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SHARP WEIGHTED BOUNDS FOR FRACTIONAL INTEGRALS VIA THE TWO-WEIGHT THEORY

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Abstract. We derive sharp weighted norm estimates for positive kernel operators on spaces of homogeneous type. Similar problems are studied for one-sided fractional integrals. Bounds of weighted norms are of mixed type. The problems are studied using the two-weight theory of positive kernel operators. As special cases, we derive sharp weighted estimates in terms of Muckenhoupt characteristics.

1. Introduction

The focus of this article is on mixed-type weighted bounds for fractional integrals on *spaces of homogeneous type* (SHT). Our main results are obtained by means of the two-weight theory of integral operators with positive kernels. We investigate a similar problem for one-sided fractional integrals. In the latter case, sharp bounds involve one-sided A_∞ characteristics of weights. Let X and Y be two Banach spaces. Given a bounded operator $T : X \to Y$, we denote the operator norm by $\|T\|_{X\to Y}$ which is defined in the standard way, that is, $\sup_{\|f\|_X\le 1} \|Tf\|_Y$. If $X = Y$, then we use the symbol $\|T\|_X$. A nonnegative locally integrable function w defined on $\mathbb{R}^n$ is said to satisfy $A_p(\mathbb{R}^n)$ condition ($w \in A_p(\mathbb{R}^n)$) for $1 < p < \infty$

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if

$$\|w\|_{A_p(\mathbb{R}^n)} := \sup_Q \Big(\frac{1}{|Q|}\int_Q w(x)\,dx\Big)\Big(\frac{1}{|Q|}\int_Q w(x)^{1-p'}\,dx\Big)^{p-1} < \infty,$$

where $p' = \frac{p}{p-1}$, and the supremum is taken over all cubes Q in $\mathbb{R}^n$ with sides parallel to the coordinate axes. We call $\|w\|_{A_p(\mathbb{R}^n)}$ the A_p *characteristic* of w.

In 1972, Muckenhoupt [21] showed that if $w \in A_p(\mathbb{R}^n)$, where $1 < p < \infty$, then the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q \big|f(y)\big|\,dy$$

is bounded in $L^p_w(\mathbb{R}^n)$. Buckley [2] investigated the sharp A_p bound for the operator M and established the inequality

$$\|M\|_{L^p_w(\mathbb{R}^n)} \le C\|w\|^{\frac{1}{p-1}}_{A_p(\mathbb{R}^n)}, \quad 1 < p < \infty. \tag{1.1}$$

Moreover, he showed that the exponent $\frac{1}{p-1}$ is best possible in the sense that we cannot replace $\|w\|^{\frac{1}{p-1}}_{A_p}$ by $\psi(\|w\|_{A_p})$ for any positive nondecreasing function ψ growing slower than $x^{\frac{1}{p-1}}$. From here it follows that for any $\lambda > 0$,

$$\sup_{w\in A_p} \frac{\|M\|_{L^p_w} \to L^p_w}{\|w\|^{\frac{1}{p-1}-\lambda}_{A_p}} = \infty.$$

In 1974, Muckenhoupt and Wheeden [22] found a necessary and sufficient condition for the one-weight inequality; namely, they proved that the Riesz potential I_α defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy,$$

or the fractional maximal operator defined by

$$M_\alpha f(x) = \sup_{x\in Q} \frac{1}{|Q|^{1-\frac{\alpha}{n}}}\int_Q \big|f(y)\big|\,dy,$$

is bounded from $L^p_{w^p}(\mathbb{R}^n)$ to $L^q_{w^q}(\mathbb{R}^n)$, where $1 < p < \infty$, $0 < \alpha < n/p$, $q = \frac{np}{n-\alpha p}$ if and only if w satisfies the so-called $A_{p,q}(\mathbb{R}^n)$ *condition* (see the definition below). Moreover, from their result it follows that there is a positive constant c depending only on p and α such that

$$\|K_\alpha\|_{L^p_{w^p}\to L^q_{w^q}} \le c\|w\|^{\beta}_{A_{p,q}}, \tag{1.2}$$

for some positive exponent β, where K_α is either I_α or M_α, and $\|w\|_{A_{p,q}}$ is the $A_{p,q}$ characteristic of w. In [15], Lacey, Moen, Pérez, and Torres proved that the best possible value of β in (1.2) is $(1-\alpha/n)p'/q$ (resp., $(1-\alpha/n)\max\{1, p'/q\}$) for M_α (resp., for I_α).

A two-weight characterization for fractional integrals was given by Sawyer [23], [24] under conditions involving the operator itself. The study of the same problem for Riesz potentials in terms of capacities is due to Maz'ya [20]. Kokilashvili and Krbec [13, Theorem 6.2.4] gave two-weight criteria for the Riesz potential I_α

under more transparent conditions. Using that result, Wheeden [29] characterized the two-weight norm inequality for the fractional maximal operator M_α. Similar problems for integral operators with positive kernels on SHT were studied in [25] and [7] under different types of conditions (see also [27], [26], [11] for the Sawyer-type result on SHT without any additional geometric conditions on SHT).

Cruz-Uribe and Moen [3] obtained the following results for the Riesz potentials and fractional maximal operators defined on $\mathbb{R}^n$.

Theorem 1.1 ([3, Theorems 2.1, 2.3])**.** *Given $0 < \alpha < n$ and $1 < p \le q < \infty$, suppose that $[v, u] \in A^\alpha_{p,q}(\mathbb{R}^n)$ and $u^{1-p'} \in A_\infty(\mathbb{R}^n)$. Then*

$$\|M_\alpha\|_{L^p_u \to L^q_v} \le c[v,u]_{A^\alpha_{p,q}(\mathbb{R}^n) A^{\exp}_\infty (u^{1-p'})^{1/q}} \tag{1.3}$$

and

$$\|M_\alpha\|_{L^p_u \to L^q_v} \le c[v,u]_{A_{p,q}(\mathbb{R}^n)} \|u^{1-p'}\|^{1/q}_{A^M_\infty(\mathbb{R}^n)}, \tag{1.4}$$

where

$$\begin{aligned}
[v,u]_{A^\alpha_{p,q}(\mathbb{R}^n)A^{\exp}_\infty(u)} &:= \sup_Q A^\alpha_{p,q}(v,u,Q) A^{\exp}_\infty(u,Q),\\
[v,u]_{A^\alpha_{p,q}(\mathbb{R}^n)} &:= \sup_Q A^\alpha_{p,q}(v,u,Q)\\
&:= \sup_Q |Q|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}} \Big(\frac{1}{|Q|}\int_Q v\Big)^{1/q} \Big(\frac{1}{|Q|}\int_Q u^{1-p'}\Big)^{1/p'},\\
\|\omega\|_{A^M_\infty(\mathbb{R}^n)} &:= \sup_Q A^M_\infty(\omega,Q) := \sup_Q \frac{1}{\omega(Q)} \int_Q M(\omega\chi_Q)(x)\,dx,\\
\|\omega\|_{A^{\exp}_\infty(\mathbb{R}^n)} &:= \sup_Q A^{\exp}_\infty(\omega,Q)\\
&:= \sup_Q \Big(\frac{1}{|Q|}\int_Q \omega(x)\,dx\Big) \exp\Big(-\frac{1}{|Q|}\int_Q \log\omega(x)\,dx\Big),
\end{aligned}$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ with sides parallel to the coordinate axes.

Theorem 1.2. *Given $0 < \alpha < n$ and $1 < p < q < \infty$, suppose that $[v, u] \in A^\alpha_{p,q}$ and $u^{1-p'}, v \in A_\infty$. Then*

$$\|I_\alpha\|_{L^p_u \to L^q_v} \le c[v,u]_{A^\alpha_{p,q}(\mathbb{R}^n) A^{\exp}_\infty(u^{1-p'})^{1/q}} + [v,u]_{A^\alpha_{q',p'}(\mathbb{R}^n) A^{\exp}_\infty(v)^{1/p'}} \tag{1.5}$$

and

$$\|I_\alpha\|_{L^p_u \to L^q_v} \le c[v, u^{1-p'}]_{A_{p,q}(\mathbb{R}^n)} \big(\|v\|^{1/p'}_{A^M_\infty(\mathbb{R}^n)} + \|u^{1-p'}\|^{1/q}_{A^M_\infty(\mathbb{R}^n)}\big), \tag{1.6}$$

where the characteristics $A^\alpha_{p,q} A^{\exp}_\infty$ and A^M_∞ are defined in the preceding theorem.

