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ABSTRACT HARMONIC ANALYSIS OF WAVE-PACKET TRANSFORMS OVER LOCALLY COMPACT ABELIAN GROUPS

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Abstract. This article presents a systematic study for abstract harmonic analysis aspects of wave-packet transforms over locally compact abelian (LCA) groups. Let H be a locally compact group, let K be an LCA group, and let $\theta : H \to \mathrm{Aut}(K)$ be a continuous homomorphism. We introduce the abstract notion of the wave-packet group generated by θ, and we study basic properties of wave-packet groups. Then we study theoretical aspects of wave-packet transforms. Finally, we will illustrate application of these techniques in the case of some well-known examples.

1. Introduction

The abstract theory of coherent states and covariant transforms is the mathematical basis of modern high frequency approximation techniques and time-frequency (resp., time-scale) analysis (see [1], [22], [26], and references therein). Over the last decades, abstract and computational aspects of coherent state and covariant transforms have achieved significant popularity in coherent-state analysis and mathematical physics (see [23]–[25], and references therein). Coherent-state transforms are obtained by a given coherent-function system. Then admissibility conditions on the coherent system imply the analyzing of functions with respect to the system by the inner-product evaluation (see [7], [8], [16]). From aspects of functional analysis, coherent structures are classically originated from representation theory of locally compact groups (see [1], [10], and references

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therein). Commonly used coherent-state transforms in mathematical analysis are *wavelet transforms* (see [4], [5], [20]), *Gabor transforms* (see [15], [27]), and *wave-packet transforms* (see [6], [12]–[14], [19], [29], [28]).

The theory of Gabor analysis is based on the coherent state generated by modulations and translations of a given window function. Theoretical aspects of Gabor analysis over *locally compact abelian* (LCA) groups are studied in depth in [15]. Wavelet analysis is a time-scale analysis which is based on the continuous affine group as the group of dilations and translations. Abstract harmonic analysis extensions of wavelet analysis are studied in [2], [3], [10], [11], and [21].

The present article focuses on abstract aspects of wave-packet transforms over LCA groups. Our aim is to further develop the concept of wave-packet transforms over LCA groups, which has not been studied as extensively as the abstract Gabor (resp., wavelet) transforms on LCA groups. We also address analytic aspects of wave-packet transforms over LCA groups as coherent-state transforms, using tools from abstract harmonic analysis and representation theory.

This article contains five sections. Section 2 is devoted to fixing notation and a brief summary of Fourier analysis on LCA groups and classical harmonic analysis on locally compact semidirect product groups. In Section 3 we assume that H is a locally compact group, K is an LCA group, and $\theta : H \to \mathrm{Aut}(K)$ is a continuous homomorphism. Then we present the abstract notion of the wave-packet group generated by θ. We also show that the group structure of the wave-packet group canonically determines a (unitary) projective-group representation of the wave-packet group which is called *wave-packet representation*. Then we introduce the abstract notion of wave-packet transforms over LCA groups, and we study basic properties of this transform from harmonic analysis aspects. It is also shown that for all nonzero window functions satisfying an admissibility condition, we can continuously reconstruct any L^2-function from wave-packet coefficients. Finally, we will illustrate application of these techniques in the case of some well-known examples.

2. Preliminaries and notation

Let R and S be locally compact groups with identity elements e_R and e_S, respectively. Let m_R (resp., n_R) be a left (resp., right) Haar measure of R and simultaneously m_S (resp., n_S) a left (resp., right) Haar measure of S. Let $\tau : R \to \mathrm{Aut}(S)$ be a homomorphism, where $\mathrm{Aut}(S)$ denotes the automorphism group of S. Then τ is called *continuous* if the map $(r, s) \mapsto \tau_r(s)$ is continuous from $R \times S$ onto S, where the automorphism $\tau_r : S \to S$ denotes the homomorphism τ applied to r. It is worthwhile to mention that there is also a natural topology, sometimes called *Braconnier topology*, turning $\mathrm{Aut}(S)$ into a Hausdorff topological group (not necessarily locally compact), which is defined by the sub-base of identity neighborhoods

$$\mathcal{B}(F, U) = \left\{\alpha \in \mathrm{Aut}(S) : \alpha(a), \alpha^{-1}(a) \in Ua \ \forall a \in F\right\},$$

where $F \subseteq S$ is compact and $U \subseteq S$ is a neighborhood of the identity. If $\mathrm{Aut}(S)$ is equipped with the Braconnier topology, then continuity of the homomorphism

$\tau : R \to \mathrm{Aut}(S)$ is equivalent to the continuity of the map $(r,s) \mapsto \tau_r(s)$ from $R \times S$ onto S (see [17], [18]).

The semidirect product $G_\tau = R \ltimes_\tau S$ is the locally compact topological group with underlying set $R \times S$ which is equipped by the product topology, and the group operation is defined by

$$(r,s) \ltimes_\tau (r',s') = \big(rr', \tau_{r'^{-1}}(s)s'\big) \qquad \text{and} \qquad (r,s)^{-1} = \big(r^{-1}, \tau_r(s^{-1})\big).$$

If $R_1 := \{(r, e_S) : r \in R\}$ and $S_1 := \{(e_R, s) : s \in S\}$, then R_1 is a closed subgroup and S_1 is a closed normal subgroup of G_τ. Under this identification, one can consider $R \cong R_1$ (resp., $S \cong S_1$) as a closed (resp., closed normal) subgroup of G_τ.

Let $\gamma = \gamma^\tau_{R,S} : R \to (0,\infty)$ be the positive continuous homomorphism which satisfies

$$dn_S\big(\tau_r(s)\big) = \gamma(r)\, dn_S(s) \quad \text{for } r \in R.$$

Existence of $\gamma(r)$ for $r \in R$ is guaranteed by the uniqueness (up to scaling) of the right Haar measure on S. Then

$$dm_{G_\tau}(r,s) = dm_R(r)\, dm_S(s) \tag{2.1}$$

is a left Haar measure of the locally compact group G_τ, and a right Haar measure of G_τ is given by

$$dn_{G_\tau}(r,s) = \gamma(r)\, dn_R(r)\, dn_S(s). \tag{2.2}$$

(For a comprehensive picture of basic results concerning the classical harmonic analysis on locally compact semidirect product groups, see [1] and [17, Section 15.26 and Section 15.29].)

By convention, we denote the operation of any abelian group by $+$ except the circle group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Let K be an LCA group. Then, by Schur's lemma, all irreducible representations of K are 1-dimensional. Thus any irreducible unitary representation $(\pi, \mathcal{H}_\pi)$ of K satisfies $\mathcal{H}_\pi = \mathbb{C}$, and hence there exists a continuous homomorphism ω of K into the circle group $\mathbb{T}$ such that, for each $k \in K$ and $z \in \mathbb{C}$, we have $\pi(k)(z) = \omega(k)z$. Such homomorphisms are called *characters of* K, and the set of all such characters of K is denoted by $\widehat{K}$. If $\widehat{K}$ is equipped with the topology of compact convergence on K which coincides with the w^*-topology that $\widehat{K}$ inherits as a subset of $L^\infty(K)$, then $\widehat{K}$ with respect to the product of characters is an LCA group which is called the *dual group of* K. The linear map $\mathcal{F}_K : L^1(K) \to \mathcal{C}(\widehat{K})$ defined by $f \mapsto \mathcal{F}_K(f) = \widehat{f}$ via

$$\mathcal{F}_K(f)(\omega) = \widehat{f}(\omega) = \int_K f(s)\overline{\omega(s)}\, dm_K(s) \tag{2.3}$$

is called the *Fourier transform* on K. It is a norm-decreasing $*$-homomorphism from $L^1(K)$ into $\mathcal{C}_0(\widehat{K})$ with a uniformly dense range in $\mathcal{C}_0(\widehat{K})$. If a Haar measure m_K on K is given and fixed, then there is a Haar measure $m_{\widehat{K}}$ on $\widehat{K}$ which is called the *normalized Plancherel measure* associated to m_K such that the Fourier transform (2.3) is an isometric transform on $L^1(K) \cap L^2(K)$, and hence it can be extended uniquely to a unitary isomorphism from $L^2(K)$ onto $L^2(\widehat{K})$ (see

[9], [17]). Then each $f \in L^1(K)$ with $\widehat{f} \in L^1(\widehat{K})$ satisfies the following Fourier inversion formula:

$$f(s) = \int_{\widehat{K}} \widehat{f}(\omega)\omega(s)\, dm_{\widehat{K}}(\omega) \quad \text{for a.e. } s \in K. \tag{2.4}$$

For $k \in K$ and $f \in L^2(K)$, the translation of f by k is defined by $T_k f(s) = f(s-k)$ for $s \in K$. The translation $T_k : L^2(K) \to L^2(K)$ is a unitary operator. For $\omega \in \widehat{K}$ and $f \in L^2(K)$, the modulation of f by ω is defined by $M_\omega f(s) = \overline{\omega(s)} f(s)$ for $s \in K$. The modulation operator $M_\omega : L^2(K) \to L^2(K)$ is unitary as well. The modulation and translation operators are connected via the Fourier transform by

$$\widehat{M_\omega f} = T_{-\omega}\widehat{f}, \qquad \widehat{T_k f} = M_k \widehat{f} \tag{2.5}$$

for all $f \in L^2(K)$, $\omega \in \widehat{K}$, and $k \in K$ (see [9], [17]).

