Lambda}(q_\gamma+\omega_\gamma)B_\gamma$, where $\tilde\Lambda\supset\Lambda$ is defined by $\{k\in\mathbf{Z}^d\mid\max_{i=1,\dots,d}|k_i|\le L/2+1\}$. We next expand the trace as $\operatorname{Tr}\chi_I(A_\omega^\Lambda)=\sum_{x\in\Lambda}\langle\delta_x,\chi_I(A_\omega^\Lambda)\delta_x\rangle$. By using the spectral measure $\mu_A^{\delta_x}(d\lambda)$ for A_ω^Λ with respect to δ_x, we have

$$\langle\delta_x,\chi_I(A_\omega^\Lambda)\delta_x\rangle=\int_I\mu_A^{\delta_x}(d\lambda)\le\frac{1}{a^2}\int_a^b\lambda^2\mu_A^{\delta_x}(d\lambda).$$

Thus, since ρ_0 and ρ_ω are bounded, we get

$$\operatorname{Tr}\chi_I(A_\omega^\Lambda)\le\frac{(M_0+M)^2}{a^2}\sum_{x\in\Lambda}\sum_{\gamma_1,\gamma_2\in\tilde\Lambda}\langle\delta_x,B_{\gamma_1}\chi_I(A_\omega^\Lambda)B_{\gamma_2}\delta_x\rangle.$$

Now, we note that

$$\nabla^*\Pi_\gamma\nabla=\sum_{j=1}^d|\partial_j^*\delta_\gamma\rangle\langle\partial_j^*\delta_\gamma|=\sum_{j=1}^d|\delta_\gamma-\delta_{\gamma-e_j}\rangle\langle\delta_\gamma-\delta_{\gamma-e_j}|.$$

Then, it is easy to see that for fixed $x\in\Lambda$,

$$\nabla^*\Pi_\gamma\nabla\delta_x=\begin{cases}d\delta_x-\sum_{j=1}^d\delta_{x-e_j} & \text{if }\gamma=x,\\ \delta_x-\delta_{x+e_j} & \text{if }\gamma=x+e_j,\ j=1,2,\dots,d,\\ 0 & \text{otherwise.}\end{cases}$$

Thus, we have $B_\gamma\delta_x=0$ if $\gamma\notin K_x$. By using

$$\begin{aligned}&\langle\delta_x,B_{\gamma_1}\chi_I(A_\omega^\Lambda)B_{\gamma_2}\delta_x\rangle\\&\quad\le\frac12\big(\langle\delta_x,B_{\gamma_1}\chi_I(A_\omega^\Lambda)B_{\gamma_1}\delta_x\rangle+\langle\delta_x,B_{\gamma_2}\chi_I(A_\omega^\Lambda)B_{\gamma_2}\delta_x\rangle\big),\end{aligned}$$

we complete the proof. □

For $j \in \mathbf{Z}^d$, we write $\mathbf{E}_{\omega_j}$ for the expectation with respect to the random variable ω_j. We also write $\omega_k^\perp := \{\omega_j\}_{j\in\mathbf{Z}^d\setminus\{k\}}$ and write $\mathbf{E}_{\omega_k^\perp}$ for the corresponding expectation.

Proof of Theorem 2.1
Let $I = [a,b] \subset [E_0, \infty)$. Taking the expectation for (2.12), we get

$$\mathbf{E}\,\mathrm{Tr}\,\chi_I(A_\omega^\Lambda) \le C \sum_{x\in\Lambda}\sum_{\gamma\in K_x} \mathbf{E}_{\omega_\gamma^\perp}\mathbf{E}_{\omega_\gamma}\langle \delta_x, B_\gamma \chi_I(A_\omega^\Lambda) B_\gamma \delta_x\rangle. \tag{2.13}$$

Now, note that $\|B_\gamma\|_{op} \le 2d$ and $\|\delta_x\| = 1$. By using Proposition 2.1 with $k = 1$ in (2.11), we can show that the right-hand side of (2.13) is less than or equal to

$$C \sum_{x\in\Lambda}\sum_{\gamma\in K_x} \mathbf{E}_{\omega_\gamma^\perp} Q_{A,\gamma}(|I|) \le C(d+1) Q_A(|I|)|\Lambda|,$$

where $Q_A(|I|) = \sup_\gamma Q_{A,\gamma}(|I|)$ and $Q_{A,\gamma}(|I|)$ is defined by (2.6). This completes the proof. □

To prove Theorem 2.2, we prepare the following lemma, which is a generalization of [9, Lemma 4.1].

LEMMA 2.2
Consider the self-adjoint operator $H_s = H_0 + sB$ on a separable Hilbert space $\mathcal{H}$, where H_0 and B are two self-adjoint operators on $\mathcal{H}$, B is a nonnegative operator satisfying $\mathrm{rank}\, B \le d$, and $s \in \mathbf{R}$. Let $P_s(J) = \chi_J(H_s)$ for an interval J, and suppose that $\mathrm{Tr}\, P_0((-\infty, c]) < \infty$ for all $c \in \mathbf{R}$. Then, given $a, b \in \mathbf{R}$ with $a < b$, we have

$$\mathrm{Tr}\, P_s\big((a,b]\big) \le d + \mathrm{Tr}\, P_t\big((a,b]\big)$$

for all $0 \le s \le t$.

Proof
Let $0 \le s \le t$. Note that for any $c \in \mathbf{R}$ we always have

$$0 \le \mathrm{Tr}\, P_s\big((-\infty, c]\big) - \mathrm{Tr}\, P_t\big((-\infty, c]\big) \le d. \tag{2.14}$$

The last inequality is a consequence of the min-max principle, proved similarly in [28, Lemma 5.22]. We can also consider (2.14) as a bound of the spectral shift function (SSF) $\xi(\lambda; H_s, H_t)$ for the pair (H_s, H_t). Since $H_s - H_t$ is an operator of finite rank, we have the bound $0 \le \xi(\lambda; H_s, H_t) \le \mathrm{rank}(H_s - H_t) \le d$. For the general theory of the SSF, refer to [6]. Thus, we obtain

$$\begin{aligned}\mathrm{Tr}\, P_s\big((a,b]\big) &= \mathrm{Tr}\, P_s\big((-\infty,b]\big) - \mathrm{Tr}\, P_s\big((-\infty,a]\big)\\ &\le \mathrm{Tr}\, P_s\big((-\infty,b]\big) - \mathrm{Tr}\, P_t\big((-\infty,a]\big)\\ &= \mathrm{Tr}\, P_s\big((-\infty,b]\big) - \mathrm{Tr}\, P_t\big((-\infty,b]\big) + \mathrm{Tr}\, P_t\big((a,b]\big)\\ &\le d + \mathrm{Tr}\, P_t\big((a,b]\big).\end{aligned}$$

□

Proof of Theorem 2.2

Let $\mathbf{1}_A(\omega)$ be the indicator function of an event A. For $n \in \mathbf{N}$, we set the events E_n and A_n by $E_n := \{\omega \mid \operatorname{Tr}\chi_I(A_\omega^\Lambda) \geq nd+1\}$ and $A_n := \{\omega \mid \prod_{k=0}^n (\operatorname{Tr}\chi_I(A_\Lambda^\omega) - kd) \geq C_{n,d}\}$, where $C_{n,d} = \prod_{k=0}^n (kd+1)$. By $E_n \subset A_n$ and Chebyshev's inequality, we have

$$\mathbf{P}\big(\operatorname{Tr}\chi_I(A_\omega^\Lambda) \geq nd+1\big) = \int_{E_n} \mathbf{1}_{A_n}(\omega)\mathbf{P}(d\omega)$$
$$\leq \mathbf{E}\Big[\prod_{k=0}^n \big(\operatorname{Tr}\chi_I(A_\omega^\Lambda) - kd\big)\mathbf{1}_{E_n}(\omega)\Big]/C_{n,d}.$$