These results, in particular, imply the Buckley-type theorems for fractional integrals established in [15]. Regarding the sharp one-weighted estimate for the fractional integral operator

$$T_\alpha f(x) := \int_X \frac{f(y)}{\mu(B(x, d(x,y)))^{1-\alpha}}\, d\mu(y), \quad 0 < \alpha < 1, \tag{1.7}$$

defined on an SHT, we have the following theorem due to Kairema [12].

Theorem 1.3 ([12, Proposition 5.5, Theorem 7.1]). *Suppose that* (X, d, μ) *is an SHT. Let* $0 < \alpha < 1$, *and let* $1 < p < 1/\alpha$. *We set* $1/p - 1/q = \alpha$. *Let* $w \in A_{p,q}(X)$. *Then*

(a)

$$\|T_\alpha\|_{L^p_{w^p}(X) \to L^{q,\infty}_{w^q}(X)} \le c\|w\|^{1-\alpha}_{A_{p,q}(X)};$$

(b)

$$\|T_\alpha\|_{L^p_{w^p}(X) \to L^q_{w^q}(X)} \le c\|w\|^{(1-\alpha)\max\{1, \frac{p'}{q}\}}_{A_{p,q}(X)}.$$

Furthermore, these estimates are sharp in the sense that $1 - \alpha$ *and* $(1 - \alpha) \times \max\{1, \frac{p'}{q}\}$ *are best possible in* (a) *and* (b), *respectively.*

In this theorem, $\|w\|_{A_{p,q}(X)}$ is the $A_{p,q}$ characteristic of the weight w defined on X (see the definition below).

Mixed $(A_p - A_\infty)$-type estimates for Calderón–Zygmund operators were established by Lerner and Moen in [16]. Recently, Hytönen, Pérez, and Rela [10] derived mixed $(A_p - A_\infty)$-type estimates for the Hardy–Littlewood maximal operator

$$M^\mu f(x) := \sup_{x \in B} \frac{1}{\mu(B)} \int_B \big|f(t)\big|\, d\mu(t)$$

defined on an SHT. In particular, they showed that

$$\|M^\mu\|_{L^p_w(X)} \le C\|w\|_{A_p(X)}\|w^{1-p'}\|_{A^M_\infty(X)}, \quad 1 < p < \infty, \tag{1.8}$$

holds, where $\|\omega\|_{A^M_\infty(X)}$ is one of the A_∞ characteristics of a weight ω (see the definition below).

The article is organized as follows. In Section 2, we give definitions of the space of homogeneous type and some characteristics of weights. We also discuss well-known two-weight criteria for positive kernel operators. Section 3 is devoted to the mixed-type weighted sharp bounds for kernel operators and fractional integrals defined on spaces of homogeneous type. In Section 4, we derive weighted sharp bounds for one-sided fractional integrals involving one-sided A_∞ characteristics of weights.

By the symbol $A \approx B$, we mean that there are positive constants c_1 and c_2 (depending on appropriate parameters) such that $c_1 A \le B \le c_2 A$; $A \ll B$ means that there is a positive constant c such that $A \le cB$. For a weight function ρ, $\rho(E) := \int_E \rho(x)\, dx$; constants (often different constants) will be denoted by c or C. We denote $N_0 := a_1(1 + 2a_0)$, where a_0 and a_1 are constants from the definition of a quasimetric (see the definition below in Preliminaries).

2. Preliminaries

Let (X, d, μ) be a quasimetric measure space. A quasimetric d is a function $d : X \times X \to [0, \infty)$ which satisfies the following conditions:

(a) $d(x, y) = 0$ for all $x \in X$;
(b) $d(x, y) > 0$ for all $x \ne y$, $x, y \in X$;
(c) there is a constant $a_0 > 0$ such that $d(x, y) < a_0 d(y, x)$ for all $x, y \in X$;

(d) there is a constant $a_1 > 0$ such that $d(x,y) < a_1(d(x,z) + d(z,y))$ for all $x, y, z \in X$.

Let $d_X = \operatorname{diam}(X) = \sup\{d(x,y) : x, y \in X\}$. Let $B(x,r) = \{y \in X : d(x,y) < r\}$ be the ball with center x and radius r. The dilation of a ball $B(x, \lambda r)$ with $\lambda > 0$ will be denoted by λB. Throughout this article, we will assume that $\mu\{x\} = 0$ for all $x \in X$. A measure μ is said to *satisfy the doubling condition* ($\mu \in DC(X)$) if there is a constant $D_\mu > 0$ such that

$$\mu B(x, 2r) \leq D_\mu \mu B(x,r)$$

for every $x \in X$ and every $0 < r < \infty$.

Definition 2.1. A *space of homogeneous type* (SHT) is the triple (X, d, μ), where X is a set, d is a quasimetric on X, and μ is a doubling measure.

Throughout this article, we will assume that $B(x, r_2) \backslash B(x, r_1) \neq \phi$ for any $x \in X$ and for all r_1, r_2 with $0 < r_1 < r_2 < d_X$.

Definition 2.2. A measure μ *satisfies the reverse doubling condition* ($\mu \in RD(X)$) if there exist constants $\theta > 1$ and $\eta > 1$ such that

$$\mu\big(B(x, \theta r)\big) \geq \eta \mu\big(B(x,r)\big)$$

for any $x \in X$ and for all $r > 0$.

Remark 2.3. It is known that if $\mu \in DC(X)$, then $\mu \in RD(X)$ (see, e.g., [28, p. 11, Lemma 20]).

Let ω be a weight on X; that is, it is μ almost everywhere positive and locally integrable on X. We denote by $L^p_\omega(X)$, $1 < p < \infty$, the set of all measurable functions $f : X \to \mathbb{R}$ for which the norm

$$\|f\|_{L^p_\omega(X)} = \Big(\int_X \big|f(x)\big|^p \omega(x)\, d\mu(x)\Big)^{\frac{1}{p}}$$

is finite. Suppose that $L^{p,\infty}_w(X)$ is the weighted weak Lebesgue space with respect to the quasinorm

$$\|f\|_{L^{p,\infty}_w(X)} = \sup_{\lambda > 0} \lambda \big[w\big(\big\{x \in X : \big|f(x)\big| > \lambda\big\}\big)\big]^{1/p},$$

where $w(E) := \int_E w\, d\mu$ for all E measurable subsets of X.

Now we give the definitions of A_p and $A_{p,q}$ classes of weights defined on an SHT.

Definition 2.4. Let $1 < p < \infty$. We say that $w \in A_p(X)$ if

$$\|w\|_{A_p(X)} := \sup_B A_p(w,B) = \sup_B \Big(\frac{1}{\mu(B)} \int_B w\, d\mu\Big)\Big(\frac{1}{\mu(B)} \int_B w^{1-p'}\, d\mu\Big)^{p-1} < \infty.$$

Definition 2.5. Let $1 < p, q < \infty$. We say that $w \in A_{p,q}(X)$ if

$$\|w\|_{A_{p,q}(X)} := \sup_B A_{p,q}(w,B) = \sup_B \Big(\frac{1}{\mu(B)} \int_B w^q\, d\mu\Big)\Big(\frac{1}{\mu(B)} \int_B w^{-p'}\, d\mu\Big)^{q/p'} < \infty.$$

By definition, the class of weights $A_\infty(X)$ is defined as $A_\infty(X) = \bigcup_{p\ge 1} A_p(X)$. For an $A_\infty(X)$ weight, it is defined as the following characteristic due to [8].