3. Abstract wave-packet groups

Throughout this paper, we assume that H is a locally compact group with a left (resp., right) Haar measure m_H (resp., n_H), K is an LCA group with a Haar measure m_K, and dual group $\widehat{K}$ has the normalized Plancherel measure $m_{\widehat{K}}$. It is also assumed that $\theta : H \to \mathrm{Aut}(K)$ is a continuous homomorphism and that $\delta : H \to (0, \infty)$ is the positive continuous homomorphism which satisfies

$$dm_K(k) = \delta(h)\, dm_K\big(\theta_h(k)\big). \tag{3.1}$$

From now on, and due to the simplicity in notation, we may use k^h instead of $\theta_h(k)$ at times.

3.1. **Structure of the wave-packet group.** The first step to extend the abstract theory of wave-packet analysis on LCA groups is the generalization of the notion of wave-packet groups. The wave-packet group is placed as the phase space of the wave-packet analysis.

Let $h \in H$. For $\omega \in \widehat{K}$, define $\omega_h \in \widehat{K}$ via

$$\omega_h(k) := \omega \circ \theta_{h^{-1}}(k) = \omega\big(\theta_{h^{-1}}(k)\big) \quad \text{for } k \in K.$$

Then, for $k \in K$ and $\omega, \omega' \in \widehat{K}$, we can write

$$\begin{aligned}(\omega + \omega')_h(k) &= (\omega + \omega') \circ \theta_{h^{-1}}(k) \\ &= (\omega + \omega')\big(\theta_{h^{-1}}(k)\big) \\ &= \omega\big(\theta_{h^{-1}}(k)\big)\omega'\big(\theta_{h^{-1}}(k)\big) \\ &= \omega_h(k)\omega'_h(k),\end{aligned}$$

which implies that $\widehat{\theta}_h : \widehat{K} \to \widehat{K}$ defined by

$$\widehat{\theta}_h(\omega) := \omega_h \quad \text{for } \omega \in \widehat{K}$$

satisfies

$$\widehat{\theta}_h(\omega + \omega') = \widehat{\theta}_h(\omega) + \widehat{\theta}_h(\omega') \quad \text{for } \omega, \omega' \in \widehat{K}.$$

Thus $\widehat{\theta}_h : \widehat{K} \to \widehat{K}$ is a group homomorphism. It is straightforward to see that the homomorphism $\widehat{\theta}_h$ is an automorphism of $\widehat{K}$, as well.

Let $\widehat{\theta} : H \to \mathrm{Aut}(\widehat{K})$ be given by $h \mapsto \widehat{\theta}_h$. If $h, h' \in H$, $\omega \in \widehat{K}$, then we have

$$\begin{aligned}
\widehat{\theta}_{hh'}(\omega) &= \omega_{hh'} \\
&= \omega \circ \theta_{(hh')^{-1}} \\
&= \omega \circ \theta_{h'^{-1}h^{-1}} \\
&= \omega \circ \theta_{h'^{-1}} \circ \theta_{h^{-1}} \\
&= \widehat{\theta}_h(\omega \circ \theta_{h'^{-1}}) \\
&= \widehat{\theta}_h\big(\widehat{\theta}_{h'}(\omega)\big),
\end{aligned}$$

which guarantees that $\widehat{\theta}$ is a group homomorphism of H into $\mathrm{Aut}(\widehat{K})$.

If $h \in H$, then the automorphism $\widehat{\theta}_h : \widehat{K} \to \widehat{K}$ induces the positive Radon measure $m_{\widehat{K}} \circ \widehat{\theta}_h$ on $\widehat{K}$ in a canonical way by

$$dm_{\widehat{K}} \circ \widehat{\theta}_h(\omega) = dm_{\widehat{K}}(\omega_h);$$

that is,

$$m_{\widehat{K}} \circ \widehat{\theta}_h(E) = m_{\widehat{K}}\big(\widehat{\theta}_h(E)\big)$$

for all Borel subsets $E \subseteq \widehat{K}$, where $\widehat{\theta}_h(E) = \{\widehat{\theta}_h(\omega) = \omega_h : \omega \in E\}$.

The following theorem indicates that the measure $m_{\widehat{K}} \circ \widehat{\theta}_h$ is connected with the normalized Plancherel measure $m_{\widehat{K}}$ via δ.

Theorem 3.1. *Let H be a locally compact group, let K be an LCA group with Haar measure m_K, and let $m_{\widehat{K}}$ be the normalized Plancherel measure on $\widehat{K}$. Let $\theta : H \to \mathrm{Aut}(K)$ be a continuous homomorphism, and let $\delta : H \to (0, \infty)$ be the positive continuous homomorphism which satisfies (3.1). Then we have*

$$m_{\widehat{K}} \circ \widehat{\theta}_h = \delta(h) \cdot m_{\widehat{K}} \quad \text{for } h \in H. \tag{3.2}$$

Proof. Let $h \in H$ be given. Then $m_{\widehat{K}} \circ \widehat{\theta}_h$ is a nonzero translation invariant measure (Haar measure) on the LCA group $\widehat{K}$. To check this, let $E \subseteq \widehat{K}$ be a Borel subset, and let $\xi \in \widehat{K}$. By the translation invariance of the normalized Plancherel measure $m_{\widehat{K}}$, we can write

$$\begin{aligned}
m_{\widehat{K}} \circ \widehat{\theta}_h(\xi + E) &= m_{\widehat{K}} \circ \widehat{\theta}_h\big(\{\xi + \omega : \omega \in E\}\big) \\
&= m_{\widehat{K}}\big(\widehat{\theta}_h\{\xi + \omega : \omega \in E\}\big) \\
&= m_{\widehat{K}}\big(\big\{\widehat{\theta}_h(\xi + \omega) : \omega \in E\big\}\big) \\
&= m_{\widehat{K}}\big(\{\xi_h + \omega_h : \omega \in E\}\big) \\
&= m_{\widehat{K}}\big(\xi_h + \{\omega_h : \omega \in E\}\big) \\
&= m_{\widehat{K}}\big(\{\omega_h : \omega \in E\}\big) \\
&= m_{\widehat{K}} \circ \widehat{\theta}_h(E).
\end{aligned}$$

Thus, by the uniqueness (up to scaling) of the Haar measure on locally compact groups, we deduce that

$$m_{\widehat{K}} \circ \widehat{\theta}_h = \beta_h \cdot m_{\widehat{K}},$$

where β_h is a positive constant which depends on h. Now we claim that $\beta_h = \delta(h)$. To prove this, let $f \in L^1(K)$. Then, using (3.1), we have $f \circ \theta_h \in L^1(K)$ with $\|f \circ \theta_h\|_{L^1(K)} = \delta(h)\|f\|_{L^1(K)}$. Thus, for $\omega \in \widehat{K}$, we get

$$\begin{aligned}
\widehat{f \circ \theta_h}(\omega) &= \int_K f \circ \theta_h(k)\overline{\omega(k)}\, dm_K(k) \\
&= \int_K f\big(\theta_h(k)\big)\overline{\omega(k)}\, dm_K(k) \\
&= \int_K f(k)\overline{\omega\big(\theta_{h^{-1}}(k)\big)}\, dm_K\big(\theta_{h^{-1}}(k)\big) \\
&= \int_K f(k)\overline{\omega_h(k)}\, dm_K\big(\theta_{h^{-1}}(k)\big) \\
&= \delta(h) \int_K f(k)\overline{\omega_h(k)}\, dm_K(k) \\
&= \delta(h)\widehat{f}(\omega_h).
\end{aligned}$$

If $f \in L^1(K) \cap L^2(K)$ is a nonzero function, then, by the Plancherel theorem, we can write

$$\begin{aligned}
\int_{\widehat{K}} \big|\widehat{f}(\omega)\big|^2\, dm_{\widehat{K}}(\omega_h) &= \int_{\widehat{K}} \big|\widehat{f}(\omega_{h^{-1}})\big|^2\, dm_{\widehat{K}}(\omega) \\
&= \delta(h)^2 \int_{\widehat{K}} \big|\widehat{f \circ \theta_{h^{-1}}}(\omega)\big|^2\, dm_{\widehat{K}}(\omega) \\
&= \delta(h)^2 \int_K \big|f \circ \theta_{h^{-1}}(k)\big|^2\, dm_K(k) \\
&= \delta(h)^2 \int_K \big|f(k)\big|^2\, dm_K\big(\theta_h(k)\big) \\
&= \delta(h) \int_K \big|f(k)\big|^2\, dm_K(k) \\
&= \delta(h) \int_{\widehat{K}} \big|\widehat{f}(\omega)\big|^2\, dm_{\widehat{K}}(\omega),
\end{aligned}$$

which implies that

$$\begin{aligned}
\beta_h\|\widehat{f}\|^2_{L^2(\widehat{K})} &= \beta_h \int_{\widehat{K}} \big|\widehat{f}(\omega)\big|^2\, dm_{\widehat{K}}(\omega) \\
&= \int_{\widehat{K}} \big|\widehat{f}(\omega)\big|^2\, dm_{\widehat{K}}(\omega_h) \\
&= \delta(h) \int_{\widehat{K}} \big|\widehat{f}(\omega)\big|^2\, dm_{\widehat{K}}(\omega) \\
&= \delta(h)\|\widehat{f}\|^2_{L^2(\widehat{K})}.
\end{aligned}$$

Since f and hence $\|\widehat{f}\|_{L^2(K)}$ are nonzero, we conclude that $\beta_h = \delta(h)$. $\square$

Now we are in the position to present the abstract notion of wave-packet groups on LCA groups. The motivation for this definition of wave-packet group is originated from the classical abstract harmonic analysis approach in Gabor and wavelet analysis on LCA groups (see [3], [10], [15], and references therein).