When $n=0$, this is just the Wegner estimate (2.5). For the reader's convenience, we next consider the case $n=1$. By using Lemma 2.1, we first get

$$\begin{aligned}(2.15)\qquad &\operatorname{Tr}\chi_I(A_\omega^\Lambda)\big(\operatorname{Tr}\chi_I(A_\omega^\Lambda) - d\big)\mathbf{1}_{E_1}(\omega)\\ &\leq C\sum_{x\in\Lambda}\sum_{\gamma\in K_x}\langle\delta_x, B_\gamma\chi_I(A_\omega^\Lambda)B_\gamma\delta_x\rangle\big(\operatorname{Tr}\chi_I(A_\omega^\Lambda) - d\big)\mathbf{1}_{E_1}(\omega),\end{aligned}$$

where $C = C(a,d,M_0,M)$ in Lemma 2.1. For $\omega = (\omega_\gamma^\perp, \omega_\gamma)$ and $\gamma \in \tilde{\Lambda}$, we define $A^\Lambda_{(\omega_\gamma^\perp, s)}$ by $A^\Lambda_{(\omega_\gamma^\perp, s)} = \sum_k q_k B_k + \sum_{k\neq\gamma}\omega_k B_k + sB_\gamma$. For each $\gamma \in \mathbf{Z}^d$, we take a random variable $\tilde{\omega}_\gamma$ with the probability distribution μ_γ, so that $\tilde{\omega}_\gamma$ is independent from all of the other random variables, and set $\tau_\gamma = \tilde{\omega}_\gamma + M - m$. Write $\tilde{\omega} = \{\tilde{\omega}_\gamma\}_{\gamma\in\mathbf{Z}^d}$. By Lemma 2.2 with $\tau_\gamma \geq M \geq \omega_\gamma$, we have that the right-hand side of (2.15) is less than or equal to

$$C\sum_{x\in\Lambda}\sum_{\gamma\in K_x}\langle\delta_x, B_\gamma\chi_I(A_\omega^\Lambda)B_\gamma\delta_x\rangle \operatorname{Tr}\chi_I(A^\Lambda_{(\omega_\gamma^\perp,\tau_\gamma)})\mathbf{1}_{E_1'}(\omega_\gamma^\perp,\tau_\gamma),$$

where we set $E_1' = \{(\omega_\gamma^\perp,\tau_\gamma) \mid \operatorname{Tr}\chi_I(A^\Lambda_{(\omega_\gamma^\perp,\tau_\gamma)}) \geq 1\}$ and used

$$(2.16)\qquad \mathbf{1}_{E_1}(\omega) \leq \mathbf{1}_{E_1'}(\omega_\gamma^\perp,\tau_\gamma)$$

for any $\tilde{\omega}_\gamma$ and ω. Since $\operatorname{Tr}\chi_I(A^\Lambda_{(\omega_\gamma^\perp,\tau_\gamma)})\mathbf{1}_{E_1'}(\omega_\gamma^\perp,\tau_\gamma)$ is independent of ω_γ, by taking the expectation with respect to ω and $\tilde{\omega}$, and using Proposition 2.1, we get

$$\begin{aligned}&\mathbf{E}_{\tilde{\omega}}\mathbf{E}_\omega\big[\operatorname{Tr}\chi_I(A_\omega^\Lambda)\big(\operatorname{Tr}\chi_I(A_\omega^\Lambda) - d\big)\mathbf{1}_{E_1}(\omega)\big]\\ &\quad\leq C\sum_{x\in\Lambda}\sum_{\gamma\in K_x}\mathbf{E}_{\tilde{\omega}_\gamma}\mathbf{E}_{\omega_\gamma^\perp}\big[\mathbf{E}_{\omega_\gamma}[\langle\delta_x, B_\gamma\chi_I(A_\omega^\Lambda)B_\gamma\delta_x\rangle]\operatorname{Tr}\chi_I(A^\Lambda_{(\omega_\gamma^\perp,\tau_\gamma)})\mathbf{1}_{E_1'}(\omega_\gamma^\perp,\tau_\gamma)\big]\\ &\quad\leq CQ_A(|I|)\sum_{x\in\Lambda}\sum_{\gamma\in K_x}\mathbf{E}_{\tilde{\omega}_\gamma}\mathbf{E}_{\omega_\gamma^\perp}[\operatorname{Tr}\chi_I(A^\Lambda_{(\omega_\gamma^\perp,\tau_\gamma)})\mathbf{1}_{E_1'}(\omega_\gamma^\perp,\tau_\gamma)],\end{aligned}$$

where $Q_A(|I|) = \sup_\gamma Q_{A,\gamma}(|I|)$ and $Q_{A,\gamma}(|I|)$ is defined in (2.6). By using the Wegner estimate (2.5), we get

$$\mathbf{E}_{\tilde{\omega}_\gamma}\mathbf{E}_{\omega_\gamma^\perp}[\operatorname{Tr}\chi_I(A^\Lambda_{(\omega_\gamma^\perp,\tau_\gamma)})\mathbf{1}_{E_1'}(\omega_\gamma^\perp,\tau_\gamma)] \leq C'(d+1)Q_A(|I|)|\Lambda|,$$

where $C' = C(a,d,M_0,2M-m)$. We remark that $\sup_\gamma\{\omega_\gamma^\perp,\tau_\gamma\} = 2M-m$. This completes the proof for $n=1$.

For the general $n \in \mathbf{N}$, we can proceed inductively as in [9]. Let us assume that for all m, M with $0 < m < M$ and for all possible probability distributions μ_γ of ω_γ satisfying $\operatorname{supp} \mu_\gamma \subset [m, M]$ for all $\gamma \in \mathbf{Z}^d$,

$$\mathbf{E}\Big[\prod_{k=0}^{n-1} \big(\operatorname{Tr} \chi_I(A_\omega^\Lambda) - kd\big) \mathbf{1}_{E_{n-1}}(\omega)\Big] \leq C_n \big(Q_A(|I|)|\Lambda|\big)^n \tag{2.17}$$

holds for $n \in \mathbf{N}$, where $C_n = C_n(a, d, M_0, M, m)$ is defined by

$$\prod_{j=0}^{n-1} (M_0 + 2^j M - 2^j m + m)^2 \frac{(d+1)^{2n}}{a^{2n}}.$$

We remark that $M \geq \sup_\gamma \omega_\gamma$ and $m \leq \inf_\gamma \omega_\gamma$. For each $\gamma \in \mathbf{Z}^d$, we take $\tilde{\omega}_\gamma$ as in the case for $n = 1$. Then, from Lemmas 2.1 and 2.2, it follows that

$$\prod_{k=0}^{n} \big(\operatorname{Tr} \chi_I(A_\omega^\Lambda) - kd\big) \mathbf{1}_{E_n}(\omega) \tag{2.18}$$

is dominated by

$$\begin{aligned} &C \sum_{x \in \Lambda} \sum_{\gamma \in K_x} \langle \delta_x, B_\gamma \chi_I(A_\omega^\Lambda) B_\gamma \delta_x \rangle \prod_{k=1}^{n} \big(\operatorname{Tr} \chi_I(A_\omega^\Lambda) - kd\big) \mathbf{1}_{E_n}(\omega) \\ &\quad \leq C \sum_{x \in \Lambda} \sum_{\gamma \in K_x} \langle \delta_x, B_\gamma \chi_I(A_\omega^\Lambda) B_\gamma \delta_x \rangle \prod_{k=0}^{n-1} \big(\operatorname{Tr} \chi_I(A^\Lambda_{(\omega_\gamma^\perp, \tau_\gamma)}) - kd\big) \mathbf{1}_{E_n'}(\omega_\gamma^\perp, \tau_\gamma), \end{aligned}$$

where we set $E_n' = \{(\omega_\gamma^\perp, \tau_\gamma) \mid \operatorname{Tr} \chi_I(A^\Lambda_{(\omega_\gamma^\perp, \tau_\gamma)}) \geq (n-1)d + 1\}$ and used

$$\mathbf{1}_{E_n}(\omega) \leq \mathbf{1}_{E_n'}(\omega_\gamma^\perp, \tau_\gamma) \tag{2.19}$$

for any $\tilde{\omega}_\gamma$ and ω. Take the expectation with respect to $\omega = \{\omega_\gamma\}_{\gamma \in \mathbf{Z}^d}$ and $\tilde{\omega} = \{\tilde{\omega}_\gamma\}_{\gamma \in \mathbf{Z}^d}$. From Proposition 2.1, it follows that the expectation of (2.18) is dominated by

$$\begin{aligned} &C Q_A(|I|) \sum_{x \in \Lambda} \sum_{\gamma \in K_x} \mathbf{E}_{\tilde{\omega}_\gamma} \mathbf{E}_{\omega_\gamma^\perp} \Big[\prod_{k=0}^{n-1} \big(\operatorname{Tr} \chi_I(A^\Lambda_{(\omega_\gamma^\perp, \tau_\gamma)}) - kd\big) \mathbf{1}_{E_{n-1}}(\omega_\gamma^\perp, \tau_\gamma)\Big] \\ &\quad \leq C Q_A(|I|) \sum_{x \in \Lambda} \sum_{\gamma \in K_x} C_n' \big(Q_A(|I|)|\Lambda|\big)^n, \end{aligned}$$

where $C_n' = C_n(a, d, M_0, 2M - m, m)$. Note that $2M - m \geq \sup_\gamma \{\omega_\gamma^\perp, \tau_\gamma\}$ and $m \leq \inf_\gamma \{\omega_\gamma^\perp, \tau_\gamma\}$. In the last line, we used the induction hypothesis (2.18) and the fact that E_n' corresponds to E_{n-1}. By $C(d+1)C_n' = C_{n+1}$, we complete the proof. □

2.3. Proof of Corollary 2.2

In this subsection, we prove that the multiplicity of eigenvalues for A_ω in some localization region is finite by using Klein and Molchanov's method [29].