Definition 2.6. For a weight $w \in A_\infty(X)$, we define

$$\begin{aligned}\|w\|_{A_\infty^{\exp}(X)} &:= \sup_B A_\infty^{\exp}(w,B)\\ &= \sup_B \Big(\frac{1}{\mu(B)}\int_B w\,d\mu\Big)\exp\Big(\frac{1}{\mu(B)}\int_B \log w^{-1}\,d\mu\Big) < \infty.\end{aligned}$$

Another type of A_∞ characteristic in terms of the maximal function was originally introduced by Fujii [6] and later investigated by Wilson [30]:

$$\|w\|_{A_\infty^M(X)} := \sup_B A_\infty^M(w,B) = \sup_B \Big(\frac{1}{w(B)}\int_B M^\mu(w\chi_B)(x)\,d\mu\Big) < \infty.$$

Furthermore, it is also known that if $X = \mathbb{R}^n$, then $\|w\|_{A_\infty^M(X)} \ll \|w\|_{A_\infty^{\exp}(X)}$ (see, e.g., [10]).

For $0<\alpha<1$, the fractional maximal operator is defined as

$$M_\alpha^\mu f(x) := \sup_{x\in B} \frac{1}{\mu(B)^{1-\alpha}}\int_B |f(t)|\,d\mu(t).$$

The two-weight characterization for M_α^μ was proved in [29, Theorem 1] for Euclidean spaces and in [7, p. 158] for an SHT.

Theorem 2.7. *Let $1<p<q<\infty$, and let $0<\alpha<1$. Suppose that v and w are weights on X. Then*

$$\|M_\alpha^\mu f\|_{L_v^q(X)} \le c\|f\|_{L_u^p(X)} \tag{2.1}$$

if and only if

$$\begin{aligned} A_{\mathrm{GK}}(v,u,\alpha,p,q) := &\sup_{x\in X, r>0} \big(u^{1-p'}\big(B(x,2N_0 r)\big)\big)^{1/p'}\\ &\times \Big(\int_{X\setminus B(x,r)} \mu\big(B\big(x,d(x,y)\big)\big)^{(\alpha-1)q} v(y)\,d\mu(y)\Big)^{1/q}\\ &< \infty. \end{aligned}\tag{2.2}$$

Moreover,

$$\|M_\alpha^\mu\|_{L_u^p\to L_v^q} \approx A_{\mathrm{GK}}(v,u,\alpha,p,q)$$

with constants depending only on p, q, and α.

Now we define a class of kernels on X^2 (see [25], [7, Chapter 3]).

Definition 2.8. We say that a positive measurable kernel $k : X^2 \to \mathbb{R}$ *belongs* to V $(k\in V)$ if there exists $c>0$ such that

$$k(x,y) \le ck(x',y)$$

for all x,y and for x' in X such that $d(x,x') \le 2N_0 d(x,y)$.

Consider the positive kernel operator defined on X

$$Kf(x) = \int_X k(x,y) f(y)\, d\mu(y), \quad x \in X.$$

Let

$$K^*f(x) = \int_X k^*(x,y) f(y)\, d\mu(y), \quad x \in X,$$

where $k^*(x,y) = k(y,x)$.

The following two-weight characterization for the positive kernel operator on an SHT was given in Chapter 3 of [7].

Theorem 2.9. *Let $1 < p < q < \infty$, and let both $k, k^* \in V$. Then there exists $c > 0$ such that*

$$\|Kf\|_{L^q_v(X)} \le c\|f\|_{L^p_u(X)} \tag{2.3}$$

holds if and only if

(i)

$$B(v,u,k,p,q) := \sup_{x\in X, r>0} \big(v\big(B(x,2N_0 r)\big)\big)^{1/q} \Big(\int_{X\setminus B(x,r)} k^{p'}(x,y) u^{1-p'}(y)\, d\mu(y)\Big)^{1/p'} < \infty; \tag{2.4}$$

(ii)

$$D(v,u,k,p,q) := \sup_{x\in X, r>0} \big(u^{1-p'}\big(B(x,2N_0 r)\big)\big)^{1/p'} \Big(\int_{X\setminus B(x,r)} k^{q}(y,x) v(y)\, d\mu(y)\Big)^{1/q} < \infty,$$

where N_0 is the constant depending only on quasimetric constants a_0 and a_1. Moreover,

$$\|K\|_{L^p_u(X)\to L^q_v(X)} \approx B(v,u,k,p,q) + D(v,u,k,p,q). \tag{2.5}$$

The following two-weight characterization for the positive kernel operator on an SHT was also given in Chapter 3 of [7].

Theorem 2.10. *Let $1 < p < q < \infty$, and let $k \in V$. Then there exists $c > 0$ such that*

$$\|Kf\|_{L^{q,\infty}_v(X)} \le c\|f\|_{L^p_u(X)} \tag{2.6}$$

holds if and only if (2.4) holds. Moreover,

$$\|K\|_{L^p_u(X)\to L^{q,\infty}_v(X)} \approx B(v,u,k,p,q). \tag{2.7}$$

The next remark follows from Theorems 2.9 and 2.10.

Remark 2.11. Let the conditions of Theorem 2.9 be satisfied. Then

$$\|K\|_{L^p_u(X)\to L^q_v(X)} \approx \|K\|_{L^p_u(X)\to L^{q,\infty}_v(X)} + \|K^*\|_{L^{q'}_{v^{1-q'}}(X)\to L^{p',\infty}_{u^{1-p'}}(X)}.$$

Let k be a positive kernel. Then by the symbols $\phi(B)$ and $\phi^*(B)$, we denote (see [25], [7, Chapter 3])

$$\phi(B) := \sup\big\{k(x,y) : x, y \in B, d(x,y) \geq cr(B)\big\},$$

$$\phi^*(B) := \sup\big\{k^*(x,y) : x, y \in B, d(x,y) \geq cr(B)\big\},$$

respectively, where $r(B)$ is the radius of the ball B, and c is a sufficiently small positive constant depending on a_1.

Let us denote

$$\begin{aligned}[v,u]_{A^{\phi}_{p,q}(X)} &:= \sup_B A^{\phi}_{p,q}(v,u,B)\\ &= \sup_B \phi(B)\Big(\int_B v(x)\,d\mu(x)\Big)^{1/q}\Big(\int_B u^{1-p'}(x)\,d\mu(x)\Big)^{1/p'}.\end{aligned}$$

It is easy to see that $[v,u]_{A^{\phi}_{p,q}(X)} = [u^{1-p'}, v^{1-q'}]_{A^{\phi}_{q',p'}(X)}$. Furthermore, if $\phi(B) = \mu(B)^{\alpha-1}$, then we denote

$$[v,u]_{A^{\phi}_{p,q}(X)} = [v,u]_{A^{\alpha}_{p,q}(X)}.$$

3. Integral operators on an SHT

Initially we prove some lemmas similar to those in [15] (see also [14]).

Lemma 3.1. *Let $1 < r < \infty$. Suppose that $\omega \in A_r(X)$. Let θ and η be as in Definition 2.2. Then for any balls $B \subset X$, the estimate*

$$\frac{\omega(B)}{\omega(\theta B)} \leq 1 - c_{\eta,r}\|w\|^{-1}_{A_r(X)} \tag{3.1}$$

holds for a constant $c_{\eta,r}$ depending only on η and r.

Proof. Let $E \subset \theta B$. We show that

$$\Big(\frac{\mu(E)}{\mu(\theta B)}\Big)^r\big(A_r(\omega,\theta B)\big)^{-1} \leq \frac{\omega(E)}{\omega(\theta B)} \tag{3.2}$$

holds. Indeed, denoting $\sigma = \omega^{1-r'}$ and using Hölder's inequality, we see that

$$\begin{aligned}\frac{\mu(E)}{\mu(\theta B)} &= \frac{\int_E \omega\omega^{-1}\,d\mu}{\mu(\theta B)}\\ &\leq \Big[\frac{\omega(E)}{\mu(\theta B)}\Big]^{1/r}\Big[\frac{\sigma(E)}{\mu(\theta B)}\Big]^{1/r'}\\ &\leq \Big[\frac{\omega(E)}{\omega(\theta B)}\Big]^{1/r}\Big[\frac{\omega(\theta B)}{\mu(\theta B)}\Big]^{1/r}\Big[\frac{\sigma(\theta B)}{\mu(\theta B)}\Big]^{1/r'}\\ &\leq \Big[\frac{\omega(E)}{\omega(\theta B)}\Big]^{1/r}\big(A_r(\omega,\theta B)\big)^{1/r}\\ &\leq \Big[\frac{\omega(E)}{\omega(\theta B)}\Big]^{1/r}\|w\|^{1/r}_{A_r(X)}.\end{aligned}$$

Since μ satisfies the doubling condition, then by Remark 2.3 we have $\mu(B) \leq \frac{1}{\eta}\mu(\theta B)$. Taking $E = \theta B - B$ in (3.2), we get the required result with $c_{\eta,r} = (1 - 1/\eta)^r$. □

Lemma 3.2. *Let $1 < p \leq q < \infty$, and let $s > 1$. Furthermore, let $v \in A_s(X)$ and $k \in V$. Then the following estimate*

$$B(v, u, k, p, q) \ll [v, u]_{A^{\phi}_{p,q}(X)} \|v\|^{1/p'}_{A_s(X)}$$

holds.