For $h \in H$, define $\Theta_h : K \times \widehat{K} \to K \times \widehat{K}$ by

$$\Theta_h(k,\omega) := \big(\theta_h(k), \widehat{\theta}_h(\omega)\big) = (k^h, \omega_h) \quad \text{for } (k,\omega) \in K \times \widehat{K}. \tag{3.3}$$

Let $(k,\omega),(k',\omega') \in K \times \widehat{K}$. Then we have

$$\begin{aligned}\Theta_h\big((k,\omega)+(k',\omega')\big) &= \Theta_h(k+k',\omega+\omega') \\ &= \big((k+k')^h, (\omega+\omega')_h\big) \\ &= (k^h + k'^h, \omega_h + \omega'_h) \\ &= (k^h, \omega_h) + (k'^h, \omega'_h) \\ &= \Theta_h(k,\omega) + \Theta_h(k',\omega'),\end{aligned}$$

which guarantees that Θ_h is a homomorphism of $K \times \widehat{K}$. Since θ_h and $\widehat{\theta}_h$ are automorphisms of K and $\widehat{K}$, respectively, we achieve that $\Theta_h \in \mathrm{Aut}(K \times \widehat{K})$.

Let $h, h' \in H$, and let $(k,\omega) \in K \times \widehat{K}$. Then we can write

$$\begin{aligned}\Theta_{hh'}(k,\omega) &= (k^{hh'}, \omega_{hh'}) \\ &= \big((k^{h'})^h, (\omega_{h'})_h\big) \\ &= \Theta_h(k^{h'}, \omega_{h'}) \\ &= \Theta_h\Theta_{h'}(k,\omega),\end{aligned}$$

which implies that the mapping $\Theta : H \to \mathrm{Aut}(K \times \widehat{K})$ given by $h \mapsto \Theta_h$ is a well-defined homomorphism.

Definition 3.2. Let H be a locally compact group, let K be an LCA group with dual group $\widehat{K}$, and let $\theta : H \to \mathrm{Aut}(K)$ be a continuous homomorphism. The semidirect product group $G_\Theta := H \ltimes_\Theta (K \times \widehat{K})$ is considered the wave-packet group associated to the continuous homomorphism $\theta : H \to \mathrm{Aut}(K)$ or the wave-packet group generated by θ.

Remark 3.3. The group operations for $(h,k,\omega),(h',k',\omega') \in G_\Theta$ are

$$\begin{aligned}(h,k,\omega) \ltimes_\Theta (h',k',\omega') &= \big(hh', \Theta_{h'^{-1}}(k,\omega) + (k',\omega')\big) \\ &= \big(hh', \big(\theta_{h'^{-1}}(k), \omega_{h'^{-1}}\big) + (k',\omega')\big) \\ &= \big(hh', \theta_{h'^{-1}}(k) + k', \omega_{h'^{-1}} + \omega'\big),\end{aligned}$$

and

$$(h,k,\omega)^{-1} = \big(h^{-1}, \Theta_h(-k,-\omega)\big) = (h^{-1}, -k^h, -\omega_h).$$

The following theorem lists basic properties of the wave-packet group from the abstract harmonic-analysis perspective.

Theorem 3.4. *Let H be a locally compact group with left (resp., right) Haar measure m_H (resp., n_H) and modular function Δ_H. Let K be an LCA group with Haar measure m_K and normalized Plancherel measure $m_{\widehat{K}}$ on $\widehat{K}$. Let $\theta : H \to \mathrm{Aut}(K)$ be a continuous homomorphism. Then the following hold.*

(1) *The product Haar measure $m_{K\times\widehat{K}} = m_K \cdot m_{\widehat{K}}$ of the LCA group $K \times \widehat{K}$ satisfies*

$$dm_{K\times\widehat{K}}(k^h, \omega_h) = dm_{K\times\widehat{K}}(k, \omega) \quad \text{for } h \in H.$$

(2) *The result $\Theta : H \to \mathrm{Aut}(K \times \widehat{K})$ given by (3.3) is a continuous homomorphism, and the semidirect product*

$$G_\Theta := H \ltimes_\Theta (K \times \widehat{K})$$

is a locally compact group with a left Haar measure

$$dm_{G_\Theta}(h, k, \omega) = dm_H(h)\, dm_K(k)\, dm_{\widehat{K}}(\omega)$$

and a right Haar measure

$$dn_{G_\Theta}(h, k, \omega) = dn_H(h)\, dm_K(k)\, dm_{\widehat{K}}(\omega).$$

(3) *The modular function $\Delta_{G_\Theta} : G_\Theta \to (0, \infty)$ is given by*

$$\Delta_{G_\Theta}(h, k, \omega) = \Delta_H(h) \quad \text{for } (h, k, \omega) \in G_\Theta.$$

Proof. (1) Due to (3.1), (3.2), and (3.3), for $h \in H$ we have

$$\begin{aligned} dm_{K\times\widehat{K}}(k^h, \omega_h) &= dm_K(k^h)\, dm_{\widehat{K}}(\omega_h) \\ &= \delta(h)^{-1}\, dm_K(k)\delta(h)\, dm_{\widehat{K}}(\omega) \\ &= dm_K(k)\, dm_{\widehat{K}}(\omega) \\ &= dm_{K\times\widehat{K}}(k, \omega). \end{aligned}$$

(2) Continuity of the homomorphism $\Theta : H \to \mathrm{Aut}(K \times \widehat{K})$ given in (3.3) is guaranteed by Theorem 26.9 of [17]. Thus the semidirect product $G_\Theta = H \ltimes_\Theta (K \times \widehat{K})$ is a locally compact group with left Haar measure

$$dm_{G_\Theta}(h, k, \omega) = dm_H(h)\, dm_K(k)\, dm_{\widehat{K}}(\omega)$$

and right Haar measure

$$dn_{G_\Theta}(h, k, \omega) = dn_H(h)\, dm_K(k)\, dm_{\widehat{K}}(\omega).$$

(3) This proof is straightforward. □

Corollary 3.5. *The wave-packet group G_Θ is unimodular if and only if H is unimodular.*

Next we state basic algebraic-analytic properties of the wave-packet group G_Θ.

Proposition 3.6. *Let H be a locally compact group, and let K be an LCA group with dual group $\widehat{K}$. Let $\theta : H \to \mathrm{Aut}(K)$ be a continuous homomorphism. Then we have the following.*

1. *The wave-packet group G_Θ contains $K \times \widehat{K}$ as a closed normal abelian subgroup.*

(2) *The closed subgroup H is normal in G_Θ if and only if θ is the identity homomorphism.*
(3) *The semidirect product groups $H \ltimes_\theta K$ and $H \ltimes_{\widehat{\theta}} \widehat{K}$ are closed nonabelian subgroups of G_Θ.*

3.2. **Structure of the wave-packet representation.** Next we will show that the structure of the wave-packet group G_Θ is firmly attached with a canonical projective unitary group representation, which is called the *wave-packet representation*.

For $h \in H$ and a function $f : K \to \mathbb{C}$, define the θ-*dilation* of f by h via

$$D_h^\theta f(k) := \delta(h)^{1/2} f\big(\theta_{h^{-1}}(k)\big) \quad \text{for } k \in K.$$

Similarly, for $h \in H$ and a function $\phi : \widehat{K} \to \mathbb{C}$, define the $\widehat{\theta}$-*dilation* of ϕ by h via

$$\widehat{D}_h^{\widehat{\theta}}\phi(\omega) := \delta(h)^{-1/2}\phi\big(\widehat{\theta}_{h^{-1}}(\omega)\big) \quad \text{for } \omega \in \widehat{K}.$$

For simplicity in notation, we may use D_h and $\widehat{D}_h$ instead of D_h^θ and $\widehat{D}_h^{\widehat{\theta}}$, respectively.

The following proposition states some fundamental analytic aspects of dilation operators.

Proposition 3.7. *Let H be a locally compact group, and let K be an LCA group. Let $\theta : H \to \mathrm{Aut}(K)$ be a continuous homomorphism. Then*
(1) *The map $\mathcal{D} : H \to \mathcal{U}(L^2(K))$ given by $h \mapsto \mathcal{D}(h) := D_h$ is a unitary representation of H on the Hilbert space $L^2(K)$.*
(2) *For $h \in H$ and $f \in L^2(K)$, we have $\widehat{D_h f} = \widehat{D}_h \widehat{f}$.*
(3) *The map $\widehat{\mathcal{D}} : H \to \mathcal{U}(L^2(\widehat{K}))$ given by $h \mapsto \widehat{\mathcal{D}}(h) := \widehat{D}_h$ is a unitary representation of H on the Hilbert space $L^2(\widehat{K})$.*

Proof. Let $\theta : H \to \mathrm{Aut}(K)$ be a continuous homomorphism. Then

(1) Let $f \in L^2(K)$ be given. For $h \in H$ and $k \in K$, we have

$$\begin{aligned}
\|D_h f\|_{L^2(K)}^2 &= \int_K \big|D_h f(k)\big|^2 \, dm_K(k) \\
&= \delta(h) \int_K \big|f\big(\theta_{h^{-1}}(k)\big)\big|^2 \, dm_K(k) \\
&= \delta(h) \int_K \big|f(k)\big|^2 \, dm_K\big(\theta_h(k)\big) \\
&= \int_K \big|f(k)\big|^2 \, dm_K(k) \\
&= \|f\|_{L^2(K)}^2.
\end{aligned}$$

Thus $D_h : L^2(K) \to L^2(K)$ is an isometric operator.