Proof of Corollary 2.2
We prove only the case for $\alpha = 1$ (see [9] for the general case). The proof is based on two results (see [29, Lemmas 1, 2]). The first one is a probabilistic estimate given by Theorem 2.2. The other is a deterministic result.

We say that $\psi \in \ell^2(\mathbf{Z}^d)$ is β-decaying if $|\psi(x)| \le C_\psi(1+|x|)^{-\beta}$ for some $C_\psi > 0$. If $\alpha = 1$ and $\beta > 5d/2$, we show that, with probability one, A_ω cannot have an eigenvalue with $d+1$ linearly independent β-decaying eigenfunctions. Pick $q > 2d$. For a scale $L > 0$, we cover an open interval I by $2([L^q|I|/2]+1) \le L^q|I|+2$ intervals of length $2L^{-q}$ so that any subinterval $J \subset I$ with length $|J| \le L^{-q}$ is contained in one of these intervals. We consider the event $\mathcal{B}_{L,I,q}$, which occurs if there exists an interval $J \subset I$ with $|J| \le L^{-q}$ such that $\operatorname{Tr}\chi_J(A_\omega^\Lambda) \ge d+1$. By Theorem 2.2 with $n = 1$, its probability can be estimated by

$$\begin{aligned} \mathbf{P}(\mathcal{B}_{L,I,q}) &\le (L^q|I|+2)C_G\big(C(2L^{-q})L^d\big)^2 \\ &\le C'(|I|+1)L^{-q+2d}, \end{aligned} \tag{2.20}$$

where C_G is the constant given in (2.7) and C, C' are some constants independent of L. Take $L_k = 2^k$, and use (2.20) and the Borel-Cantelli lemma. It follows that if $q > 2d$, then for $\mathbf{P}$ almost every ω there exists $k(q,\omega) < \infty$ such that the event $\mathcal{B}_{L_k,I,q}$ does not occur if $k \ge k(q,\omega)$.

Now, suppose that for some ω there exists $E \in I$ which is an eigenvalue of A_ω with $d+1$ linearly independent β-decaying eigenfunctions. As in the proof of [29, Lemma 1], it follows that for L large enough A_ω^Λ has at least $d+1$ eigenvalues in the interval $J_{E,L} = [E-\epsilon_L, E+\epsilon_L]$, where $\epsilon_L = C_0L^{-\beta+d/2}$ for an appropriate constant C_0 independent of L. By $\beta > 5d/2$, we can pick q satisfying $2d < q < \beta - d/2$. Hence, we have $\epsilon_L \le L^{-q}$ for all large L. However, with probability one, this is not possible since the event $\mathcal{B}_{L_k,I,q}$ does not occur for large L_k. □

3. Random acoustic operator Γ_ω

3.1. Generalized eigenvalue-counting estimate for Γ_ω

In this section, we consider a simple acoustic model including (1.2). Our second model Γ_ω is the random self-adjoint operator defined by

$$\Gamma_\omega := \frac{1}{\sqrt{\rho_\omega}} H_0 \frac{1}{\sqrt{\rho_\omega}}$$

on $\ell^2(\mathbf{Z}^d)$, where H_0 is a nonnegative bounded self-adjoint operator. The multiplicative operator $1/\sqrt{\rho_\omega}$ is defined as in (2.2). We also take the random variables $\omega = \{\omega_\gamma\}_{\gamma\in\mathbf{Z}^d}$ as before. Since H_0 is any bounded operator, we may choose $H_0 = \nabla^*(1/b)\nabla$ as in (2.1). Of course, this model includes the case

$$H_0 = -\Delta \tag{3.1}$$

where $-\Delta$ is the discrete Laplacian defined by

$$(-\Delta f)(x) := \sum_{y:|x-y|=1} \big(f(x) - f(y)\big). \tag{3.2}$$

Faris [18]–[20] studied this random operator and Combes, Hislop, and Tip [13] studied the continuous version of Γ_ω.

We define the restriction of Γ_ω to the cube Λ by $\Gamma_\omega^\Lambda = \chi_\Lambda \Gamma_\omega \chi_\Lambda$. We also denote the IDS for Γ_ω by N_Γ as in the definition of N_A for A_ω. We first have the following Wegner estimate and its corollary. This is an extension of the result by Faris [18]–[20] to the case for general probability measures.

THEOREM 3.1

Fix $E_0 \in (0,\infty)$. Let I be any bounded interval in $[E_0,\infty)$. Then we have the bound

$$\mathbf{E}\,\mathrm{Tr}\,\chi_I(\Gamma_\omega^\Lambda) \le Q_\Gamma(|I|)|\Lambda|, \tag{3.3}$$

where $Q_\Gamma(|I|) = \sup_\gamma Q_{\Gamma,\gamma}(|I|)$ and

$$Q_{\Gamma,\gamma}(|I|) = \begin{cases} |g_\gamma|_\infty |I| M/E_0 & \text{if } \mu_\gamma \text{ has a bounded density } g_\gamma, \\ 4(1+M/E_0)S_{\mu_\gamma}(|I|) & \text{otherwise.} \end{cases} \tag{3.4}$$

COROLLARY 3.1

Assume (2.4), (2.9), and (3.1) for Γ_ω. Then the IDS for Γ_ω is α-Hölder continuous.

Next, we obtain the following generalized eigenvalue-counting estimate for Γ_ω.

THEOREM 3.2

Fix $E_0 \in (0,\infty)$. Let I be any bounded interval in $[E_0,\infty)$. For $n = 1,2,3,\dots,$ we have the bound

$$\mathbf{P}\big(\mathrm{Tr}\,\chi_I(\Gamma_\omega^\Lambda) \ge n\big) \le \frac{1}{n!}\big(Q_\Gamma(|I|)|\Lambda|\big)^n. \tag{3.5}$$

From Theorem 3.2, we have the following, which is proven as in Corollary 2.2.

COROLLARY 3.2

Assume (2.4), (2.9), (2.10), and (3.1) for Γ_ω. Then, with probability one, every eigenvalue of Γ_ω in I has multiplicity less than or equal to $[\alpha^{-1}]$. In particular, if $\alpha > 1/2$, with probability one, every eigenvalue of Γ_ω in I is simple.

REMARK 4

By using the multiscale analysis, Faris [18]–[20] showed the localization for Γ_ω near the top of the spectrum under some assumptions on the distribution of $\{\omega_\gamma\}_{\gamma\in\mathbf{Z}^d}$ (see also our example in Appendix C).

3.2. Auxiliary random Schrödinger operator

In this subsection, to prove Theorems 3.1 and 3.2, we introduce an auxiliary Schrödinger-type operator defined by $H_\omega(E) = H_0 - E\rho_\omega$. In fact, $H_\omega(E)$ is an Anderson model if (3.1) holds. The main statement is the following proposition.

PROPOSITION 3.1

Suppose $0 \notin I_\epsilon := [E-\epsilon, E+\epsilon]$. *Then we have*

$$\operatorname{Tr}\chi_{[-m\epsilon,m\epsilon]}\big(H_\omega^\Lambda(E)\big) \le \operatorname{Tr}\chi_{I_\epsilon}(\Gamma_\omega^\Lambda) \le \operatorname{Tr}\chi_{[-M\epsilon,M\epsilon]}\big(H_\omega^\Lambda(E)\big). \tag{3.6}$$

REMARK 5

Let N_H be the IDS for $H_\omega(E)$. Proposition 3.1 implies

$$N_H(m\epsilon)-N_H(-m\epsilon) \le N_\Gamma(E+\epsilon)-N_\Gamma(E-\epsilon) \le N_H(M\epsilon)-N_H(-M\epsilon), \tag{3.7}$$

which shows upper and lower bounds of the density of states for Γ_ω in terms of that for $H_\omega(E)$.

This idea was already used to study the IDS for continuous random Schrödinger operators. For example, Hislop and Klopp [26] used the Birman-Schwinger–type operator $\Xi_\omega(E)$ as an auxiliary operator. Let us consider $H_\omega := -\triangle + V_\omega$ on $L^2(\mathbf{R}^d)$, where $V_\omega(x) := \sum_{\gamma\in\mathbf{Z}^d}\omega_\gamma u_\gamma(x)$ is a standard Anderson-type random potential and the sign of single site potential $u_\gamma(x)$ is indefinite. For $E<0$, they used the following inequality:

$$\operatorname{Tr}\chi_{[E-\epsilon,E+\epsilon]}(H_\omega^\Lambda) \le \operatorname{Tr}\chi_{[-1-c\epsilon,-1+c\epsilon]}\big(\Xi_\omega^\Lambda(E)\big),$$

where c is some constant independent of Λ, ϵ, and ω. The Birman-Schwinger–type operator $\Xi_\omega(E)$ is defined by $\Xi_\omega(E) := (-\triangle - E)^{-1/2}V_\omega(-\triangle - E)^{-1/2}$. Thus, they estimated the expectation of the number of eigenvalues near -1 (for some application to random magnetic fields, see, e.g., [41, Lemma 2.1]).