Proof. Let $s > 0$, and let θ be the same as in Definition 2.2. By Lemma 3.1 we have

$$\begin{aligned}
&\big(v\big(B(x, N_0 r)\big)\big)^{1/q}\Big(\int_{X\setminus B(x,r)} k^{p'}(x,y)u^{1-p'}(y)\,d\mu(y)\Big)^{1/p'} \\
&= c\big(v\big(B(x, N_0 r)\big)\big)^{1/q}\Big(\sum_{k=0}^{\infty}\int_{B(x,\theta^k N_0 r)\setminus B(x,\theta^{k-1}N_0 r)} k^{p'}(x,y)u^{1-p'}(y)\,d\mu(y)\Big)^{1/p'} \\
&= c\Big[\sum_{k=0}^{\infty}\Big(\frac{v(B(x,N_0 r))}{v(B(x,\theta^k N_0 r)}\Big)^{p'/q}\phi\big(B(x,\theta^k N_0 r)\big)^{p'}\Big(\int_{B(x,\theta^k N_0 r)} v\,d\mu\Big)^{p'/q} \\
&\quad \times\Big(\int_{B(x,\theta^k N_0 r)} u^{1-p'}\,d\mu\Big)\Big]^{1/p'} \\
&\leq c[v,u]_{A^{\phi}_{p,q}(X)}\Big[\sum_{k=0}^{\infty}\big(1 - c\|v\|^{-1}_{A_s(X)}\big)^{p'k/q}\Big]^{1/p'} \\
&\leq c[v,u]_{A^{\phi}_{p,q}(X)}\|v\|^{1/p'}_{A_s(X)}.
\end{aligned}$$

Therefore, we have

$$\frac{1}{1-(1-\lambda^{-1})^{p'/q}} = O(\lambda), \quad \lambda \to \infty.$$

□

Theorem 3.3. *Let $r, s > 1$. Let p, q, k, k^* be as in Theorem 2.9. Furthermore, suppose that $v \in A_r(X)$ and $u^{1-p'} \in A_s(X)$. Then there exists $c > 0$ depending only on r, s, μ, and quasimetric d such that the inequalities*

$$\|K\|_{L^p_u(X)\to L^{q,\infty}_v(X)} \leq c[v,u]_{A^{\phi}_{p,q}}\|v\|^{1/p'}_{A_r(X)}, \tag{3.3}$$

$$\|K\|_{L^p_u(X)\to L^q_v(X)} \leq c\big([v,u]_{A^{\phi}_{p,q}}\|v\|^{1/p'}_{A_r(X)} + [v,u]_{A^{\phi^*}_{p,q}}\|u^{1-p'}\|^{1/q}_{A_s(X)}\big) \tag{3.4}$$

hold.

Proof. By Theorem 2.10, we have that

$$\|Kf\|_{L^p_u(X)\to L^{q,\infty}_v(X)} \ll B(v, u, k, p, q).$$

Now using Lemma 3.2, we find that estimate (3.3) holds. Estimate (2.5), Remark 2.11, Lemma 3.2, and the relation $[v,u]_{A^{\phi^*}_{p,q}(X)} = [u^{1-p'}, v^{1-q'}]_{A^{\phi^*}_{q',p'}(X)}$ imply (3.4). □

From the two-weight estimate in Theorem 3.3, we can derive the sharp weighted bounds for the potential operator T_α (see (1.7) for the definition).

Corollary 3.4. *Let $r, s > 1$. Let $0 < \alpha < 1$ and $1 < p < q < \infty$. Suppose that $[v, u] \in A^\alpha_{p,q}(X)$. Further suppose that $v \in A_r(X)$ and $u^{1-p'} \in A_s(X)$. Then*

(a)

$$\|T_\alpha\|_{L^p_u(X)\to L^{q,\infty}_v(X)} \ll [v, u]_{A^\alpha_{p,q}(X)} \|v\|^{1/p'}_{A_r(X)};$$

(b)

$$\|T_\alpha\|_{L^p_u(X)\to L^q_v(X)} \ll [v, u]_{A^\alpha_{p,q}(X)} \big(\|v\|^{1/p'}_{A_r(X)} + \|u^{1-p'}\|^{1/q}_{A_s(X)}\big).$$

Proof. First, note that for $k(x, y) = \mu(B(x, d(x, y)))^{\alpha-1}$, the doubling and reverse doubling conditions imply that $\phi(B) \approx \mu(B)^{\alpha-1}$ and $k^*(x, y) \approx k(y, x)$ for $x, y \in X$. Then by taking these estimates into account and using Theorem 3.3, we have the required inequalities. □

Corollary 3.5. *Let $r, s > 1$. Let $0 < \alpha < 1$ and $1 < p < q < \infty$. Suppose that $w \in A_{p,q}(X)$. Further suppose that $w^q \in A_r(X)$ and $w^{-p'} \in A_s(X)$. Then*

(a)

$$\|T_\alpha\|_{L^p_{w^p}(X)\to L^{q,\infty}_{w^q}(X)} \le c\|w\|^{1/q}_{A_{p,q}(X)} \|w^q\|^{1/p'}_{A_r(X)};$$

(b)

$$\|T_\alpha\|_{L^p_{w^p}(X)\to L^q_{w^q}(X)} \le c\|w\|^{1/q}_{A_{p,q}(X)} \big(\|w^q\|^{1/p'}_{A_r(X)} + \|w^{-p'}\|^{1/q}_{A_s(X)}\big).$$

Proof. Taking $v = w^q$ and $u = w^p$, and taking into account that $[w^q, w^p]_{A^\alpha_{p,q}(X)} = \|w\|^{1/q}_{A_{p,q}(X)}$ in Theorem 3.3, we have the required estimate. □

Observe that by taking $r = 1 + \frac{q}{p'}$ and $s = 1 + \frac{p'}{q}$ in Corollary 3.5, we have the sharp estimate given in Theorem 1.3. In this case, $\|w^q\|_{A_r(X)} = \|w\|_{A_{p,q}(X)}$, $\|w^{-p'}\|_{A_s(X)} = \|w\|^{p'/q}_{A_{p,q}(X)}$.

Now we investigate sharp bounds for the fractional maximal operator M^μ_α using the two-weight theory. First, observe now that

$$A_{\text{GK}}(v, u, \alpha, p, q) = D(v, u, k, p, q),$$

where $k(x, y) = (\mu B(x, d(x, y)))^{\alpha-1}$.