For $h, h' \in H$ and $k \in K$, we have

$$\begin{aligned}
D_{hh'} f(k) &= \delta(hh')^{1/2} f\big(\theta_{(hh')^{-1}}(k)\big) \\
&= \delta(h)^{1/2}\delta(h')^{1/2} f\big(\theta_{(hh')^{-1}}(k)\big)
\end{aligned}$$

$$
\begin{aligned}
&= \delta(h)^{1/2}\delta(h')^{1/2} f\big(\theta_{h'^{-1}h^{-1}}(k)\big) \\
&= \delta(h)^{1/2}\delta(h')^{1/2} f\big(\theta_{h'^{-1}}\theta_{h^{-1}}(k)\big) \\
&= \delta(h)^{1/2} D_{h'} f\big(\theta_{h^{-1}}(k)\big) \\
&= D_h D_{h'} f(k),
\end{aligned}
$$

which implies that $D_{hh'} = D_h D_{h'}$. Therefore, we get $D_h D_{h^{-1}} = D_{h^{-1}} D_h = I$, where I is the identity operator on $L^2(K)$. Thus we deduce that $D_h : L^2(K) \to L^2(K)$ is invertible, and hence it is surjective. Since D_h is an isometric operator, we derive that D_h is unitary as well, and so the map $\mathcal{D} : H \to \mathcal{U}(L^2(K))$ given by $h \mapsto \mathcal{D}(h) = D_h$ is a unitary representation of H on the Hilbert space $L^2(K)$.

(2) Let $f \in L^2(K)$, and let $h \in H$. For $\omega \in \widehat{K}$, we have

$$
\begin{aligned}
\widehat{D_h f}(\omega) &= \int_K D_h f(k)\overline{\omega(k)}\, dm_K(k) \\
&= \delta(h)^{1/2} \int_K f\big(\theta_{h^{-1}}(k)\big)\overline{\omega(k)}\, dm_K(k) \\
&= \delta(h)^{1/2} \int_K f(k)\overline{\omega\big(\theta_h(k)\big)}\, dm_K\big(\theta_h(k)\big) \\
&= \delta(h)^{-1/2} \int_K f(k)\overline{\omega\big(\theta_h(k)\big)}\, dm_K(k) \\
&= \delta(h)^{-1/2}\widehat{f}(\omega\circ\theta_h) \\
&= \delta(h)^{-1/2}\widehat{f}(\omega_{h^{-1}}) \\
&= \delta(h)^{-1/2}\widehat{f}\big(\widehat{\theta}_{h^{-1}}(\omega)\big) \\
&= \widehat{D}_h\widehat{f}(\omega).
\end{aligned}
$$

(3) This can be proved in a way similar to that used in (1). □

The following proposition briefly summarizes commuting relations of basic operators in wave-packet analysis.

Proposition 3.8. *Let H be a locally compact group, and let K be an LCA group. Let $\theta : H \to \mathrm{Aut}(K)$ be a continuous homomorphism. Then*

(1) *For $(h,k) \in H \times K$, we have $D_h T_k = T_{k^h} D_h$.*
(2) *For $(h,\omega) \in H \times \widehat{K}$, we have $D_h M_\omega = M_{\omega_h} D_h$.*
(3) *For $(k,\omega) \in K \times \widehat{K}$, we have $T_k M_\omega = \omega(k) M_\omega T_k$.*

Proof. Let $f \in L^2(K)$, and let $s \in K$. Then

(1) For $(h,k) \in H \times K$, we have

$$
\begin{aligned}
[T_{k^h} D_h f](s) &= D_h f(s - k^h) \\
&= \delta(h)^{1/2} f\big(\theta_{h^{-1}}(s - k^h)\big) \\
&= \delta(h)^{1/2} f\big(\theta_{h^{-1}}(s) - \theta_{h^{-1}}(k^h)\big) \\
&= \delta(h)^{1/2} f\big(\theta_{h^{-1}}(s) - k\big) \\
&= \delta(h)^{1/2} [T_k f]\big(\theta_{h^{-1}}(s)\big) \\
&= [D_h T_k f](s).
\end{aligned}
$$

(2) For $(h,\omega)\in H\times\widehat{K}$, we have

$$\begin{aligned}[M_{\omega_h}D_hf](s)&=\overline{\omega_h(s)}[D_hf](s)\\&=\delta(h)^{1/2}\overline{\omega_h(s)}f\big(\theta_{h^{-1}}(s)\big)\\&=\delta(h)^{1/2}\overline{\omega\big(\theta_{h^{-1}}(s)\big)}f\big(\theta_{h^{-1}}(s)\big)\\&=\delta(h)^{1/2}[M_\omega f]\big(\theta_{h^{-1}}(s)\big)\\&=[D_hM_\omega f](s).\end{aligned}$$

(3) For $(k,\omega)\in K\times\widehat{K}$, we have

$$\begin{aligned}[T_kM_\omega f](s)&=[M_\omega f](s-k)\\&=\overline{\omega(s-k)}f(s-k)\\&=\omega(k)\overline{\omega(s)}f(s-k)\\&=\omega(k)\overline{\omega(s)}[T_kf](s)\\&=\omega(k)[M_\omega T_kf](s).\end{aligned}$$

□

For $(h,k,\omega)\in G_\Theta$, define the linear operator $\Gamma(h,k,\omega):L^2(K)\to L^2(K)$ by

$$\Gamma(h,k,\omega):=D_hT_kM_\omega. \tag{3.4}$$

Thus, for $f\in L^2(K)$ and $s\in K$, we get

$$\begin{aligned}\big[\Gamma(h,k,\omega)f\big](s)&=D_hT_kM_\omega f(s)\\&=\delta(h)^{1/2}T_kM_\omega f\big(\theta_{h^{-1}}(s)\big)\\&=\delta(h)^{1/2}M_\omega f\big(\theta_{h^{-1}}(s)-k\big)\\&=\delta(h)^{1/2}\omega(k)\overline{\omega_h(s)}f\big(\theta_{h^{-1}}(s)-k\big).\end{aligned}$$

Remark 3.9. It should be mentioned that restriction of the wave-packet representation to the closed subgroup $K\times\widehat{K}$ is unitarily equivalent to the projective Schrödinger representation of the group $K\times\widehat{K}$ on $L^2(K)$ (see [15] and references therein), and, similarly, restriction of the wave-packet representation to the closed subgroup $H\times K$ is unitarily equivalent to the quasiregular representation of the group $H\rtimes_\theta K$ on $L^2(K)$ (see [2], [3], and references therein).

In the following theorem, we show that $(h,k,\omega)\mapsto\Gamma(h,k,\omega)$ given by (3.4) defines a unitary projective group representation of the wave-packet group G_Θ on the Hilbert space $L^2(K)$.

Theorem 3.10. *Let H be a locally compact group, and let K be an LCA group with dual group $\widehat{K}$. Let $\theta:H\to\mathrm{Aut}(K)$ be a continuous homomorphism. Then $\Gamma:G_\Theta\to\mathcal{U}(L^2(K))$ given by $\Gamma(h,k,\omega)=D_hT_kM_\omega$ is a unitary irreducible projective group representation of the locally compact group G_Θ on the Hilbert space $L^2(K)$.*

Proof. It is evident to check that $\Gamma(e_H,e_K,e_{\widehat{K}})=I$. Then the operator $\Gamma(h,k,\omega)=D_hT_kM_\omega$ is a unitary operator on $L^2(K)$ for all $(h,k,\omega)\in G_\Theta$ because it is the composition of three unitary operators, namely D_h, T_k, and M_ω. Let $(h,k,\omega),(h',k',\omega')\in G_\Theta$. Then we have

$$\begin{aligned}
D_{hh'}T_{\theta_{h'^{-1}}(k)+k'}M_{\omega_{h'^{-1}}+\omega'} &= D_h(D_{h'}T_{\theta_{h'^{-1}}(k)})T_{k'}M_{\omega_{h'^{-1}}}M_{\omega'}\\
&= D_h(T_kD_{h'})T_{k'}M_{\omega_{h'^{-1}}}M_{\omega'}\\
&= D_hT_kD_{h'}(T_{k'}M_{\omega_{h'^{-1}}})M_{\omega'}\\
&= \omega_{h'^{-1}}(k')D_hT_kD_{h'}(M_{\omega_{h'^{-1}}}T_{k'})M_{\omega'}\\
&= \omega_{h'^{-1}}(k')D_hT_k(D_{h'}M_{\omega_{h'^{-1}}})T_{k'}M_{\omega'}\\
&= \omega_{h'^{-1}}(k')D_hT_k(M_\omega D_{h'})T_{k'}M_{\omega'}\\
&= \omega_{h'^{-1}}(k')(D_hT_kM_\omega)(D_{h'}T_{k'}M_{\omega'}).
\end{aligned}$$