Proof of Proposition 3.1

We prove only the second inequality. The same argument also applies to the first inequality.

To estimate the number of eigenvalues of Γ_ω^Λ in the interval I_ϵ, we use the Birman-Schwinger principle (see, e.g., [38]). Suppose that ϕ_λ is an eigenfunction of Γ_ω^Λ with the eigenvalue $\lambda\neq 0$ and $\lambda\in I_\epsilon$. Then the function ψ defined by $\psi = (1/\sqrt{\rho_\omega})\phi_\lambda$ satisfies $H_\omega^\Lambda(\lambda)\psi = 0$, where we set $H_\omega(\lambda) = H_0 - \lambda\rho_\omega$. Therefore, we know that $0\in\sigma_p\big(H_\omega^\Lambda(\lambda)\big)$ if and only if $\lambda\in\sigma_p(\Gamma_\omega^\Lambda)$. By the assumption of ρ_ω, we have $H_\omega^\Lambda(\lambda) - M\epsilon \le H_\omega^\Lambda(E) \le H_\omega^\Lambda(\lambda) + M\epsilon$. Let $\mu_n(\lambda)$ be the nth eigenvalue of $H_\omega^\Lambda(\lambda)$. Similarly, let $k_n(E)$ be the nth eigenvalue of $H_\omega^\Lambda(E)$. By using the min-max principle, we have

$$\mu_n(\lambda) - M\epsilon \le k_n(E) \le \mu_n(\lambda) + M\epsilon. \tag{3.8}$$

Now, take $\lambda_n\in I_\epsilon$ satisfying $\mu_n(\lambda_n) = 0$. By (3.8), we have $-M\epsilon \le k_n(E) \le M\epsilon$. Hence, we obtain

$$\#\big\{n \bigm| \mu_n(\lambda) = 0 \quad \text{for some } \lambda\in I_\epsilon\big\} \le \#\big\{n \bigm| k_n(E)\in[-M\epsilon, M\epsilon]\big\}. \qquad \square$$

As in Section 2, we first prove the Wegner estimate by using Proposition 3.1.

Proof of Theorem 3.1
We set $I = [E-\epsilon, E+\epsilon]$ and $\tilde{I} = [-M\epsilon, M\epsilon]$. From (3.6), it follows that $\mathbf{E}\,\mathrm{Tr}\,\chi_I(\Gamma_\omega^\Lambda)$ is dominated by

$$\mathbf{E}\,\mathrm{Tr}\,\chi_{\tilde{I}}\big(H_\omega^\Lambda(E)\big) = \sum_{\gamma\in\Lambda} \mathbf{E}\big\langle \delta_\gamma, \chi_{\tilde{I}}\big(H_\omega^\Lambda(E)\big)\delta_\gamma\big\rangle.$$

Now, use Proposition 2.1 with $B = E\Pi_\varphi$, $k = M$, and $\varphi = \delta_\gamma$; then we have the bound

$$\tag{3.9} \begin{aligned} &\int \big\langle \delta_\gamma, \chi_{\tilde{I}}\big(H_\omega^\Lambda(E)\big)\delta_\gamma\big\rangle \mu_\gamma\,(d\omega_\gamma) \\ &\quad\le \begin{cases} |g_\gamma|_\infty |\tilde{I}|/E & \text{if } \mu_\gamma \text{ has a bounded density } g_\gamma, \\ 4(1+M/E) S_{\mu_\gamma}(|\tilde{I}|/M) & \text{otherwise.} \end{cases} \end{aligned}$$

By (3.9) and $|\tilde{I}| = M|I|$, we complete the proof. □

We next prove the generalized eigenvalue-counting estimate for Γ_ω.

Proof of Theorem 3.2
We can prove Theorem 3.2 as we proved Theorem 2.2. Set $I = [E-\epsilon, E+\epsilon]$ and $\tilde{I} = [-M\epsilon, M\epsilon]$. We also set

$$E_n = \big\{\omega \,\big|\, \mathrm{Tr}\,\chi_I(\Gamma_\omega^\Lambda) \ge n\big\} \quad \text{and} \quad \tilde{E}_n = \big\{\omega \,\big|\, \mathrm{Tr}\,\chi_{\tilde{I}}\big(H_\omega^\Lambda(E)\big) \ge n\big\},$$

where we write $H_\omega(E) = H_0 - E\rho_\omega$ as in Proposition 3.1 and $H_\omega^\Lambda(E)$ is its restriction to Λ. Then, by Proposition 3.1, we have $E_n \subset \tilde{E}_n$. From Chebyshev's inequality and Proposition 3.1, it follows that $\mathbf{P}(\mathrm{Tr}\,\chi_I(\Gamma_\omega^\Lambda) \ge n)$ is dominated by

$$\tag{3.10} \frac{1}{n!}\mathbf{E}\Big[\prod_{k=0}^{n-1}\big(\mathrm{Tr}\,\chi_I(\Gamma_\omega^\Lambda) - k\big)\mathbf{1}_{E_n}\Big] \le \frac{1}{n!}\mathbf{E}\Big[\prod_{k=0}^{n-1}\big(\mathrm{Tr}\,\chi_{\tilde{I}}(H_\omega^\Lambda(E)) - k\big)\mathbf{1}_{\tilde{E}_n}\Big].$$

Then, since $H_\omega(E)$ is an Anderson model, the right-hand side of (3.10) can be estimated similarly as in [9] from Lemma 2.2 and the spectral averaging (3.9). We proceed by the induction on $n \in \mathbf{N}$ and complete the proof. □

3.3. Poisson statistics for eigenvalues of Γ_ω

As an application of Theorem 3.2, we state Poisson statistics for eigenvalues of Γ_ω^Λ. Following Molchanov [33] and Minami [34], we study a point process defined by the rescaled eigenvalues of Γ_ω^Λ.

We first introduce an important point process. A point process ξ_ω on $\mathbf{R}$ is said to be the Poisson point process with intensity measure ν if it satisfies the following two conditions.

(a) For each bounded Borel set $J \subset \mathbf{R}$, $\xi_\omega(J)$ obeys the Poisson distribution with parameter $\nu(J)$, namely,

$$\mathbf{P}\big(\xi_\omega(J) = r\big) = e^{-\nu(J)}\frac{\nu(J)^r}{r!}, \quad r \ge 0.$$

(b) If $J_1, \ldots, J_n$ are disjoint, then $\xi_\omega(J_1), \ldots, \xi_\omega(J_n)$ are independent random variables.

Now, let $E^\Lambda_{\omega,k}$ be kth eigenvalue of Γ^Λ_ω $(k = 1, 2, \ldots, |\Lambda|)$. For $E \in \sigma(\Gamma_\omega)$ and $E > 0$, we define a point process $\xi^\Lambda_{\omega,E}$ on $\mathbf{R}$ by the rescaled eigenvalue of Γ^Λ_ω near E:

$$\xi^\Lambda_{\omega,E}(dt) = \sum_{k=1}^{|\Lambda|} \delta_{|\Lambda|(E^\Lambda_{\omega,k} - E)}(dt). \tag{3.11}$$

We always assume (2.4) and (3.1); that is, the random variables $\{\omega_\gamma\}_{\gamma \in \mathbf{Z}^d}$ are independent and identically distributed (i.i.d.) by μ and $H_0 = -\Delta$. Moreover, we introduce the following assumptions

(H1) The common distribution μ has a bounded density g.

(H2) There are finite constants $C, D > 0$, $s \in (0,1)$, and $I \subset \mathbf{R}$ such that

$$\mathbf{E}\big|\langle \delta_x, (\Gamma^\Lambda_\omega - z)^{-1}\delta_y\rangle\big|^s \le C e^{-D|x-y|} \tag{3.12}$$

for all $z \in \mathbf{C}$ satisfying $\Re z \in I$, $\Im z \neq 0$ and for all cubes $\Lambda \subset \mathbf{Z}^d$; (3.12) holds also for Γ_ω.

(H3) The IDS N_Γ is differentiable at $E \in I$ with $n(E) := N'_\Gamma(E) > 0$.

By Minami's method in [34], we can show the following statement.

THEOREM 3.3

Assume (H1), (H2), and (H3). Then, the point process $\xi^\Lambda_{\omega,E}(dt)$ converges weakly, as $|\Lambda| \to \infty$, to a Poisson point process with intensity measure $n(E)\,dt$.