Theorem 3.6. *Let $s > 1$. Let $0 < \alpha < 1$ and $1 < p < q < \infty$. Suppose that $[v, u] \in A^\alpha_{p,q}(X)$. Further suppose that $u^{1-p'} \in A_s(X)$. Then*

$$\|M^\mu_\alpha\|_{L^p_u(X)\to L^q_v(X)} \ll [v, u]_{A^\alpha_{p,q}(X)} \|u^{1-p'}\|^{1/q}_{A_s(X)}.$$

Proof. By Corollary 3.4 and simple observations, we have that

$$\begin{aligned}\|M^\mu_\alpha\|_{L^p_u(X)\to L^q_v(X)} &\ll [u^{1-p'}, v^{1-q'}]_{A^\alpha_{q',p'}(X)} \|u^{1-p'}\|^{1/q}_{A_s(X)} \\ &\ll [v, w]_{A^\alpha_{p,q}(X)} \|u^{1-p'}\|^{1/q}_{A_s(X)}.\end{aligned}$$

□

Corollary 3.7. *Let $s > 1$. Let $0 < \alpha < 1$ and $1 < p < q < \infty$. Suppose that $[v, u] \in A^{\alpha}_{p,q}(X)$. Further suppose that $u^{1-p'} \in A_s(X)$. Then*

$$\|M^{\mu}_{\alpha}\|_{L^p_u(X)\to L^q_v(X)} \ll [v, u]_{A^{\alpha}_{p,q}(X)} \|u^{1-p'}\|^{1/q}_{A_s(X)}.$$

Corollary 3.8. *Let $s > 1$. Let $0 < \alpha < 1$ and $1 < p < q < \infty$. Suppose that $w \in A_{p,q}(X)$. Further suppose that $w^{-p'} \in A_s(X)$. Then*

$$\|M^{\mu}_{\alpha}\|_{L^p_{w^p}(X)\to L^q_{w^q}(X)} \ll \|w\|^{1/q}_{A_{p,q}(X)} \|w^{-p'}\|^{1/q}_{A_s(X)}.$$

Finally, we have the following.

Theorem 3.9. *Let $0 < \alpha < 1$, and let $1 < p < \frac{1}{\alpha}$. Suppose that $w \in A_{p,q}(X)$. We set $q = \frac{p}{1-\alpha p}$. Then the following estimate holds:*

$$\|M^{\mu}_{\alpha}\|_{L^p_{w^p}(X)\to L^q_{w^q}(X)} \ll \|w\|^{(1-\alpha)p'/q}_{A_{p,q}(X)}.$$

Moreover, the exponent $(1-\alpha)p'/q$ is sharp.

Proof. Taking $s = 1 + p'/q$ in Corollary 3.8 and observing that $\|w^{-p'}\|_{A_s(X)} = \|w\|^{p'/q}_{A_{p,q}(X)}$, we have the desired result.

The sharpness follows from Theorem 1.3(a) and the observation

$$\|M^{\mu}_{\alpha}\|_{L^p_{w^p}(X)\to L^q_{w^q}(X)} \approx \|T_{\alpha}\|_{L^{q'}_{w^{-q'}}(X)\to L^{p'}_{w^{-p'}}(X)}. \qquad \square$$

4. One-sided fractional integrals

Despite the fact that the two-weight problem for one-sided fractional integrals has been studied under different types of conditions (see [18], [17], Section 2.2 in [5]), it is nonetheless useful to have the sharp estimates for these operator norms under the so-called *Muckenhoupt characteristics*. This section can be considered as a continuation of the investigation carried out in [14], where we established the sharp weighted estimates for one-sided operators in terms of the so-called *one-sided Muckenhoupt* and *Muckenhoupt–Wheeden-type characteristics*.

Let $0 < \alpha < 1$, and let

$$\mathcal{W}_{\alpha} f(x) = \int_x^{\infty} \frac{f(t)}{(t-x)^{1-\alpha}}\, dt, \qquad \mathcal{R}_{\alpha} f(x) = \int_{-\infty}^{x} \frac{f(t)}{(x-t)^{1-\alpha}}\, dt, \quad x \in \mathbb{R},$$

be one-sided fractional integrals. The corresponding one-sided fractional maximal operators are given by

$$M^{+}_{\alpha} f(x) = \sup_{h>0} \frac{1}{h^{1-\alpha}} \int_x^{x+h} |f(t)|\, dt,$$
$$M^{-}_{\alpha} f(x) = \sup_{h>0} \frac{1}{h^{1-\alpha}} \int_{x-h}^{h} |f(t)|\, dt, \quad x \in \mathbb{R}.$$

If $\alpha = 0$, then the $M^{\pm}_{\alpha}$ are one-sided Hardy–Littlewood maximal operators.

Sometimes we will use the notation $w(I)$ for a weight w and an interval $I \subset \mathbb{R}$.

For the following definition, we refer to [1] and [14].

Definition 4.1. Let $1 < p, q < \infty$. We say that a weight function w defined on $\mathbb{R}$ satisfies the $A^{+}_{p,q}(\mathbb{R})$ condition ($w \in A^{+}_{p,q}(\mathbb{R})$) if

$$\|w\|_{A^{+}_{p,q}(\mathbb{R})} := \sup_{\substack{x\in\mathbb{R}\\ h>0}} \Big(\frac{1}{h}\int_{x-h}^{x} w^{q}(t)\,dt\Big)\Big(\frac{1}{h}\int_{x}^{x+h} w^{-p'}(t)\,dt\Big)^{q/p'} < \infty,$$

and we say that a weight function w satisfies the $A^{-}_{p,q}(\mathbb{R})$ condition ($w \in A^{-}_{p,q}(\mathbb{R})$) if

$$\|w\|_{A^{-}_{p,q}(\mathbb{R})} := \sup_{\substack{x\in\mathbb{R}\\ h>0}} \Big(\frac{1}{h}\int_{x}^{x+h} w^{q}(t)\,dt\Big)\Big(\frac{1}{h}\int_{x-h}^{x} w^{-p'}(t)\,dt\Big)^{q/p'} < \infty.$$

We are also interested in the so-called *two-weighted $A^{\pm}_{p,q,\alpha}$ characteristics of Muckenhoupt type* (cf. [3]).

Definition 4.2. Let $1 < p, q < \infty$. We say that a weight pair (v, w) defined on $\mathbb{R}$ satisfies the $A^{+}_{p,q,\alpha}$ condition ($w \in A^{+}_{p,q,\alpha}$) if

$$A^{+}_{p,q,\alpha}(v,u) := \sup_{\substack{x\in\mathbb{R}\\ h>0}} h^{\alpha+1/q-1/p}\Big(\frac{1}{h}\int_{x-h}^{x} v(t)\,dt\Big)^{1/q}\Big(\frac{1}{h}\int_{x}^{x+h} u^{1-p'}(t)\,dt\Big)^{1/p'} < \infty,$$

and we say that a weight function w satisfies the $A^{-}_{p,q,\alpha}$ condition ($w \in A^{-}_{p,q,\alpha}$) if

$$A^{-}_{p,q,\alpha}(v,u) := \sup_{\substack{x\in\mathbb{R}\\ h>0}} h^{\alpha+1/q-1/p}\Big(\frac{1}{h}\int_{x}^{x+h} v(t)\,dt\Big)^{1/q}\Big(\frac{1}{h}\int_{x-h}^{x} u^{1-p'}(t)\,dt\Big)^{1/p'} < \infty.$$

If $0 < \alpha < 1/p$, $q = \frac{p}{1-\alpha p}$, $v = w^{q}$, and $u = w^{p}$, then the $A^{\pm}_{p,q,\alpha}$ coincide with $A^{\pm}_{p,q}$ characteristics. In our recent work [14], we proved the sharp weighted bounds for these operators in terms of $A^{\pm}_{p,q}$ characteristics; in particular, we proved the following Buckley-type statements.

Theorem 4.3. *Suppose that $0 < \alpha < 1$, $1 < p < 1/\alpha$ and that q is such that $1/p - 1/q - \alpha = 0$. Then*

(i) *there exists a positive constant c depending only on p and α such that*

$$\|M^{+}_{\alpha}\|_{L^{p}_{w^p}\to L^{q}_{w^q}} \le c\|w\|^{\frac{p'}{q}(1-\alpha)}_{A^{+}_{p,q}(\mathbb{R})}, \tag{4.1}$$

and moreover, the exponent $\frac{p'}{q}(1-\alpha)$ is best possible;

(ii) *there exists a positive constant c depending only on p and α such that*

$$\|M^{-}_{\alpha}\|_{L^{p}_{w^p}\to L^{q}_{w^q}} \le c\|w\|^{\frac{p'}{q}(1-\alpha)}_{A^{-}_{p,q}(\mathbb{R})}, \tag{4.2}$$

and moreover, the exponent $\frac{p'}{q}(1-\alpha)$ is best possible.