Thus, invoking the group law of the wave-packet group G_Θ (see Remark 3.3), we get

$$\begin{aligned}
\Gamma\big((h,k,\omega)\rtimes(h',k',\omega')\big) &= \Gamma\big(hh',\theta_{h'^{-1}}(k)+k',\omega_{h'^{-1}}+\omega'\big)\\
&= D_{hh'}T_{\theta_{h'^{-1}}(k)+k'}M_{\omega_{h'^{-1}}+\omega'}\\
&= \omega_{h'^{-1}}(k')(D_hT_kM_\omega)(D_{h'}T_{k'}M_{\omega'})\\
&= \omega_{h'^{-1}}(k')\Gamma(h,k,\omega)\Gamma(h',k',\omega'),
\end{aligned}$$

which implies that $\Gamma : G_\Theta \to \mathcal{U}(L^2(K))$ is a unitary projective group representation of the locally compact group G_Θ on the Hilbert space $L^2(K)$. Using Remark 3.9 and the fact that the projective Schrödinger representation of $K\times\widehat{K}$ is irreducible on $L^2(K)$, we deduce that Γ is a unitary irreducible projective group representation of the locally compact group G_Θ on the Hilbert space $L^2(K)$, as well. $\square$

4. Abstract wave-packet transforms

In this section, we present the abstract theory of wave-packet transforms over LCA groups. Let $\psi\in L^2(K)$ be a window function. The wave-packet transform of $f\in L^2(K)$ with respect to the window function ψ is given by the voice transform associated to the wave-packet representation; that is,

$$\mathcal{V}_\psi f(h,k,\omega) := \big\langle f,\Gamma(h,k,\omega)\psi\big\rangle_{L^2(K)} = \langle f, D_hT_kM_\omega\psi\rangle_{L^2(K)} \tag{4.1}$$

for $(h,k,\omega)\in G_\Theta$.

Evidently $f\mapsto\mathcal{V}_\psi f$ is linear, and we have

$$\begin{aligned}
\langle f, D_hT_kM_\omega\psi\rangle_{L^2(K)} &= \langle D_{h^{-1}}f, T_kM_\omega\psi\rangle_{L^2(K)}\\
&= \langle T_{-k}D_{h^{-1}}f, M_\omega\psi\rangle_{L^2(K)}\\
&= \langle M_{-\omega}T_{-k}D_{h^{-1}}f, \psi\rangle_{L^2(K)}.
\end{aligned}$$

Since

$$D_hT_kM_\omega = T_{k^h}D_hM_\omega = T_{k^h}M_{\omega_h}D_h,$$

we achieve

$$\mathcal{V}_\psi f(h,k,\omega) = \langle f, T_{\theta_h(k)}M_{\omega_h}D_h\psi\rangle_{L^2(K)}. \tag{4.2}$$

The following proposition presents interesting representations of the wave-packet transform.

Proposition 4.1. *Let $\psi \in L^2(K)$ be a window function, let $f \in L^2(K)$, and let $(h,k,\omega) \in G_\Theta$. Then we have*

(1) $\mathcal{V}_\psi f(h,k,\omega) = \langle \widehat{D_{h^{-1}} f}, \widehat{T_k M_\omega \psi}\rangle_{L^2(\widehat{K})}$.
(2) $\mathcal{V}_\psi f(h,k,\omega) = \langle \widehat{D}_{h^{-1}} \widehat{f}, M_k \widehat{M_\omega \psi}\rangle_{L^2(\widehat{K})}$.
(3) $\mathcal{V}_\psi f(h,k,\omega) = \langle \widehat{D}_{h^{-1}} \widehat{f}, M_k T_{-\omega} \widehat{\psi}\rangle_{L^2(\widehat{K})}$.
(4) $\mathcal{V}_\psi f(h,k,\omega) = \mathcal{F}_{\widehat{K}}(\widehat{D}_{h^{-1}} \widehat{f} \cdot T_{-\omega}\overline{\widehat{\psi}})(-k)$.
(5) $\mathcal{V}_\psi f(h,k,\omega) = \mathcal{F}_{\widehat{K}}(\overline{T_{-\omega_h} \widehat{D}_h \widehat{\psi}} \cdot \widehat{f})(-\theta_h(k))$.

Proof. (1), (2), and (3) are straightforward using the Plancherel formula and basic properties of wave-packet operators.

(4) Using the Plancherel formula and (1), we can write

$$\begin{aligned}
\mathcal{V}_\psi f(h,k,\omega) &= \langle \widehat{D}_{h^{-1}} \widehat{f}, M_k T_{-\omega} \widehat{\psi}\rangle_{L^2(\widehat{K})} \\
&= \int_{\widehat{K}} \widehat{D}_{h^{-1}} \widehat{f}(\xi) \overline{M_k T_{-\omega} \widehat{\psi}(\xi)}\, dm_{\widehat{K}}(\xi) \\
&= \int_{\widehat{K}} \widehat{D}_{h^{-1}} \widehat{f}(\xi) \overline{\overline{\xi(k)} T_{-\omega} \widehat{\psi}(\xi)}\, dm_{\widehat{K}}(\xi) \\
&= \int_{\widehat{K}} \widehat{D}_{h^{-1}} \widehat{f}(\xi) T_{-\omega} \overline{\widehat{\psi}}(\xi) \xi(k)\, dm_{\widehat{K}}(\xi) \\
&= \int_{\widehat{K}} \widehat{D}_{h^{-1}} \widehat{f}(\xi) T_{-\omega} \overline{\widehat{\psi}}(\xi) \overline{\xi(-k)}\, dm_{\widehat{K}}(\xi) \\
&= \mathcal{F}_{\widehat{K}}(\widehat{D}_{h^{-1}} \widehat{f} \cdot T_{-\omega}\overline{\widehat{\psi}})(-k).
\end{aligned}$$

(5) By (4.2) and the Plancherel formula, we get

$$\begin{aligned}
\mathcal{V}_\psi f(h,k,\omega) &= \langle f, T_{\theta_h(k)} M_{\omega_h} D_h \psi\rangle_{L^2(K)} \\
&= \langle T_{-\theta_h(k)} f, M_{\omega_h} D_h \psi\rangle_{L^2(K)} \\
&= \langle \widehat{T_{-\theta_h(k)} f}, \widehat{M_{\omega_h} D_h \psi}\rangle_{L^2(\widehat{K})} \\
&= \langle M_{-\theta_h(k)} \widehat{f}, T_{-\omega_h} \widehat{D_h \psi}\rangle_{L^2(\widehat{K})} \\
&= \langle M_{-\theta_h(k)} \widehat{f}, T_{-\omega_h} \widehat{D}_h \widehat{\psi}\rangle_{L^2(\widehat{K})} \\
&= \int_{\widehat{K}} M_{-\theta_h(k)} \widehat{f}(\xi) \overline{T_{-\omega_h} \widehat{D}_h \widehat{\psi}(\xi)}\, dm_{\widehat{K}}(\xi) \\
&= \int_{\widehat{K}} \overline{T_{-\omega_h} \widehat{D}_h \widehat{\psi}(\xi)}\, \widehat{f}(\xi) \overline{\xi\big(-\theta_h(k)\big)}\, dm_{\widehat{K}}(\xi) \\
&= \mathcal{F}_{\widehat{K}}(\overline{T_{-\omega_h} \widehat{D}_h \widehat{\psi}} \cdot \widehat{f})\big(-\theta_h(k)\big).
\end{aligned}$$

□

Let m_H (resp., n_H) be a left (resp., right) Haar measure on H. For $\psi \in L^2(K)$, let $\alpha_\psi, \beta_\psi : \widehat{K} \to \mathbb{C}$ be given by

$$\begin{aligned}\alpha_\psi(\omega) &:= \int_H \big|\widehat{D}_h\widehat{\psi}(\omega)\big|^2 \, dm_H(h) \\ &= \int_H \big|\widehat{\psi}(\omega\circ\theta_h)\big|^2 \delta(h)^{-1} \, dm_H(h) \quad \text{for } \omega \in \widehat{K},\end{aligned}$$

and

$$\beta_\psi(\omega) := \int_H \big|\widehat{D}_h\widehat{\psi}(\omega)\big|^2 \, dn_H(h) = \int_H \big|\widehat{\psi}(\omega\circ\theta_h)\big|^2 \delta(h)^{-1} \, dn_H(h) \quad \text{for } \omega \in \widehat{K}.$$

The function $\psi \in L^2(K)$ is called *left* Γ*-admissible* (resp., *right* Γ*-admissible*) if and only if $\alpha_\psi \in L^1(\widehat{K})$ (resp., $\beta_\psi \in L^1(\widehat{K})$). Then we put

$$\begin{aligned}a_\psi &:= \int_{\widehat{K}} \alpha_\psi(\omega)\, dm_{\widehat{K}}(\omega) = \int_{\widehat{K}}\int_H \big|\widehat{D}_h\widehat{\psi}(\omega)\big|^2 \, dm_H(h)\, dm_{\widehat{K}}(\omega), \\ b_\psi &:= \int_{\widehat{K}} \beta_\psi(\omega)\, dm_{\widehat{K}}(\omega) = \int_{\widehat{K}}\int_H \big|\widehat{D}_h\widehat{\psi}(\omega)\big|^2 \, dn_H(h)\, dm_{\widehat{K}}(\omega).\end{aligned}$$

In the following theorem, we present a Plancherel formula for the wave-packet transform.