REMARK 6

The above point process $\xi^\Lambda_{\omega,E}$ converges weakly to a Poisson point process in the following sense. For any given disjoint bounded intervals $J_1, \ldots, J_m$ in $\mathbf{R}$, we have

$$\lim_{|\Lambda| \to \infty} \mathbf{P}\big(\xi^\Lambda_{\omega,E}(J_1) = r_1, \ldots, \xi^\Lambda_{\omega,E}(J_m) = r_m\big) = \prod_{s=1}^m e^{-n(E)|J_s|} \frac{(n(E)|J_s|)^{r_s}}{r_s!}$$

for all $r_1, \ldots, r_m \in \mathbf{N}$. For the reader's convenience, in Appendix C we present an example, and we prove that there exists an interval $I \subset \sigma(\Gamma_\omega)$ such that the condition (H2) is satisfied.

For the proof of Theorem 3.3, we use Minami's methods in [34] for our acoustic model Γ_ω. We proceed as in [32, Appendix A]; hence, we need to prove two propositions for Γ_ω.

We fix $\alpha \in (0,1)$ and divide $\Lambda = \Lambda_L(0)$ into n_L boxes $\Lambda_j = \Lambda_\ell(k_j)$ of side $\ell \sim L^\alpha$ centered at $k_j \in \mathbf{Z}^d$, that is, $\Lambda = \bigcup_{j=1}^{n_L} \Lambda_j$. Note that $n_L = |\Lambda|/|\Lambda_j| \sim L^{(1-\alpha)d}$.

For each j, we define point processes

$$\xi^{\Lambda_j}_{\omega,E}(dt) = \sum_{k=1}^{|\Lambda_j|} \delta_{|\Lambda_j|(E^{\Lambda_j}_{\omega,k}-E)}(dt),$$

where $E^{\Lambda_j}_{\omega,k}$ is the kth eigenvalue of $\Gamma^{\Lambda_j}_{\omega}$. Note that from our assumptions it follows that $\{\xi^{\Lambda_j}_{\omega,E}\}_{j=1}^{n_L}$ are i.i.d. point processes. We consider the point process defined by the superposition

$$\tilde{\xi}^{\Lambda}_{\omega,E}(dt) = \sum_{j=1}^{n_L} \xi^{\Lambda_j}_{\omega,E}(dt).$$

Moreover, we consider the random measure $\theta^{\Lambda}_{\omega,E}$ defined by

$$\theta^{\Lambda}_{\omega,E}(J) := \operatorname{Tr} \chi_\Lambda \chi_J\big(|\Lambda|(\Gamma_\omega - E)\big)\chi_\Lambda$$

for a Borel set $J \subset \mathbf{R}$. We first prepare the following proposition, which corresponds to [32, Proposition A.2] or [10, Lemma 6.1].

PROPOSITION 3.2

Assume (H1) and (H2). Then, for every $f \in \mathcal{C}_0^\infty(\mathbf{R}, dt)$, we have

$$\lim_{|\Lambda|\to\infty} \mathbf{E}\Big[\Big|\int f(t)\xi^{\Lambda}_{\omega,E}(dt) - \int f(t)\tilde{\xi}^{\Lambda}_{\omega,E}(dt)\Big|\Big] = 0 \tag{3.13}$$

and

$$\lim_{|\Lambda|\to\infty} \mathbf{E}\Big[\Big|\int f(t)\xi^{\Lambda}_{\omega,E}(dt) - \int f(t)\theta^{\Lambda}_{\omega,E}(dt)\Big|\Big] = 0. \tag{3.14}$$

To prove Proposition 3.2, we first recall a spectral averaging result for $\Gamma^{\Lambda}_{\omega}$. It is known that

$$\int \langle \delta_\gamma, \chi_J(\Gamma^{\Lambda}_{\omega})\delta_\gamma\rangle \frac{d\omega_\gamma}{\omega_\gamma} \le \int \chi_J(\lambda)\frac{d\lambda}{\lambda} \tag{3.15}$$

for a Borel set $J \subset (0,\infty)$, $\Lambda \subset \mathbf{Z}^d$, and $\gamma \in \Lambda$. We remark that (3.15) also gives another proof of Theorem 3.1 for the case when μ has bounded density. We also note that (3.15) holds for Γ_ω and $\gamma \in \mathbf{Z}^d$ (see [18], [20] for details). For every Borel set $J \subset \mathbf{R}$, it follows from (3.15) that

$$\mathbf{E}\xi^{\Lambda}_{\omega,E}(J) \le C|g|_\infty |J|.$$

Similarly, we have

$$\mathbf{E}\tilde{\xi}^{\Lambda}_{\omega,E}(J) \le C|g|_\infty |J| \qquad \text{and} \qquad \mathbf{E}\theta^{\Lambda}_{\omega,E}(J) \le C|g|_\infty |J|.$$

We next recall the Helffer-Sjöstrand formula (see, e.g., [27, Appendix B] for details). For $f \in \mathcal{C}^\infty(\mathbf{R})$ and $n \in \mathbf{N}$, we set

$$\{\{f\}\}_n := \sum_{r=0}^{n} \int_{\mathbf{R}} du |f^{(r)}(u)|(1+|u|^2)^{(r-1)/2}.$$

If $\{\{f\}\}_n < \infty$ for some $n \geq 2$, then for any self-adjoint operator T we have

$$f(T) = \int_{\mathbf{R}^2} d\tilde{f}(z)(T-z)^{-1}, \tag{3.16}$$

where $z = x + iy$, $\tilde{f}(z)$ is an almost analytic extension of f to the complex plane in the sense that it satisfies $\partial_{\bar{z}}\tilde{f}(z) = 0$ for $z \in \mathbf{R}$, and $d\tilde{f}(z) := (1/2\pi)\partial_{\bar{z}}\tilde{f}(z)\,dx\,dy$ with $\partial_{\bar{z}} := \partial_x + i\partial_y$. Moreover, for all $p \geq 0$, we have

$$\int_{\mathbf{R}^2} |d\tilde{f}(z)||\Im z|^{-p} \leq c_p\big\{\{f\}\big\}_n < \infty \tag{3.17}$$

for $n \geq p+1$ with a constant c_p, where $|d\tilde{f}(z)| := (1/2\pi)|\partial_{\bar{z}}\tilde{f}(z)|\,dx\,dy$.

Proof of Proposition 3.2
We prove only (3.13) since (3.14) will be proven similarly. As in [32], let $v_L = \beta \log L$, where $\beta > 0$ is a fixed, big enough constant to be taken later. We set $\mathrm{int}(\Lambda_j) := \{x \in \Lambda_j \mid \mathrm{dist}(x, \partial\Lambda_j) \geq v_L\}$ and $\mathrm{wall}(\Lambda_j) := \Lambda_j \setminus \mathrm{int}(\Lambda_j)$. Then, we have

$$\mathbf{E}\Big[\Big|\int f(t)\,d\xi_\omega^{\Lambda,E}(t) - \int f(t)\,d\tilde{\xi}_\omega^{\Lambda,E}(t)\Big|\Big] \leq \mathrm{I}_1 + \mathrm{I}_2,$$

where

$$\mathrm{I}_1 = \sum_{j=1}^{n_L}\sum_{x\in \mathrm{wall}(\Lambda_j)} \mathbf{E}\big[\big|\big\langle \delta_x, f\big(|\Lambda|(\Gamma_\omega^\Lambda - E)\big)\delta_x\big\rangle - \big\langle \delta_x, f\big(|\Lambda|(\Gamma_\omega^{\Lambda_j} - E)\big)\delta_x\big\rangle\big|\big]$$

and

$$\mathrm{I}_2 = \sum_{j=1}^{n_L}\sum_{x\in \mathrm{int}(\Lambda_j)} \mathbf{E}\big[\big|\big\langle \delta_x, f\big(|\Lambda|(\Gamma_\omega^\Lambda - E)\big)\delta_x\big\rangle - \big\langle \delta_x, f\big(|\Lambda|(\Gamma_\omega^{\Lambda_j} - E)\big)\delta_x\big\rangle\big|\big].$$

To estimate I_1, we use (3.15). Then, this yields that there is a constant $C > 0$ independent of L such that $\mathrm{I}_1 \leq C|\Lambda|^{-1}\sum_{j=1}^{n_L}|\mathrm{wall}(\Lambda_j)|$, and this converges to zero as $L \to \infty$.