Theorem 4.4. *Let $1 < p < \frac{1}{\alpha}$, where $0 < \alpha < 1$. We set $q = \frac{p}{1-\alpha p}$. Then*

(a)

$$\|\mathcal{R}_{\alpha}\|_{L^{p}_{w^p}\to L^{q,\infty}_{w^q}} \le c\|w\|^{1-\alpha}_{A^{-}_{p,q}(\mathbb{R})}, \tag{4.3}$$

where the positive constant c depends only on p and α, and

(b)

$$\|\mathcal{W}_\alpha\|_{L^p_{w^p}\to L^{q,\infty}_{w^q}} \le c\|w\|^{1-\alpha}_{A^+_{p,q}(\mathbb{R})}, \tag{4.4}$$

where the positive constant c depends only on p and α.

Theorem 4.5. *Let $0<\alpha<1$, $1<p<1/\alpha$, and let q satisfy $q=\frac{p}{1-\alpha p}$. Then*

(a) *there is a positive constant c depending only on p and α such that*

$$\|\mathcal{R}_\alpha\|_{L^p_{w^p}\to L^q_{w^q}} \le c\|w\|^{(1-\alpha)\max\{1,p'/q\}}_{A^-_{p,q}(\mathbb{R})}, \tag{4.5}$$

and moreover, this estimate is sharp;

(b) *there is a positive constant c depending only on p and α such that*

$$\|\mathcal{W}_\alpha\|_{L^p_{w^p}\to L^q_{w^q}} \le c\|w\|^{(1-\alpha)\max\{1,p'/q\}}_{A^+_{p,q}(\mathbb{R})}, \tag{4.6}$$

and moreover, this estimate is sharp.

The proofs of these statements were based on the two-weight theory for one-sided fractional integrals, and in particular, on the two-weight criteria of Gabidzashvili–Kokilashvili type (see [5], [14]; see also Section 2.2 of [4]). We give these criteria below.

Theorem 4.6. *Let $1<p<q<\infty$, and let $0<\alpha<1$. Suppose that v and w are weight functions on $\mathbb{R}$. Then $\mathcal{W}_\alpha$ is bounded from $L^p_w(\mathbb{R})$ to $L^{q,\infty}_v(\mathbb{R})$ if and only if*

$$[v,w]^+_{\mathrm{Glo}}(p,q) := \sup_{\substack{a\in\mathbb{R}\\ h>0}} \Big(\int_{a-h}^{a+h} v(t)\,dt\Big)^{1/q} \times \Big(\int_{a+h}^{\infty} (t-a)^{(\alpha-1)p'} w^{1-p'}(t)\,dt\Big)^{1/p'} < \infty.$$

Moreover, $\|\mathcal{W}_\alpha\|_{L^p_w\to L^{q,\infty}_v} \approx [v,w]^+_{\mathrm{Glo}}(p,q)$.

Theorem 4.7. *Let $1<p<q<\infty$, and let $0<\alpha<1$. Suppose that v and w are weight functions on $\mathbb{R}$. Then $\mathcal{R}_\alpha$ is bounded from $L^p_w(\mathbb{R})$ to $L^{q,\infty}_v(\mathbb{R})$ if and only if*

$$[v,w]^-_{\mathrm{Glo}}(p,q) := \sup_{\substack{a\in\mathbb{R}\\ h>0}} \Big(\int_{a-h}^{a+h} v(t)\,dt\Big)^{1/q} \times \Big(\int_{-\infty}^{a-h} (a-t)^{(\alpha-1)p'} w^{1-p'}(t)\,dt\Big)^{1/p'} < \infty.$$

Moreover, $\|\mathcal{R}_\alpha\|_{L^p_w\to L^{q,\infty}_v} \approx [v,w]^-_{\mathrm{Glo}}(p,q)$.

The strong-type results read as follows.

Theorem 4.8. *Let $1<p<q<\infty$, and let $0<\alpha<1$. Suppose that v and w are weight functions on $\mathbb{R}$. Then $\mathcal{W}_\alpha$ is bounded from $L^p_w(\mathbb{R})$ to $L^q_v(\mathbb{R})$ if and only if*

(i) *$[v,w]^+_{\mathrm{Glo}}(p,q)<\infty$;*

(ii)

$$A^{+}_{\mathrm{GK}}(v,w,p,q) := \sup_{\substack{a\in\mathbb{R}\\ h>0}} \Big(\int_{a-h}^{a+h} w^{1-p'}(t)\,dt\Big)^{1/p'} \times \Big(\int_{-\infty}^{a-h} \frac{v(y)}{(a-y)^{(1-\alpha)q}}\,dy\Big)^{1/q} < \infty. \tag{4.7}$$

Moreover, $\|\mathcal{W}_\alpha\|_{L^p_w\to L^q_v} \approx [v,w]^{+}_{\mathrm{Glo}}(p,q) + A^{+}_{\mathrm{GK}}(v,w,p,q)$.

Theorem 4.9. *Let* $1<p<q<\infty$*, and let* $0<\alpha<1$*. Suppose that* v *and* w *are weight functions on* $\mathbb{R}$*. Then* $\mathcal{R}_\alpha$ *is bounded from* $L^p_w(\mathbb{R})$ *to* $L^q_v(\mathbb{R})$ *if and only if*

(i) $[v,w]^{-}_{\mathrm{Glo}}(p,q)<\infty$;

(ii)

$$A^{-}_{\mathrm{GK}}(v,w,p,q) := \sup_{\substack{a\in\mathbb{R}\\ h>0}} \Big(\int_{a-h}^{a+h} w^{1-p'}(t)\,dt\Big)^{1/p'} \times \Big(\int_{a+h}^{\infty} \frac{v(y)}{(y-a)^{(1-\alpha)q}}\,dy\Big)^{1/q} < \infty. \tag{4.8}$$

Moreover, $\|\mathcal{R}_\alpha\|_{L^p_w\to L^q_v} \approx [v,w]^{-}_{\mathrm{Glo}}(p,q) + A^{-}_{\mathrm{GK}}(v,w,p,q)$.

Theorem 4.10. *Suppose that* $1<p<q<\infty$ *and that* $0<\alpha<1$*. Then we have the following.*

(i) M^{+}_α *is bounded from* $L^p_w(\mathbb{R})$ *to* $L^q_v(\mathbb{R})$ *if and only if (4.7) holds. Moreover,* $\|M^{+}_\alpha\|_{L^p_w\to L^q_v} \approx A^{+}_{\mathrm{GK}}(v,w,p,q)$.

(ii) M^{-}_α *is bounded from* $L^p_w(\mathbb{R})$ *to* $L^q_v(\mathbb{R})$ *if and only if (4.8) holds. Moreover,* $\|M^{-}_\alpha\|_{L^p_w\to L^q_v} \approx A^{-}_{\mathrm{GK}}(v,w,p,q)$.

Let us recall the $A^{\pm}_p$ characteristics for a locally integrable weight ω:

$$\|\omega\|_{A^{+}_p(\mathbb{R})} := \sup_{\substack{x\in\mathbb{R}\\ h>0}} A^{+}_p(\omega,x,h) := \sup_{\substack{x\in\mathbb{R}\\ h>0}} \Big(\frac{1}{h}\int_{x-h}^{x} \omega(t)\,dt\Big)\Big(\frac{1}{h}\int_{x}^{x+h} \omega^{1-p'}(t)\,dt\Big)^{p-1}, \tag{4.9}$$

$$\|\omega\|_{A^{-}_p(\mathbb{R})} := \sup_{\substack{x\in\mathbb{R}\\ h>0}} A^{-}_p(\omega,x,h) := \sup_{\substack{x\in\mathbb{R}\\ h>0}} \Big(\frac{1}{h}\int_{x}^{x+h} \omega(t)\,dt\Big)\Big(\frac{1}{h}\int_{x-h}^{x} \omega^{1-p'}(t)\,dt\Big)^{p-1}. \tag{4.10}$$