Theorem 4.2. *Let m_H (resp., n_H) be a left (resp., right) Haar measure on H, and let $f \in L^2(K)$. Then we have the following.*

(1) *For any left Γ-admissible window function $\psi \in L^2(K)$, we have*

$$\int_H\int_K\int_{\widehat{K}} \big|\mathcal{V}_\psi f(h,k,\omega)\big|^2 \, dm_H(h)\, dm_K(k)\, dm_{\widehat{K}}(\omega) = a_\psi \int_K \big|f(k)\big|^2 \, dm_K(k).$$

(2) *For any right Γ-admissible window function $\psi \in L^2(K)$, we have*

$$\int_H\int_K\int_{\widehat{K}} \big|\mathcal{V}_\psi f(h,k,\omega)\big|^2 \, dn_H(h)\, dm_K(k)\, dm_{\widehat{K}}(\omega) = b_\psi \int_K \big|f(k)\big|^2 \, dm_K(k).$$

Proof. (1) By Proposition 4.1 and the Plancherel formula, we have

$$\begin{aligned}\int_K \big|\mathcal{V}_\psi f(h,k,\omega)\big|^2 \, dm_K(k) &= \int_K \big|\mathcal{F}_{\widehat{K}}(\overline{T_{-\omega_h}\widehat{D}_h\widehat{\psi}}.\widehat{f})\big(-\theta_h(k)\big)\big|^2 \, dm_K(k) \\ &= \int_K \big|\mathcal{F}_{\widehat{K}}(\overline{T_{-\omega_h}\widehat{D}_h\widehat{\psi}}\cdot\widehat{f})(-k)\big|^2 \, dm_K\big(\theta_{h^{-1}}(k)\big) \\ &= \delta(h)\int_K \big|\mathcal{F}_{\widehat{K}}(\overline{T_{-\omega_h}\widehat{D}_h\widehat{\psi}}\cdot\widehat{f})(-k)\big|^2 \, dm_K(k) \\ &= \delta(h)\int_K \big|\mathcal{F}_{\widehat{K}}(\overline{T_{-\omega_h}\widehat{D}_h\widehat{\psi}}\cdot\widehat{f})(k)\big|^2 \, dm_K(-k) \\ &= \delta(h)\int_K \big|\mathcal{F}_{\widehat{K}}(\overline{T_{-\omega_h}\widehat{D}_h\widehat{\psi}}\cdot\widehat{f})(k)\big|^2 \, dm_K(k) \\ &= \delta(h)\int_{\widehat{K}} \big|(\overline{T_{-\omega_h}\widehat{D}_h\widehat{\psi}}\cdot\widehat{f})(\xi)\big|^2 \, dm_{\widehat{K}}(\xi)\end{aligned}$$

$$
\begin{aligned}
&= \delta(h) \int_{\widehat{K}} \left|T_{-\omega_h} \widehat{D}_h \widehat{\psi}(\xi)\right|^2 \left|\widehat{f}(\xi)\right|^2 dm_{\widehat{K}}(\xi) \\
&= \delta(h) \int_{\widehat{K}} \left|\widehat{D}_h \widehat{\psi}(\xi + \omega_h)\right|^2 \left|\widehat{f}(\xi)\right|^2 dm_{\widehat{K}}(\xi).
\end{aligned}
$$

Then we can write

$$
\begin{aligned}
&\|\mathcal{V}_\psi f\|^2_{L^2(G_\Theta, m_{G_\Theta})} \\
&= \int_H \int_K \int_{\widehat{K}} \left|\langle f, \Gamma(h,k,\omega)\psi\rangle_{L^2(K)}\right|^2 dm_H(h)\, dm_K(k)\, dm_{\widehat{K}}(\omega) \\
&= \int_H \int_{\widehat{K}} \Big(\int_K \left|\langle f, \Gamma(h,k,\omega)\psi\rangle_{L^2(K)}\right|^2 dm_K(k)\Big)\, dm_H(h)\, dm_{\widehat{K}}(\omega) \\
&= \int_H \int_{\widehat{K}} \Big(\int_{\widehat{K}} \left|\widehat{D}_h \widehat{\psi}(\xi+\omega_h)\right|^2 \left|\widehat{f}(\xi)\right|^2 dm_{\widehat{K}}(\xi)\Big) \delta(h)\, dm_H(h)\, dm_{\widehat{K}}(\omega) \\
&= \int_H \int_{\widehat{K}} \Big(\int_{\widehat{K}} \left|\widehat{D}_h \widehat{\psi}(\xi+\omega_h)\right|^2 dm_{\widehat{K}}(\omega)\Big) \left|\widehat{f}(\xi)\right|^2 dm_{\widehat{K}}(\xi) \delta(h)\, dm_H(h) \\
&= \int_H \int_{\widehat{K}} \Big(\int_{\widehat{K}} \left|\widehat{D}_h \widehat{\psi}(\xi+\omega)\right|^2 dm_{\widehat{K}}(\omega_{h^{-1}})\Big) \left|\widehat{f}(\xi)\right|^2 dm_{\widehat{K}}(\xi) \delta(h)\, dm_H(h) \\
&= \int_H \int_{\widehat{K}} \Big(\int_{\widehat{K}} \left|\widehat{D}_h \widehat{\psi}(\xi+\omega)\right|^2 dm_{\widehat{K}}(\omega)\Big) \left|\widehat{f}(\xi)\right|^2 dm_{\widehat{K}}(\xi)\, dm_H(h) \\
&= \int_H \int_{\widehat{K}} \Big(\int_{\widehat{K}} \left|\widehat{D}_h \widehat{\psi}(\omega)\right|^2 dm_{\widehat{K}}(\omega)\Big) \left|\widehat{f}(\xi)\right|^2 dm_{\widehat{K}}(\xi)\, dm_H(h) \\
&= \Big(\int_{\widehat{K}} \int_H \left|\widehat{D}_h \widehat{\psi}(\omega)\right|^2 dm_H(h)\, dm_{\widehat{K}}(\omega)\Big) \Big(\int_{\widehat{K}} \left|\widehat{f}(\xi)\right|^2 dm_{\widehat{K}}(\xi)\Big) \\
&= \Big(\int_{\widehat{K}} \alpha_\psi(\omega)\, dm_{\widehat{K}}(\omega)\Big) \Big(\int_{\widehat{K}} \left|\widehat{f}(\xi)\right|^2 dm_{\widehat{K}}(\xi)\Big) \\
&= a_\psi \|\widehat{f}\|^2_{L^2(\widehat{K})} \\
&= a_\psi \|f\|^2_{L^2(K)}.
\end{aligned}
$$

(2) This can be proved by an argument similar to that used in (1). □

As an immediate consequence of Theorem 4.2, we deduce the following orthogonality relation concerning the wave-packet transform.

Corollary 4.3. *Let m_H (resp., n_H) be a left (resp., right) Haar measure on H. The wave-packet transform satisfies the following orthogonality relations.*

(1) *For any left Γ-admissible window function $\psi \in L^2(K)$, we have*

$$\langle \mathcal{V}_\psi f, \mathcal{V}_\psi g\rangle_{L^2(G_\Theta, m_{G_\Theta})} = a_\psi \langle f, g\rangle_{L^2(K)} \quad \text{for } f, g \in L^2(K).$$

(2) *For any right Γ-admissible window function $\psi \in L^2(K)$, we have*

$$\langle \mathcal{V}_\psi f, \mathcal{V}_\psi g\rangle_{L^2(G_\Theta, n_{G_\Theta})} = b_\psi \langle f, g\rangle_{L^2(K)} \quad \text{for } f, g \in L^2(K).$$

The next result is an inversion (reconstruction) formula for the wave-packet transform defined by (4.1).

Theorem 4.4. *Let m_H (resp., n_H) be a left (resp., right) Haar measure on H, and let $f \in L^2(K)$. Then we have the following.*

(1) *For any left Γ-admissible window function $\psi \in L^2(K)$, we have*

$$f = a_\psi^{-1} \int_H \int_K \int_{\widehat{K}} \mathcal{V}_\psi f(h,k,\omega)\Gamma(h,k,\omega)\psi \, dm_H(h)\, dm_K(k)\, dm_{\widehat{K}}(\omega). \tag{4.3}$$

(2) *For any right Γ-admissible window function $\psi \in L^2(K)$, we have*

$$f = b_\psi^{-1} \int_H \int_K \int_{\widehat{K}} \mathcal{V}_\psi f(h,k,\omega)\Gamma(h,k,\omega)\psi \, dn_H(h)\, dm_K(k)\, dm_{\widehat{K}}(\omega). \tag{4.4}$$

Proof. (1) Let $\psi \in L^2(K)$ be a left Γ-admissible window function. For $f \in L^2(K)$, define

$$f_{(\psi)} := \int_H \int_K \int_{\widehat{K}} \mathcal{V}_\psi f(h,k,\omega)\Gamma(h,k,\omega)\psi \, dm_H(h)\, dm_K(k)\, dm_{\widehat{K}}(\omega)$$

in $L^2(K)$ in the weak sense. Then, for all $g \in L^2(K)$, we have

$$\begin{aligned}
\langle f_{(\psi)}, g\rangle &= \int_H \int_K \int_{\widehat{K}} \mathcal{V}_\psi f(h,k,\omega)\big\langle \Gamma(h,k,\omega)\psi, g\big\rangle \, dm_H(h)\, dm_K(k)\, dm_{\widehat{K}}(\omega) \\
&= \int_H \int_K \int_{\widehat{K}} \mathcal{V}_\psi f(h,k,\omega)\overline{\mathcal{V}_\psi g(h,k,\omega)} \, dm_H(h)\, dm_K(k)\, dm_{\widehat{K}}(\omega) \\
&= \langle \mathcal{V}_\psi f, \mathcal{V}_\psi g\rangle_{L^2(G_\Theta, m_{G_\Theta})} \\
&= a_\psi \langle f, g\rangle_{L^2(K)}.
\end{aligned}$$

Then $f_{(\psi)} \in L^2(K)$ and $f_{(\psi)} = a_\psi f$ in $L^2(K)$, implying the reconstruction formula (4.3).