To estimate I_2, we proceed as in [10]. By (3.16), I_2 is dominated by

$$\sum_{j=1}^{n_L}\sum_{x\in \mathrm{int}(\Lambda_j)} |\Lambda|^{-1}\int_{\mathbf{R}^2} |d\tilde{f}(z)|\mathbf{E}\big[\big|\langle \delta_x, (\Gamma_\omega^\Lambda - z_L)^{-1}\delta_x\rangle - \langle \delta_x, (\Gamma_\omega^{\Lambda_j} - z_L)^{-1}\delta_x\rangle\big|\big],$$

where we set $z_L = E + z/|\Lambda|$. We recall the geometric resolvent identity for Γ_ω:

$$\begin{aligned}&\langle \delta_x, (\Gamma_\omega^\Lambda - z_L)^{-1}\delta_x\rangle - \langle \delta_x, (\Gamma_\omega^{\Lambda_j} - z_L)^{-1}\delta_x\rangle \\ &\quad = \sum_{(y,y')} \frac{1}{\sqrt{\omega_y\omega_{y'}}}\langle \delta_x, (\Gamma_\omega^\Lambda - z_L)^{-1}\delta_y\rangle\langle \delta_{y'}, (\Gamma_\omega^{\Lambda_j} - z_L)^{-1}\delta_x\rangle,\end{aligned} \tag{3.18}$$

where the sum is over all pairs (y, y') with $y \in \Lambda \setminus \Lambda_j$, $y' \in \Lambda_j$, and $|y - y'| = 1$. From (3.18), $\omega_\gamma \geq m$, and the Hölder inequality, it follows that for $s \in (0,1)$, I_2

is dominated by

$$(3.19)\qquad \begin{aligned} &m^{-1}|\Lambda|^{-1}\sum_{j=1}^{n_L}\sum_{x\in\mathrm{int}(\Lambda_j)}\sum_{(y,y')}\int_{\mathbf{R}^2}|d\tilde{f}(z)||\Im z_L|^{-2+s} \\ &\quad\times\big(\mathbf{E}\big|\langle\delta_x,(\Gamma_\omega^\Lambda-z_L)^{-1}\delta_y\rangle\big|^s\big)^{1/2}\big(\mathbf{E}\big|\langle\delta_{y'},(\Gamma_\omega^{\Lambda_j}-z_L)^{-1}\delta_x\rangle\big|^s\big)^{1/2}, \end{aligned}$$

where we use the bound $|\langle\delta_x,(\Gamma_\omega^\Lambda-z_L)^{-1}\delta_y\rangle\langle\delta_{y'},(\Gamma_\omega^{\Lambda_j}-z_L)^{-1}\delta_x\rangle|\le|\Im z_L|^{-2}$. Note that $|\Im z_L|^{-1}=|\Lambda||\Im z|^{-1}$ and that there are $O(L^{\alpha(d-1)})$ pairs (y,y') for each Λ_j. Then, by (H2) and (3.17), we have $\mathrm{I}_2\le O(L^{d(2-s)+\alpha(d-1)}e^{-Dv_L})$. Hence, if we choose $v_L=\beta\log L$ and $\beta>D^{-1}(d(2-s)+\alpha(d-1))$, I_2 converges to zero as $L\to\infty$. □

Because of Proposition 3.2, it suffices to prove the following proposition to complete the proof of Theorem 3.3. This corresponds to [32, Proposition A.3].

PROPOSITION 3.3
Assume (H1), (H2), and (H3). Then, the point process $\tilde{\xi}_{\omega,E}^\Lambda(dt)$ converges weakly, as $|\Lambda|\to\infty$, to a Poisson point process with intensity measure $n(E)\,dt$.

Proof
By standard results from the theory of point processes (see, e.g., [14, Theorem 9.2.V], [15, Theorem 11.2.V], [32, Theorem 2.3]), the weak convergence of $\tilde{\xi}_{\omega,E}^\Lambda$ to the Poisson point process is equivalent to verifying the following three conditions for all bounded intervals I:

$$(3.20)\qquad \lim_{L\to\infty}\max_{j=1,\dots,n_L}\mathbf{P}\big(\xi_{\omega,E}^{\Lambda_j}(I)\ge 1\big)=0,$$

$$(3.21)\qquad \lim_{L\to\infty}\sum_{j=1}^{n_L}\mathbf{P}\big(\xi_{\omega,E}^{\Lambda_j}(I)\ge 1\big)=n(E)|I|,$$

$$(3.22)\qquad \lim_{L\to\infty}\sum_{j=1}^{n_L}\mathbf{P}\big(\xi_{\omega,E}^{\Lambda_j}(I)\ge 2\big)=0.$$

Since $\xi_{\omega,E}^{\Lambda_j}(I)=\mathrm{Tr}\,\chi_{E+I/|\Lambda|}(\Gamma_\omega^{\Lambda_j})$, (3.20) follows immediately from Theorem 3.1. In addition, from Theorem 3.2 we have $\sum_{j=1}^{n_L}\mathbf{P}(\xi_{\omega,E}^{\Lambda_j}(I)\ge 2)\le C|I|^2 n_L^{-1}$; hence, (3.22) follows. Thus, the proof is finished if we verify the condition (3.21). By using Theorem 3.2 with $n=2$, we first have

$$\lim_{L\to\infty}\sum_{j=1}^{n_L}\mathbf{P}\big(\xi_{\omega,E}^{\Lambda_j}(I)\ge 1\big)=\lim_{L\to\infty}\mathbf{E}[\tilde{\xi}_{\omega,E}^\Lambda(I)].$$

By (3.14) in Proposition 3.2, we next have

$$\lim_{L\to\infty}\mathbf{E}[\tilde{\xi}_{\omega,E}^\Lambda(I)]=\lim_{L\to\infty}\mathbf{E}[\theta_{\omega,E}^\Lambda(I)]=\lim_{L\to\infty}|\Lambda|\int_{E+I/|\Lambda|}n(\lambda)\,d\lambda.$$

Finally, by (H3) and Lebesgue differentiation theorem, we complete the proof (see [10], [32], [34] for details). □

Appendix A: Simple proof of Lifshitz tails for Γ_ω

In this appendix, as an application of Proposition 3.1, we discuss an asymptotic behavior of the IDS for Γ_ω. We always assume (3.1), that is, $H_0 = -\Delta$. To guarantee the existence of the IDS, we also assume (2.4), that is, that $\{\omega_\gamma\}_{\gamma\in\mathbf{Z}^d}$ are i.i.d. by μ. Recall $\operatorname{supp}\mu \subset [m, M]$ and $m, M \in \operatorname{supp}\mu$. Then we have $\sigma(\Gamma_\omega) = [0, E_\infty]$ almost surely, where $E_\infty = 4d/m$. Our aim is to show

$$1 - N_\Gamma(E) = c_1 \exp\bigl(-(c_2 + o(1))(E_\infty - E)^{-d/2}\bigr)$$

as $E \uparrow E_\infty$, which is known as Lifshitz tails. See [8], [28], [37] for the Anderson model. Note that the internal Lifshitz tails (see, e.g., [30]) for the continuous version of A_ω were studied by Najar [35], [36].

THEOREM A.1

The IDS N_Γ satisfies

$$\limsup_{\epsilon\downarrow 0} \frac{\log\bigl(-\log(1 - N_\Gamma(E_\infty - \epsilon))\bigr)}{\log\epsilon} \le -\frac{d}{2}. \tag{A.1}$$

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Moreover, if we assume that $\mu([m, m+\kappa]) \ge c\kappa^\delta$ for some $c, \delta > 0$ and sufficiently small $\kappa > 0$, then we can use the lower bound in Proposition 3.1 to get

$$\liminf_{\epsilon\downarrow 0} \frac{\log\bigl(-\log(1 - N_\Gamma(E_\infty - \epsilon))\bigr)}{\log\epsilon} \ge -\frac{d}{2}. \tag{A.2}$$

Hence, by (A.1) and (A.2), we can determine Lifshitz exponent

$$\lim_{\epsilon\downarrow 0} \frac{\log\bigl(-\log(1 - N_\Gamma(E_\infty - \epsilon))\bigr)}{\log\epsilon} = -\frac{d}{2}.$$

Proof of Theorem A.1

Choose $E = E_\infty$ in (3.7). Then we have

$$N_H(m\epsilon) - N_H(-m\epsilon) \le N_\Gamma(E_\infty + \epsilon) - N_\Gamma(E_\infty - \epsilon) \le N_H(M\epsilon) - N_H(-M\epsilon),$$

where $H_\omega(E_\infty) = -\Delta - E_\infty\rho_\omega$ and N_H is the corresponding IDS. This auxiliary Schrödinger operator $H_\omega(E_\infty)$ is a nonpositive, bounded, and $\mathbf{Z}^d$-ergodic self-adjoint operator, and we know that $\sigma(H_\omega(E_\infty)) = [-E_\infty M, 0]$ almost surely. By $N_\Gamma(E_\infty + \epsilon) = N_H(m\epsilon) = N_H(M\epsilon) = 1$, we get

$$1 - N_H(-m\epsilon) \le 1 - N_\Gamma(E_\infty - \epsilon) \le 1 - N_H(-M\epsilon).$$

This implies

$$\limsup_{\epsilon\downarrow 0} \frac{\log\bigl(-\log(1 - N_\Gamma(E_\infty - \epsilon))\bigr)}{\log\epsilon} \tag{A.3}$$

$$= \limsup_{\epsilon \downarrow 0} \frac{\log\bigl(-\log(1 - N_H(-\epsilon))\bigr)}{\log \epsilon}.$$

Since $H_\omega(E_\infty)$ is an Anderson model, the right-hand side of (A.3) can be estimated by standard arguments (see, e.g., [8, Theorem VI.2.7], [28]). □

Appendix B: Relation between A_ω and Γ_ω

In this appendix, let us consider the simplest 1-dimensional model. Our aim is to compare $A_\omega := \partial^*(1/\rho_\omega)\partial$ with $\Gamma_\omega := (1/\sqrt{\rho_\omega})(\partial^*\partial)(1/\sqrt{\rho_\omega})$; $1/\rho_\omega$ and $1/\sqrt{\rho_\omega}$ are defined as before. For the sake of simplicity, we assume (2.4), that is, that $\{\omega_\gamma\}_{\gamma \in \mathbf{Z}}$ are i.i.d. by μ. Hence, we know that $\sigma(A_\omega) = \sigma(\Gamma_\omega) = [0, E_\infty]$ almost surely, where $E_\infty = 4/m$.