It is clear that for fixed $x\in\mathbb{R}$ and $h>0$,

$$\lim_{p\to\infty} A^{+}_p(\omega,x,h) = \Big(\frac{1}{h}\int_{x-h}^{x} \omega(t)\,dt\Big)\exp\Big(\frac{1}{h}\int_{x}^{x+h} \log\frac{1}{w(t)}\,dt\Big)$$

and

$$\lim_{p\to\infty} A_p^-(\omega, x, h) = \Big(\frac{1}{h}\int_x^{x+h} \omega(t)\,dt\Big) \exp\Big(\frac{1}{h}\int_{x-h}^{x} \log\frac{1}{w(t)}\,dt\Big).$$

Therefore, a question naturally arises regarding estimates of the one-sided operator norm by the mixed-type $(A^{\pm}_{p,q,\alpha}, A^{\pm}_{\infty,\exp})$ characteristics (see [9], [3]), where $A^{\pm}_{\infty,\exp}$ is a one-sided Hruščev-type characteristic defined as follows:

$$\begin{aligned}\|\sigma\|_{A^+_{\infty,\exp}} &:= \sup_{a\in\mathbb{R}, r>0} A^+_{\infty,\exp}(\sigma, a, r)\\ &= \sup_{a,r}\Big(\frac{1}{r}\int_{a-r}^{a} \sigma(t)\,dt\Big)\exp\Big(\frac{1}{r}\int_a^{a+r}\log\frac{1}{\sigma(t)}\,dt\Big);\\ \|\sigma\|_{A^-_{\infty,\exp}} &:= \sup_{a\in\mathbb{R}, r>0} A^-_{\infty,\exp}(\sigma, a, r)\\ &= \sup_{a,r}\Big(\frac{1}{r}\int_{a}^{a+r} \sigma(t)\,dt\Big)\exp\Big(\frac{1}{r}\int_{a-r}^{a}\log\frac{1}{\sigma(t)}\,dt\Big).\end{aligned}$$

We say that $w \in A^{\pm}_\infty$ if $w \in A^{\pm}_r$, for some r. (For the definition and properties of one-sided A_∞ weights, we refer to [19].) By Jensen's inequality, we have

$$\|w\|_{A^{\pm}_{\infty,\exp}} \le \|w\|_{A^{\pm}_r}, \quad r > 1. \tag{4.11}$$

The main statements of this section read as follows.

Theorem 4.11. *Suppose that $0 < \alpha < 1$, and let $1 < p < q < \infty$. Then the following statements hold.*

(i) *If $u^{1-p'} \in A^-_\infty$, then*

$$\|M^+_\alpha\|_{L^p_u \to L^q_v} \le c_{p,q,\alpha} A^+_{p,q,\alpha}(v,u)\|u^{1-p'}\|^{1/q}_{A^-_{\infty,\exp}}. \tag{4.12}$$

(ii) *If $u^{1-p'} \in A^+_\infty$, then*

$$\|M^-_\alpha\|_{L^p_u \to L^q_v} \le c_{p,q,\alpha} A^-_{p,q,\alpha}(v,u)\|u^{1-p'}\|^{1/q}_{A^+_{\infty,\exp}}. \tag{4.13}$$

Theorem 4.12. *Let $0 < \alpha < 1$, and let $1 < p < q < \infty$.*

(a) *If $v \in A^-_\infty$, then*

$$\|\mathcal{R}_\alpha\|_{L^p_u \to L^{q,\infty}_v} \le c A^-_{p,q,\alpha}(v,u)\|v\|^{1/p'}_{A^-_{\infty,\exp}}, \tag{4.14}$$

where the positive constant c depends only on p, q, and α.

(b) *If $v \in A^+_\infty$, then*

$$\|\mathcal{W}_\alpha\|_{L^p_u \to L^{q,\infty}_v} \le c A^+_{p,q,\alpha}(v,u)\|v\|^{1/p'}_{A^+_{\infty,\exp}}, \tag{4.15}$$

where the positive constant c depends only on p, q, and α.

Theorem 4.13. *Let $0 < \alpha < 1$, and let $1 < p < q < \infty$. Then*

(a) *if $v \in A^-_\infty$ and $u^{1-p'} \in A^+_\infty$, then there is a positive constant c depending only on p, q, and α such that*

$$\|\mathcal{R}_\alpha\|_{L^p_u \to L^q_v} \le c A^-_{p,q,\alpha}(v,u)\big(\|v\|^{1/p'}_{A^-_{\infty,\exp}} + \|u^{1-p'}\|^{1/q}_{A^+_{\infty,\exp}}\big); \tag{4.16}$$

(b) *if* $v \in A_\infty^+$ *and* $u^{1-p'} \in A_\infty^-$, *then there is a positive constant* c *depending only on* p, q, *and* α *such that*

$$\|\mathcal{W}_\alpha\|_{L_u^p \to L_v^q} \le cA_{p,q,\alpha}^+(v,u)\big(\|v\|_{A_{\infty,\exp}^+}^{1/p'} + \|u^{1-p'}\|_{A_{\infty,\exp}^-}^{1/q}\big). \tag{4.17}$$

To show that these statements are true, we need to prove some lemmas.

Lemma 4.14. *Let* ω *be a locally integrable function.*

(i) *If* $\omega \in A_\infty^-$, *then for every* $a \in \mathbb{R}$, $h > 0$, *and* $r > 1$,

$$\omega\big((a, a+h)\big) \le A_r^-(\omega, a, h)\omega\big((a-h, a)\big).$$

Hence,

$$\omega\big((a, a+h)\big) \le A_{\infty,\exp}^-(\omega, a, h)\omega\big((a-h, a)\big).$$

(ii) *Let* $\omega \in A_\infty^+$. *Then for every* $a \in \mathbb{R}$, $h > 0$, *and* $r > 1$,

$$\omega\big((a-h, a)\big) \le A_r^+(\omega, a, h)\omega\big((a, a+h)\big).$$

Hence,

$$\omega\big((a-h, a)\big) \le A_{\infty,\exp}^+(\omega, a, h)\omega\big((a, a+h)\big).$$

Proof. (i) Let $\omega \in A_\infty^-$, and let $r > 1$. Following the proof of Lemma 3.8 of [14], we see that for $r > 0$,

$$\begin{aligned}\omega\big((a, a+h)\big) &\le \Big(\int_a^{a+h} \omega(x)\,dx\Big)h^{-r}\Big(\int_{a-h}^a \omega(x)\,dx\Big)\Big(\int_{a-h}^a \omega^{1-r'}(x)\,dx\Big)^{r-1} \\ &\le h^{-1}\Big(\int_a^{a+h} \omega(x)\,dx\Big)\Big(\int_{a-h}^a \omega(x)\,dx\Big)\Big(\frac{1}{h}\int_{a-h}^a \omega^{1-r'}(x)\,dx\Big)^{r-1} \\ &\le A_r^-(\omega, x, h)\omega\big((a-h, a)\big).\end{aligned}$$

Passing now to the limit when $r \to \infty$, we have the desired result. The remaining part of the lemma is proved analogously. ☐

This lemma immediately implies (see also the proof of Lemma 3.9 in [14]) the next statement.

Lemma 4.15. *Let* $\omega \in A_\infty^-$. *Then for all* $a \in \mathbb{R}$, $h > 0$, *and* $r > 1$, *we have that*

$$\frac{\omega((a-h, a))}{\omega((a-2h, a))} \le \frac{A_r^-(\omega, a-h, h)}{A_r^-(\omega, a-h, h) + 1}.$$

Consequently,

$$\frac{\omega((a-h, a))}{\omega((a-2h, a))} \le \frac{A_{\infty,\exp}^-(\omega, a-h, h)}{A_{\infty,\exp}^-(\omega, a-h, h) + 1}.$$

Furthermore, let $\omega \in A_\infty^+$. *Then for all* $a \in \mathbb{R}$, $h > 0$, *and* $r > 1$, *we have*

$$\frac{\omega((a, a+h))}{\omega((a, a+2h))} \le \frac{A_r^+(\omega, a+h, h)}{A_r^+(\omega, a+h, h) + 1}.$$

Consequently,

$$\frac{\omega((a,a+h))}{\omega((a,a+2h))} \le \frac{A^+_{\infty,\exp}(\omega,a+h,h)}{A^+_{\infty,\exp}(\omega,a+h,h)+1}.$$

We also need the following lemma.