(2) The same argument implies (4.4). □

The next theorem can be considered as a criterion for the existence of Γ-admissible functions/vectors.

Theorem 4.5. *Let H be a locally compact group, and let K be an LCA group with dual group $\widehat{K}$. Let $\theta : H \to \mathrm{Aut}(K)$ be a continuous homomorphism. There exists a nonzero left Γ-admissible (resp., right Γ-admissible) in $L^2(K)$ if and only if H is compact.*

Proof. Let H be a compact group. Then it is easy to check that each nonzero function in $L^2(K)$ is both left and right Γ-admissible. Conversely, let $\psi \in L^2(K)$ be a nonzero left Γ-admissible function. Then, by Fubini's theorem, we get $a_\psi = m_H(H)\|\psi\|^2_{L^2(K)}$, and hence we deduce that $m_H(H)$ is finite. Thus H is compact. The same argument works if $\psi \in L^2(K)$ is a nonzero right Γ-admissible function. □

5. Examples

Throughout this section, we will illustrate the application of the abstract theory of wave-packet transforms in the case of some well-known examples.

5.1. **Wave-packet transform on $\mathbb{R}$.** Let $H = \mathbb{R}^+ = (0, +\infty)$, and let $K = \mathbb{R}$. Let $\theta : H \to \mathrm{Aut}(K)$ be given by $a \mapsto \theta_a$, where $\theta_a(x) = ax$ for all $x \in \mathbb{R}$. Then the continuous homomorphism $\delta : H \to (0, \infty)$ is given by $\delta(a) = a^{-1}$ for all $a \in H = \mathbb{R}^+$. Identifying $\widehat{\mathbb{R}}$ with $\mathbb{R}$ via $\omega(x) = \langle x, \omega\rangle = e^{2\pi i\omega x}$ for each $\omega \in \widehat{\mathbb{R}}$ and the continuous homomorphism $\widehat{\theta} : H \to \mathrm{Aut}(\widehat{K})$ is given by $a \mapsto \widehat{\theta}_a$ via the duality notation

$$\begin{aligned}\langle x, \omega_a\rangle &= \left\langle x, \widehat{\theta}_a(\omega)\right\rangle \\ &= \left\langle \theta_{a^{-1}}(x), \omega\right\rangle \\ &= \langle a^{-1}x, \omega\rangle \\ &= e^{2\pi i\omega a^{-1}x}.\end{aligned}$$

Then the continuous homomorphism $\Theta : \mathbb{R}^+ \to \mathrm{Aut}(\mathbb{R} \times \mathbb{R})$ is given by

$$\Theta_a(x, \omega) = \big(\theta_a(x), \widehat{\theta}_a(\omega)\big) = (ax, a^{-1}\omega)$$

for all $a \in \mathbb{R}^+$ and $(x, \omega) \in \mathbb{R} \times \widehat{\mathbb{R}}$. Thus the wave-packet group G_Θ has the underlying manifold

$$\mathbb{R}^+ \times \mathbb{R} \times \widehat{\mathbb{R}} = \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R},$$

which is equipped with the following group law:

$$(a, x, \omega) \rtimes_\Theta (a', x', \omega') = (aa', {a'}^{-1}x + x', a'\omega + \omega')$$

for all $(a, x, \omega), (b, y, \zeta) \in G_\Theta = \mathbb{R}^+ \rtimes_\Theta (\mathbb{R}\times\mathbb{R})$. Then $dm_{G_\Theta}(a, x, \omega) = a^{-1}\, da\, dx\, d\omega$ is a Haar measure for the wave-packet group G_Θ. The wave-packet representation

$$\Gamma : G_\Theta = \mathbb{R}^+ \rtimes_\Theta (\mathbb{R} \times \mathbb{R}) \to \mathcal{U}\big(L^2(\mathbb{R})\big)$$

is given by $\Gamma(a, x, \omega) = D_a T_x M_\omega$ for all $(a, x, \omega) \in G_\Theta$. Let $\psi \in L^2(\mathbb{R})$ be a window function. The wave-packet transform of $f \in L^2(\mathbb{R})$ with respect to the window function ψ is given by

$$\mathcal{V}_\psi f(a, x, \omega) = \big\langle f, \Gamma(a, x, \omega)\psi\big\rangle_{L^2(\mathbb{R})} = \langle f, D_a T_x M_\omega \psi\rangle_{L^2(\mathbb{R})}$$

for all $(a, x, \omega) \in G_\Theta$. In integral terms, we have

$$\mathcal{V}_\psi f(a, x, \omega) = a^{-1/2}\overline{\omega(x)} \int_{-\infty}^{\infty} f(y) e^{2\pi i a^{-1} y\omega} \overline{\psi(a^{-1}y - x)}\, dy.$$

For $\psi \in L^2(\mathbb{R})$, the function $\alpha_\psi : \widehat{\mathbb{R}} \to \mathbb{C}$ for $\omega \in \widehat{\mathbb{R}}$ is given by

$$\alpha_\psi(\omega) = \int_0^\infty \left|D_a\widehat{\psi}(\omega)\right|^2 a^{-1}\, da = \int_0^\infty \left|\widehat{\psi}(a\omega)\right|^2 da.$$

Thus $\psi \in L^2(\mathbb{R})$ is Γ-admissible if and only if $\alpha_\psi \in L^1(\widehat{\mathbb{R}})$, which means that

$$a_\psi := \int_{-\infty}^{+\infty} \alpha_\psi(\omega)\, d\omega = \int_{-\infty}^{+\infty} \int_0^\infty \left|\widehat{\psi}(a\omega)\right|^2 da\, d\omega < \infty.$$

Then we deduce that any nonzero $\psi \in L^2(\mathbb{R})$ is not Γ-admissible.

5.2. **Wave-packet transform on $\mathbb{R}^n$, $n > 1$.** Let $n \in \mathbb{N}$ with $n > 1$. Let $H = \mathrm{SO}(n)$, and let $K = \mathbb{R}^n$. Let $\theta : H \to \mathrm{Aut}(K)$ be given by $A \mapsto \theta_A$, where $\theta_A(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Since H is compact, the continuous homomorphism $\delta : H \to (0, \infty)$ is 1. Identifying $\widehat{\mathbb{R}^n}$ with $\mathbb{R}^n$ via $\mathbf{w}(\mathbf{x}) = \langle \mathbf{x}, \mathbf{w}\rangle = e^{2\pi i \mathbf{w}.\mathbf{x}}$ for each $\mathbf{w} \in \widehat{\mathbb{R}^n}$ and the continuous homomorphism $\widehat{\theta} : H \to \mathrm{Aut}(\widehat{K})$ is given by $A \mapsto \widehat{\theta}_A$ via the duality notation

$$\begin{aligned}\langle \mathbf{x}, \mathbf{w}_A\rangle &= \big\langle \mathbf{x}, \widehat{\theta}_A(\mathbf{w})\big\rangle \\ &= \big\langle \theta_{A^{-1}}(\mathbf{x}), \mathbf{w}\big\rangle \\ &= \langle A^{-1}\mathbf{x}, \mathbf{w}\rangle \\ &= e^{2\pi i \mathbf{w}.A^{-1}\mathbf{x}} \\ &= e^{2\pi i \mathbf{w}.A^{*}\mathbf{x}}.\end{aligned}$$

Then the continuous homomorphism $\Theta : \mathrm{SO}(n) \to \mathrm{Aut}(\mathbb{R}^n \times \widehat{\mathbb{R}^n})$ is given by

$$\Theta_A(\mathbf{x}, \mathbf{w}) = \big(\theta_A(\mathbf{x}), \widehat{\theta}_A(\mathbf{w})\big) = (A\mathbf{x}, A^{-1}\mathbf{w})$$

for all $A \in \mathrm{SO}(n)$ and $(\mathbf{x}, \mathbf{w}) \in \mathbb{R}^n \times \widehat{\mathbb{R}^n}$. Thus the wave-packet group G_Θ has the underlying manifold

$$\mathrm{SO}(n) \times \mathbb{R}^n \times \widehat{\mathbb{R}^n} = \mathrm{SO}(n) \times \mathbb{R}^n \times \mathbb{R}^n,$$

which is equipped with the following group law:

$$(A, \mathbf{x}, \mathbf{w}) \rtimes_\Theta (A', \mathbf{x}', \mathbf{w}') = (AA', {A'}^{-1}\mathbf{x} + \mathbf{x}', A'\mathbf{w} + \mathbf{w}')$$

for all $(A, \mathbf{x}, \mathbf{w}), (A', \mathbf{x}', \mathbf{w}') \in G_\Theta = \mathrm{SO}(n) \rtimes_\Theta (\mathbb{R}^n \times \mathbb{R}^n)$. Then $dm_{G_\Theta}(A, \mathbf{x}, \mathbf{w}) = dA\, d\mathbf{x}\, d\mathbf{w}$ is a Haar measure for the wave-packet group G_Θ. The wave-packet representation