In the 1-dimensional case, each operator is represented by a matrix: $A_\omega = T^*T$ and $\Gamma_\omega = TT^*$. The matrix T is defined by $T = (1/\sqrt{\rho_\omega})\partial$, where

$$\frac{1}{\sqrt{\rho_\omega}} = \begin{pmatrix} \ddots & & & 0 \\ & 1/\sqrt{\omega_0} & & \\ & & 1/\sqrt{\omega_1} & \\ 0 & & & \ddots \end{pmatrix} \quad \text{and} \quad \partial = \begin{pmatrix} \ddots & & & 0 \\ \ddots & 1 & & \\ & -1 & 1 & \\ & & -1 & \ddots \\ 0 & & & \ddots \end{pmatrix}.$$

We note that the 1-dimensional discrete Laplacian (3.2) satisfies $-\Delta = \partial^*\partial = \partial\partial^*$. For $L \in \mathbf{N}$, set $\Lambda := \{k \in \mathbf{Z} \,|\, |k| \le L\} \subset \mathbf{Z}$. We first define the $((2L+1) \times (2L+1))$-matrix by $T^\Lambda = (1/\sqrt{\rho_\omega^\Lambda})\partial^\Lambda$, where

$$\frac{1}{\sqrt{\rho_\omega^\Lambda}} = \begin{pmatrix} 1/\sqrt{\omega_{-L}} & & 0 \\ & \ddots & \\ 0 & & 1/\sqrt{\omega_L} \end{pmatrix} \quad \text{and} \quad \partial^\Lambda = \begin{pmatrix} 1 & & & -1 \\ -1 & \ddots & 0 & \\ & \ddots & \ddots & \\ 0 & & -1 & 1 \end{pmatrix}.$$

We imposed the periodic boundary condition since we need to guarantee $\partial^\Lambda(\partial^\Lambda)^* = (\partial^\Lambda)^*\partial^\Lambda = -\Delta^\Lambda$, where $-\Delta^\Lambda$ is the discrete Laplacian (3.2) with the periodic boundary condition, that is,

$$-\Delta^\Lambda = \begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & 0 & \\ & \ddots & \ddots & \ddots & \\ & 0 & -1 & 2 & -1 \\ -1 & & & -1 & 2 \end{pmatrix}.$$

Now, we quote a lemma in [16, Theorem 2].

LEMMA B.1

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two separable Hilbert spaces. Let T be a densely defined closed operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. Then, T^*T on $\mathcal{H}_1$ and TT^* on $\mathcal{H}_2$ are nonnegative *self-adjoint operators. These satisfy*

$$\sigma_p(T^*T) \setminus \{0\} = \sigma_p(TT^*) \setminus \{0\}.$$

Moreover, for each $\lambda \in \sigma_p(T^*T) \setminus \{0\}$*, it holds that*

$$\dim \ker(T^*T - \lambda) = \dim \ker(TT^* - \lambda).$$

We first have the following statements.

THEOREM B.1

Fix $E_0 \in (0, \infty)$*. Let* I *be any bounded interval in* $[E_0, \infty)$*. Then we have*

$$\mathbf{E}\,\mathrm{Tr}\,\chi_I(A_\omega^\Lambda) = \mathbf{E}\,\mathrm{Tr}\,\chi_I(\Gamma_\omega^\Lambda) \le Q(|I|)|\Lambda|$$

and, for all $n \in \mathbf{N}$*,*

$$\mathbf{P}\big(\mathrm{Tr}\,\chi_I(A_\omega^\Lambda) \ge n\big) = \mathbf{P}\big(\mathrm{Tr}\,\chi_I(\Gamma_\omega^\Lambda) \ge n\big) \le \frac{1}{n!}\big(Q(|I|)|\Lambda|\big)^n,$$

where

$$Q(|I|) = \begin{cases} |g|_\infty |I| M/E_0 & \textit{if } \mu \textit{ has a bounded density } g, \\ 4(1 + M/E_0) S_\mu(|I|) & \textit{otherwise.} \end{cases}$$

Proof

Suppose $0 \notin I$, and use Lemma B.1. Then we have

$$\mathrm{Tr}\,\chi_I(A_\omega^\Lambda) = \mathrm{Tr}\,\chi_I(\Gamma_\omega^\Lambda), \tag{B.1}$$

where $A_\omega^\Lambda = (T^\Lambda)^* T^\Lambda$ and $\Gamma_\omega^\Lambda = T^\Lambda (T^\Lambda)^*$. Thus, because of (B.1), as in the proof of Theorem 3.1 and Theorem 3.2, we can analyze Γ_ω^Λ. □

We next have the following Lifshitz tail for A_ω.

THEOREM B.2

The IDS N_A *in the* 1*-dimensional case satisfies*

$$\limsup_{\epsilon \downarrow 0} \frac{\log\big(-\log(1 - N_A(E_\infty - \epsilon))\big)}{\log \epsilon} \le -\frac{1}{2}.$$

Proof of Theorem B.2

From (B.1), it follows that

$$N_A(E + \epsilon) - N_A(E - \epsilon) = N_\Gamma(E + \epsilon) - N_\Gamma(E - \epsilon).$$

Therefore, we choose $E = E_\infty$, and we obtain

$$\frac{\log\big(-\log(1 - N_A(E_\infty - \epsilon))\big)}{\log \epsilon} = \frac{\log\big(-\log(1 - N_\Gamma(E_\infty - \epsilon))\big)}{\log \epsilon}. \tag{B.2}$$

Use Theorem A.1 and (B.2) to complete the proof. □

For the multidimensional case, it seems to be more difficult to apply the same approach via Lemma B.1 since we need to study a random operator on the space of vector-valued functions. Recall that the Maxwell operator M_ω is a random operator defined on $\ell^2(\mathbf{Z}^3;\mathbf{C}^3)$. As $T=(1/\sqrt{\rho_\omega})\nabla$ is the operator from $\ell^2(\mathbf{Z}^d;\mathbf{C})$ to $\ell^2(\mathbf{Z}^d;\mathbf{C}^d)$, then we will have to analyze the following random operator: $TT^*=(1/\sqrt{\tilde\rho_\omega})\tilde H_0(1/\sqrt{\tilde\rho_\omega})$, where

$$\tilde H_0=\nabla\nabla^*=\begin{pmatrix}\partial_1\partial_1^* & \partial_1\partial_2^* & \dots & \partial_1\partial_d^*\\ \partial_2\partial_1^* & \partial_2\partial_2^* & \dots & \partial_2\partial_d^*\\ & & \ddots & \\ \partial_d\partial_1^* & \partial_d\partial_2^* & \dots & \partial_d\partial_d^*\end{pmatrix}$$

and $1/\sqrt{\tilde\rho_\omega}:=\sum_\gamma(1/\sqrt{\omega_\gamma})\tilde\Pi_\gamma$, where $\tilde\Pi_\gamma=\sum_{j=1}^d|\delta_\gamma^j\rangle\langle\delta_\gamma^j|$ for the standard orthonormal basis $\{\delta_\gamma^j\}_{\gamma\in\mathbf{Z}^d}^{j=1,\dots,d}$ of $\ell^2(\mathbf{Z}^d;\mathbf{C}^d)$. Hence, if we proceed as in Proposition 3.1, we need to study the random operator $\tilde H_\omega(E)=\tilde H_0-E\tilde\rho_\omega$ on $\ell^2(\mathbf{Z}^d;\mathbf{C}^d)$. If $d\ge 2$, it is not clear how the IDS for $\tilde H_\omega(E)$ behaves near the edge of its spectrum. Note that $\tilde H_0$ is not elliptic if $d\ge 2$. However, our techniques in this paper work for $\tilde H_\omega^\Lambda(E):=\tilde\chi_\Lambda\tilde H_\omega(E)\tilde\chi_\Lambda$, where $\tilde\chi_\Lambda:=\sum_{\gamma\in\Lambda}\tilde\Pi_\gamma$, since the rank of $\tilde\Pi_\gamma$ is just d and we can use Proposition 2.1. Therefore, we can obtain the eigenvalue-counting estimates for $A_\omega^\Lambda:=\nabla^*(1/\sqrt{\rho_\omega})\chi_\Lambda(1/\sqrt{\rho_\omega})\nabla$ corresponding to $\tilde H_\omega^\Lambda(E)$.