Lemma 4.16. *Let* $0<\alpha<1$*, and let* $1<p<\infty$*. Then*

$$[v,u]^-_{\mathrm{Glo}}(p,q) \ll A^-_{p,q,\alpha}(v,u)\|v\|^{1/p'}_{A^-_{\infty,\exp}}; \tag{4.18}$$

$$[v,u]^+_{\mathrm{Glo}}(p,q) \ll A^+_{p,q,\alpha}(v,u)\|v\|^{1/p'}_{A^+_{\infty,\exp}}. \tag{4.19}$$

Proof. Let us show (4.18). The proof for (4.19) is similar. Denote $\sigma = u^{1-p'}$. Following the proof of Lemma 3.10 of [14] and taking Lemma 4.15 into account, we have, for $a\in\mathbb{R}$ and $h>0$,

$$\begin{aligned}
&(v\big((a-2h,a)\big)^{1/q}\Big(\int_{-\infty}^{a-2h}(a-x-h)^{(\alpha-1)p'}\sigma(x)\,dx\Big)^{1/p'}\\
&\le c\Big(v\big((a-2h,a)\big)^{1/q}\Big(\sum_{j=1}^{\infty}(2^jh)^{(\alpha-1)p'}\sigma(a-2^{j+1}h,a-2^jh)\Big)\Big)^{1/p'}\\
&= c[\sum_{j=1}^{\infty}(2^jh)^{(\alpha-1)p'}\big(v\big((a-2h,a)\big)^{p'/q}\big(\sigma(a-2^{j+1}h,a-2^jh)\big)\big)^{1/p'}\\
&= c\Big[\sum_{j=1}^{\infty}\Big(\frac{v((a-2h,))}{v((a-2^jh,a))}\Big)^{p'/q}\Big(\frac{1}{2^jh}\int_{a-2^jh}^{a}v(x)\,dx\Big)^{p'/q}\\
&\quad\times\Big(\frac{1}{2^jh}\int_{a-2^{j+1}h}^{a-2^jh}\sigma(x)\,dx\Big)\Big]^{1/p'}\\
&\le cA^-_{p,q,\alpha}(v,u)\Big(\sum_{j=1}^{\infty}\Big(\frac{v((a-2h,a))}{v((a-2^jh,a))}\Big)^{p'/q}\Big)^{1/p'}\\
&\le cA^-_{p,q,\alpha}(v,u)\Big(\sum_{j=0}^{\infty}\Big(\frac{\|v\|_{A^-_{\infty,\exp}}}{1+\|v\|_{A^-_{\infty,\exp}}}\Big)^{p'j/q}\Big)^{1/p'}\\
&= cA^-_{p,q,\alpha}(v,u)\Big(\frac{1}{1-(\frac{\|v\|_{A^-_{\infty,\exp}}}{1+\|v\|_{A^-_{\infty,\exp}}})^{p'/q}}\Big)^{1/p'} = cA_{p,q,\alpha}(v,u)\|v\|^{1/p'}_{A^-_{\infty,\exp}}.
\end{aligned}$$

□

Now, using this lemma, the proofs of Theorems 4.11–4.13 follow easily. For example, the proof of Theorem 4.11(i) can be derived from Lemma 4.16 and Theorem 4.10 by observing that

$$\begin{aligned}
A^+_{\mathrm{GK}}(v,u,p,q) &= [u^{1-p'},v^{1-q'}]^-_{\mathrm{Glo}}(q',p') \le cA^-_{q',p',\alpha}(u^{1-p'},v^{1-q'})\|u^{1-p'}\|^{1/q}_{A^-_{\infty,\exp}}\\
&= cA^+_{p,q,\alpha}(v,u)\|u^{1-p'}\|^{1/q}_{A^-_{\infty,\exp}}.
\end{aligned}$$

Similarly, we conclude that Theorem 4.13(a) is a consequence of Lemma 4.16 and Theorem 4.9. Taking $0 < \alpha < 1/p$ and $q = \frac{p}{1-\alpha p}$, $v = w^q$ and $u = w^p$, Theorems 4.11–4.13 yield the following corollaries.

Corollary 4.17. *Suppose that $1 < p < \infty$, $0 < \alpha < 1/p$, and $q = \frac{p}{1-\alpha p}$. Then the following statements hold.*

(i) *There is a positive constant c depending only on p and α such that*

$$\|M_\alpha^+\|_{L_u^p \to L_v^q} \le c\|w\|_{A_{p,q}^+}^{1/q} \|w^{-p'}\|_{A_{\infty,\exp}^-}^{1/q}. \tag{4.20}$$

(ii) *There is a positive constant c depending only on p and α such that*

$$\|M_\alpha^-\|_{L_u^p \to L_v^q} \le c\|w\|_{A_{p,q}^-}^{1/q} \|w^{-p'}\|_{A_{\infty,\exp}^+}^{1/q}. \tag{4.21}$$

Corollary 4.18. *Let $1 < p < \infty$ and $0 < \alpha < 1/p$. We set $q = \frac{p}{1-\alpha p}$. Then*

(a)

$$\|\mathcal{R}_\alpha\|_{L_u^p \to L_v^{q,\infty}} \le c\|w\|_{A_{p,q}^-}^{1/q} \|w^q\|_{A_{\infty,\exp}^-}^{1/p'}, \tag{4.22}$$

where the positive constant c depends only on p and α;

(b) *if $v \in A_\infty^+$, then*

$$\|\mathcal{W}_\alpha\|_{L_u^p \to L_v^{q,\infty}} \le c\|w\|_{A_{p,q}^+}^{1/q} \|w^q\|_{A_{\infty,\exp}^+}^{1/p'}, \tag{4.23}$$

where the positive constant c depends only on p and α.

Corollary 4.19. *Let $1 < p < \infty$ and $0 < \alpha < 1/p$. We set $q = \frac{p}{1-\alpha p}$. Then*

(a) *there is a positive constant c depending only on p and α such that*

$$\|\mathcal{R}_\alpha\|_{L_u^p \to L_v^q} \le c\|w\|_{A_{p,q}^-}^{1/q} \big(\|w^q\|_{A_{\infty,\exp}^-}^{1/p'} + \|w^{-p'}\|_{A_{\infty,\exp}^+}^{1/q}\big); \tag{4.24}$$

(b) *if $v \in A_\infty^+$ and $u^{1-p'} \in A_\infty^-$, then there is a positive constant c depending only on p and α such that*

$$\|\mathcal{W}_\alpha\|_{L_u^p \to L_v^q} \le c\|w\|_{A_{p,q}^+}^{1/q} \big(\|w^q\|_{A_{\infty,\exp}^+}^{1/p'} + \|w^{-p'}\|_{A_{\infty,\exp}^-}^{1/q}\big). \tag{4.25}$$

Remark 4.20. One-weighted sharp estimates (see Theorems 4.3–4.5) now follow from Corollaries 4.17–4.19. For example, Theorem 4.5(a) can be derived from Corollary 4.19(a). Indeed, by Jensen's inequality we have that $\|v\|_{A_{\infty,\exp}}^- \le \|v\|_{A_s}^-$ for any $s > 1$, and $\|u^{1-p'}\|_{A_{\infty,\exp}}^+ \le \|u^{1-p'}\|_{A_r}^+$ for any $r > 1$. Taking $s = 1 + q/p'$ and $r = 1 + p'/q$, we have that

$$\begin{aligned}\|\mathcal{R}_\alpha\|_{L_{w^p}^p \to L_{w^q}^q} &\le c\|w\|_{A_{p,q}^-}^{1/q} \big(\|w^q\|_{A_s^-}^{1/p'} + \|w^{p(1-p')}\|_{A_r^+}^{\frac{p'}{q^2}}\big)\\ &= c\|w\|_{A_{p,q}}^{1/q} \big(\|w\|_{A_{p,q}^-}^{1/p'} + \|w\|_{A_{p,q}^-}^{p'/q^2}\big)\\ &= c\|w\|^{(1-\alpha)\max\{1,p'/q\}}.\end{aligned}$$

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