$$\Gamma : G_\Theta = \mathrm{SO}(n) \rtimes_\Theta (\mathbb{R}^n \times \mathbb{R}^n) \to \mathcal{U}\big(L^2(\mathbb{R}^n)\big)$$

is given by $\Gamma(A, \mathbf{x}, \mathbf{w}) = D_A T_{\mathbf{x}} M_{\mathbf{w}}$ for all $(A, \mathbf{x}, \mathbf{w}) \in G_\Theta$. Let $\psi \in L^2(\mathbb{R}^n)$ be a window function. The wave-packet transform of $f \in L^2(\mathbb{R}^n)$ with respect to the window function ψ is given by

$$\mathcal{V}_\psi f(A, \mathbf{x}, \mathbf{w}) = \big\langle f, \Gamma(A, \mathbf{x}, \mathbf{w})\psi\big\rangle_{L^2(\mathbb{R}^n)} = \langle f, D_A T_{\mathbf{x}} M_{\mathbf{w}}\psi\rangle_{L^2(\mathbb{R}^n)}$$

for all $(A, \mathbf{x}, \mathbf{w}) \in G_\Theta$. In integral terms, we have

$$\mathcal{V}_\psi f(A, \mathbf{x}, \mathbf{w}) = \overline{\mathbf{w}(\mathbf{x})} \int_{\mathbb{R}^n} f(\mathbf{y}) e^{2\pi i \mathbf{w}.A^{-1}\mathbf{y}} \overline{\psi(A^{-1}\mathbf{y} - \mathbf{x})}\, d\mathbf{y}.$$

For $\psi \in L^2(\mathbb{R}^n)$, the function $\alpha_\psi : \widehat{\mathbb{R}^n} \to \mathbb{C}$ for $\mathbf{w} \in \widehat{\mathbb{R}^n}$ is given by

$$\alpha_\psi(\mathbf{w}) = \int_{\mathrm{SO}(n)} \big|D_A \widehat{\psi}(\mathbf{w})\big|^2\, dA = \int_{\mathrm{SO}(n)} \big|\widehat{\psi}(A\mathbf{w})\big|^2\, dA.$$

Thus $\psi \in L^2(\mathbb{R}^n)$ is Γ-admissible if and only if $\alpha_\psi \in L^1(\widehat{\mathbb{R}^n})$, which means that

$$a_\psi := \int_{\mathbb{R}^n} \alpha_\psi(\mathbf{w})\, d\mathbf{w} = \int_{\mathbb{R}^n} \int_{\mathrm{SO}(n)} \big|\widehat{\psi}(A\mathbf{w})\big|^2\, dA\, d\mathbf{w} < \infty.$$

Theorem 4.2 guarantees the following Plancherel formula:

$$\int_{\mathrm{SO}(n)}\int_{\mathbb{R}^n}\int_{\widehat{\mathbb{R}^n}}\left|\left\langle f,\Gamma(A,\mathbf{x},\mathbf{w})\psi\right\rangle_{L^2(\mathbb{R}^n)}\right|^2 dA\,d\mathbf{x}\,d\mathbf{w}=a_\psi\int_{\mathbb{R}^n}\left|f(\mathbf{y})\right|^2 d\mathbf{w};$$

or, equivalently, it guarantees the following reconstruction formula in $L^2(\mathbb{R})$:

$$f=a_\psi^{-1}\int_{\mathrm{SO}(n)}\int_{\mathbb{R}^n}\int_{\widehat{\mathbb{R}^n}}\mathcal{V}_\psi f(A,\mathbf{x},\mathbf{w})\Gamma(A,\mathbf{x},\mathbf{w})\psi\,dA\,d\mathbf{x}\,d\mathbf{w}.$$

5.3. **Wave-packet transform on finite cyclic groups.** Let N be a positive integer, and let $K=\mathbb{Z}_N$. Then K is a finite abelian additive group with respect to addition module N and the counting measure as the Haar measure. Evidently, in this case we have $\mathbb{C}^N=L^2(K)$. The character group $\widehat{\mathbb{Z}_N}$ of $\mathbb{Z}_N$ is isomorphic with $\mathbb{Z}_N$ via the group isomorphism $\ell\mapsto\omega_\ell$, where $\omega_\ell(k)=e^{2\pi i\ell k/N}$. Let $H:=\mathrm{Aut}(\mathbb{Z}_N)$. Then $H=\{m\in\mathbb{Z}_N:1\le m\le N,\gcd(m,N)=1\}$, and also H is a finite abelian multiplicative group of order $\varphi(N)$ with respect to multiplication module N and the counting measure as the Haar measure, where φ is Euler's totient function. For $m\in H$, define $\theta_m:K\to K$ by $\theta_m(s)=ms$ for all $s\in K$. Then $\theta_m\in\mathrm{Aut}(K)$, and $\theta:H\to\mathrm{Aut}(K)$ given by $m\mapsto\theta_m$ is a well-defined homomorphism. For $m\in H$, the dilation operator $D_m:\mathbb{C}^N\to\mathbb{C}^N$ is given by $D_m\mathbf{x}(s)=\mathbf{x}(m_Ns)$ for all $\mathbf{x}\in\mathbb{C}^N$ and $0\le s\le N-1$, where m_N is the multiplicative inverse of $m\in H$ (i.e., an element $m_N\in H$ with $mm_N\overset{N}{\equiv}m_Nm\overset{N}{\equiv}1$) which satisfies $m_Nm+nN=1$ for some $n\in\mathbb{Z}$, which can be done by the Bezout lemma. Then the finite wave-packet group G_Θ is the underlying set

$$H\times\{0,1,\dots,N-1\}\times\{0,1,\dots,N-1\}$$

equipped with the following group operations:

$$(m,k,\ell)\rtimes(m',k',\ell')=(mm',m'_Nk+k',m'\ell+\ell')\quad\text{for }(m,k,\ell),(m',k',\ell')\in G_\Theta,$$
$$(m,k,\ell)^{-1}=\big(m_N,m\cdot(N-k),m_N\cdot(N-\ell)\big)\quad\text{for }(m,k,\ell)\in G_\Theta,$$

where m_N,m'_N are multiplicative inverses of m,m' in H. Then G_Θ is a finite nonabelian group of order $N^2\varphi(N)$ with the identity element $(1,0,0)$. The map $\Gamma:G_\Theta\to\mathcal{U}(\mathbb{C}^N)$ defined by $(m,k,\ell)\mapsto\Gamma(m,k,\ell)=D_mT_kM_\ell$ is the wave-packet representation of the finite wavelet-packet group G_Θ on the finite-dimensional Hilbert space $\mathbb{C}^N$ (see [14], [12]). Let $\mathbf{y}\in\mathbb{C}^N$ be a window signal. Then the wave-packet transform of a given function/vector $\mathbf{x}\in\mathbb{C}^N$ with respect to the window vector $\mathbf{y}$ is a vector defined on the wave-packet group G_Θ by

$$\mathcal{V}_{\mathbf{y}}\mathbf{x}(m,k,\ell)=\sum_{s=0}^{N-1}\mathbf{x}(s)e^{2\pi i\ell(m_Ns-k)/N}\overline{\mathbf{y}(m_Ns-k)}\quad\text{for }(m,k,\ell)\in G_\Theta.$$

Let $\mathbf{y}\in\mathbb{C}^N$ be a window vector. Then $\alpha_{\mathbf{y}}:\widehat{\mathbb{Z}_N}\to\mathbb{C}$ given by

$$\alpha_{\mathbf{y}}(\omega)=\sum_{m\in H}\left|D_m\widehat{\mathbf{y}}(\omega)\right|^2\quad\text{for }\omega\in\widehat{\mathbb{Z}_N}$$

belongs to $\mathbb{C}^N = L^1(\widehat{\mathbb{Z}_N})$. Hence we achieve

$$\begin{aligned}
a_{\mathbf{y}} &= \sum_{\omega=0}^{N-1} \alpha_{\mathbf{y}}(\omega) \\
&= \sum_{\omega=0}^{N-1} \sum_{m\in H} \left|D_m \widehat{\mathbf{y}}(\omega)\right|^2 \\
&= \sum_{m\in H} \sum_{\omega=0}^{N-1} \left|D_m \widehat{\mathbf{y}}(\omega)\right|^2 \\
&= \sum_{m\in H} \|D_m \widehat{\mathbf{y}}\|_2^2 \\
&= \sum_{m\in H} \|\widehat{\mathbf{y}}\|_2^2 \\
&= \varphi(N) \cdot \|\widehat{\mathbf{y}}\|_2^2 \\
&= \varphi(N) \cdot N \cdot \|\mathbf{y}\|_2^2.
\end{aligned}$$

Thus, if $\mathbf{x} \in \mathbb{C}^N$, then we have

$$\sum_{m\in H} \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-1} \left|\mathcal{V}_{\mathbf{y}}\mathbf{x}(m,k,\ell)\right|^2 = N \cdot \varphi(N) \cdot \|\mathbf{y}\|_2^2 \|\mathbf{x}\|_2^2,$$

and also

$$\mathbf{x}(s) = \frac{\|\mathbf{y}\|_2^{-2}}{N \cdot \varphi(N)} \sum_{m\in H} \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-1} \mathcal{V}_{\mathbf{y}}\mathbf{x}(m,k,\ell) D_m T_k M_\ell \mathbf{y}(s) \quad \text{for } 0 \le s \le N-1.$$

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