Appendix C: Fractional moment bound and Localization

In this appendix, we check the condition (H2) in Theorem 3.3. Let us consider (1.2) with (3.1), that is, Γ_ω with $H_0=-\Delta$. We assume that (2.4) and its common probability measure μ are distributed uniformly on $[m,m+1]$ for some small parameter $m>0$. We know that $\sigma(\Gamma_\omega)=[0,E_\infty]$ almost surely, where $E_\infty=4d/m$. Since we proved the Wegner estimate (3.3) and the Lifshitz tail (A.1) near $E_\infty=4d/m$, we can show that Anderson localization occurs near E_∞ by using the multiscale analysis as in [18]–[20]. However, another method is well known. This is called fractional moment analysis, which was started by Aizenman and Molchanov [1], [3]. For continuous models, refer to [2] and [7]. By this approach, we will show the following exponential decay estimate for the fractional moment of the Green's function $G:=(\Gamma_\omega-z)^{-1}(x,y)$. The first key idea is to look at powers of $|G|^s$ with $0<s<1$. For the sake of simplicity, we take $s=1/2$. We proceed as in [38, Chapter 13] (see also [17, Section 5.1.2]).

THEOREM C.1
There exist finite constants $c,C,D>0$ such that

$$\mathbf{E}|\langle\delta_x,(\Gamma_\omega^\Lambda-z)^{-1}\delta_y\rangle|^{1/2}\le Ce^{-D|x-y|}\tag{C.1}$$

for all $z\in\mathbf{C}$ satisfying $|z|\ge 16d^2/c^2$ and $\Im z\ne 0$, for all cube $\Lambda\subset\mathbf{Z}^d$, and for all $x,y\in\Lambda$. Note that c is a universal constant, and C,D depend only on d,m.

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(C.1) holds also for Γ_ω.

Proof of Theorem C.1
By

$$\langle \delta_x, (\Gamma_\omega^\Lambda - z)^{-1}\delta_y\rangle = \sqrt{\omega_x\omega_y}\langle \delta_x, (-\Delta - z\rho_\omega^\Lambda)^{-1}\delta_y\rangle, \tag{C.2}$$

it is enough to study $\langle \delta_x, (-\Delta - z\rho_\omega^\Lambda)^{-1}\delta_y\rangle$. Since we can write $-\Delta = 2dI - h_0$, where I is the identity operator and h_0 is defined by $(h_0 f)(i) = \sum_{|j|=1} f(i+j)$, then we get

$$\langle \delta_x, \delta_y\rangle = (2d - z\omega_x)\langle \delta_x, (-\Delta - z\rho_\omega^\Lambda)^{-1}\delta_y\rangle - \langle h_0\delta_x, (-\Delta - z\rho_\omega^\Lambda)^{-1}\delta_y\rangle. \tag{C.3}$$

By the resolvent identity, there are $\alpha, \beta \in \mathbf{C}$, which are independent of ω_γ, such that

$$\langle \delta_x, (-\Delta - z\rho_\omega^\Lambda)^{-1}\delta_y\rangle = \frac{\alpha}{\beta - z\omega_x}. \tag{C.4}$$

By the decoupling lemma (see [38, Lemma 13.3], [17, Lemma 5.1.14]), there exists a universal constant $c > 0$ such that

$$\begin{aligned} &\int_m^{m+1} \left|(2d - z\omega_x)\frac{\alpha}{\beta - z\omega_x}\right|^{1/2} d\omega_x \\ &\qquad \ge c|z|^{1/2} \int_m^{m+1} \left|\frac{\alpha}{\beta - z\omega_x}\right|^{1/2} d\omega_x. \end{aligned} \tag{C.5}$$

Note that $(a+b)^s \le a^s + b^s$ if $s \in (0,1)$. By (C.3), (C.4), and (C.5), we get

$$\begin{aligned} &\mathbf{E}\left|\langle \delta_x, (-\Delta - z\rho_\omega^\Lambda)^{-1}\delta_y\rangle\right|^{1/2} \\ &\qquad \le (c|z|^{1/2})^{-1}\Big(\langle \delta_x, \delta_y\rangle + \sum_{|j|=1} \mathbf{E}\left|\langle \delta_{x+j}, (-\Delta - z\rho_\omega^\Lambda)^{-1}\delta_y\rangle\right|^{1/2}\Big). \end{aligned} \tag{C.6}$$

Now, we set $\delta_y(x) := \langle \delta_x, \delta_y\rangle$ and $\eta(x) := \mathbf{E}|\langle \delta_x, (-\Delta - z\rho_\omega^\Lambda)^{-1}\delta_y\rangle|^{1/2}$. By (C.6), we have

$$\big(1 - (c|z|^{1/2})^{-1}h_0\big)\eta(x) \le (c|z|^{1/2})^{-1}\delta_y(x).$$

Note that $0 < 2d(c|z|^{1/2})^{-1} < 1$. Since $(1 - (c|z|^{1/2})^{-1}h_0)^{-1}$ is positivity preserving (see [38, Lemma 13.4]), thus, we get

$$\begin{aligned} \eta(x) &\le (c|z|^{1/2})^{-1}\big(1 - (c|z|^{1/2})^{-1}h_0\big)^{-1}\delta_y(x) \\ &\le (c|z|^{1/2} - 2d)^{-1}\big(2d(c|z|^{1/2})^{-1}\big)^{|x-y|}. \end{aligned}$$

By (C.2) and $\omega_\gamma \le m+1$, we obtain

$$\mathbf{E}\left|\langle \delta_x, (\Gamma_\omega^\Lambda - z)^{-1}\delta_y\rangle\right|^{1/2} \le Ce^{-D|x-y|}. \qquad \square$$

Finally, let us state a relation between the localization and Theorem C.1.

THEOREM C.2

Assume that the disorder parameter m satisfies $0 < m < c^2/4d$. Then Γ_ω has only pure point spectrum in $[16d^2/c^2, E_\infty]$ almost surely.

Proof

We set $\langle \delta_0, (\Gamma_\omega - E - i0)^{-2}\delta_0\rangle := \lim_{\epsilon\searrow 0}\langle \delta_0, (\Gamma_\omega - E - i\epsilon)^{-2}\delta_0\rangle$. Theorem 3 in [18], which is called the Simon-Wolff criterion (see [39]), implies that if, with probability one,

$$\langle \delta_0, (\Gamma_\omega - E - i0)^{-2}\delta_0\rangle < \infty \tag{C.7}$$

for almost every $E \in I$, then, with probability one, Γ_ω has only pure point spectrum in I. By using (C.1), we can check (C.7). Set $I = [16d^2/c^2, E_\infty]$. For $z = E + i\epsilon$ with $E \in I$ and $\epsilon > 0$, we get

$$\begin{aligned} &\mathbf{E}\big|\langle \delta_0, (\Gamma_\omega - E - i\epsilon)^{-2}\delta_0\rangle\big|^{1/4} \\ &\quad \le \sum_{x\in\mathbf{Z}^d} \mathbf{E}\big[|\langle \delta_0, (\Gamma_\omega - E - i\epsilon)^{-1}\delta_x\rangle|^{1/4}|\langle \delta_x, (\Gamma_\omega - E - i\epsilon)^{-1}\delta_0\rangle|^{1/4}\big]. \end{aligned} \tag{C.8}$$

By using the Hölder inequality, (C.8) is dominated by

$$\sum_{x\in\mathbf{Z}^d} \big(\mathbf{E}|\langle \delta_0, (\Gamma_\omega - E - i\epsilon)^{-1}\delta_x\rangle|^{1/2}\big)^{1/2}\big(\mathbf{E}|\langle \delta_x, (\Gamma_\omega - E - i\epsilon)^{-1}\delta_0\rangle|^{1/2}\big)^{1/2}.$$

By (C.1) for Γ_ω, we know that $\mathbf{E}|\langle \delta_0, (\Gamma_\omega - E - i\epsilon)^{-2}\delta_0\rangle|^{1/4}$ is bounded. This completes the proof. □

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