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# POINTWISE ENTANGLED ERGODIC THEOREMS FOR DUNFORD–SCHWARTZ OPERATORS

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Abstract. We investigate pointwise convergence of entangled ergodic averages of Dunford–Schwartz operators $T_0, T_1, \dots, T_m$ on a Borel probability space. These averages take the form

$$\frac{1}{N^k} \sum_{1 \le n_1, \dots, n_k \le N} T_m^{n_{\alpha(m)}} A_{m-1} T_{m-1}^{n_{\alpha(m-1)}} \cdots A_2 T_2^{n_{\alpha(2)}} A_1 T_1^{n_{\alpha(1)}} f,$$

where $f \in L^p(X, \mu)$ for some $1 \le p < \infty$, and $\alpha : \{1, \dots, m\} \to \{1, \dots, k\}$ encodes the entanglement. We prove that, under some joint boundedness and twisted compactness conditions on the pairs $(A_i, T_i)$, convergence holds almost everywhere for all $f \in L^p$. We also present an extension to polynomial powers in the case $p = 2$, in addition to a continuous version concerning Dunford–Schwartz $C_0$-semigroups.

## 1. Introduction

Entangled ergodic averages were first introduced by Accardi, Hashimoto, and Obata in [1], who saw them as a key ingredient in providing an analogue of the central limit theorem for quantum probability models. Entangled ergodic averages take the general form

$$\frac{1}{N^k} \sum_{1 \le n_1, \dots, n_k \le N} T_m^{n_{\alpha(m)}} A_{m-1} T_{m-1}^{n_{\alpha(m-1)}} \cdots A_2 T_2^{n_{\alpha(2)}} A_1 T_1^{n_{\alpha(1)}},$$


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where $A_i$ $(1 \le i \le m-1)$ and $T_i$ $(1 \le i \le m)$ are operators on a Banach space $E$, and $\alpha : \{1, \dots, m\} \to \{1, \dots, k\}$ is a surjective map. The operators $A_i$ act as transitions between the actions of the operators $T_i$, which iteratively govern the dynamics, whereas the entanglement map $\alpha$ provides a coupling between the stages.

Subsequent work on the subject initially focused on strong convergence of these Cesàro averages (see Liebscher [19], Fidaleo [9]–[11], and Eisner and Kunszenti-Kovács [7]). In Eisner and Kunszenti-Kovács [8] and Kunszenti-Kovács [18], attention was turned to pointwise almost-everywhere convergence in the context of the $T_i$'s being operators on function spaces $E = L^p(X, \mu)$ $(1 \le p < \infty)$, where $(X, \mu)$ is a standard probability space (i.e., a compact metrizable space with a Borel probability measure), with partial results for the case $p \neq 2$. The former paper [8] focuses on the case $k = 1$ with the $T_i$'s being Dunford–Schwartz operators, whereas the latter [18] allows for multiparameter entanglement, but at the price of only dealing with Koopman operators.

In this article, we deal with the full case of general entanglement maps $\alpha$ and Dunford–Schwartz operators $T_i$, and we show convergence almost everywhere on the whole $L^p$ space for all $1 \le p < \infty$ under weakened assumptions, significantly improving on previous results. We introduce a formalism for the iterated function splittings used in the proofs in order to make them more concise, better highlighting what the main steps are and where the different assumptions of the statements come into play. We also provide results concerning polynomial and time-continuous versions of the ergodic theorems considered.

Recall that a Dunford–Schwartz operator $T$ is an operator acting as a contraction on all $L^p$ spaces over a probability space $(X, \mu)$, so that when introducing such an operator we will only specify the probability space it "acts" on, whereas the $p$ we consider will be fixed thereafter. Also, in what follows, $\mathbb{N}$ will denote the set of positive integers, and $\mathbb{T}$ will denote the unit circle in $\mathbb{C}$.

Our main result is as follows.

**Theorem 1.1.** *Let $m > 1$ and $k$ be positive integers, let $\alpha : \{1, \dots, m\} \to \{1, \dots, k\}$ be a not-necessarily surjective map, and let $T_1, T_2, \dots, T_m$ be Dunford–Schwartz operators on a Borel probability space $(X, \mu)$. Let $p \in [1, \infty)$, let $E := L^p(X, \mu)$, and let $E = E_{j;r} \oplus E_{j;s}$ be the Jacobs–de Leeuw–Glicksberg decomposition corresponding to $T_j$ $(1 \le j \le m)$. Furthermore, let $A_j \in \mathcal{L}(E)$ $(1 \le j < m)$ be bounded operators. For a function $f \in E$ and an index $1 \le j \le m-1$, write $\mathscr{A}_{j,f} := \{A_j T_j^n f | n \in \mathbb{N}\}$. Suppose that the following conditions hold.*

- (A1) *(Twisted compactness) For any function $f \in E$, index $1 \le j \le m-1$, and $\varepsilon > 0$, there exists a decomposition $E = \mathcal{U} \oplus \mathcal{R}$ with $0 < \dim \mathcal{U} < \infty$ such that*
$$P_{\mathcal{R}} \mathscr{A}_{j,f} \subset B_\varepsilon\big(0, L^\infty(X, \mu)\big),$$
*with $P_{\mathcal{R}}$ denoting the projection along $\mathcal{U}$ onto $\mathcal{R}$.*
- (A2) *(Joint $\mathcal{L}^\infty$-boundedness) There exists a constant $C > 0$ such that we have*
$$\{A_j T_j^n | n \in \mathbb{N}, 1 \le j \le m-1\} \subset B_C(0, \mathcal{L}\big(L^\infty(X, \mu)\big).$$

*Then we have the following:*

(1) *for each* $f \in E_{1;s}$,

$$\frac{1}{N^k} \sum_{1 \le n_1, \ldots, n_k \le N} |T_m^{n_{\alpha(m)}} A_{m-1} T_{m-1}^{n_{\alpha(m-1)}} \cdots A_2 T_2^{n_{\alpha(2)}} A_1 T_1^{n_{\alpha(1)}} f| \to 0$$

*pointwise almost everywhere;*

(2) *for each* $f \in E_{1;r}$,

$$\frac{1}{N^k} \sum_{1 \le n_1, \ldots, n_k \le N} T_m^{n_{\alpha(m)}} A_{m-1} T_{m-1}^{n_{\alpha(m-1)}} \cdots A_2 T_2^{n_{\alpha(2)}} A_1 T_1^{n_{\alpha(1)}} f$$

*converges pointwise almost everywhere.*

*Remark.* Note that it was proved in [8] that the Volterra operator $V$ on $L^2([0,1])$ defined through

$$(Vf)(x) := \int_0^x f(z)\,\mathrm{d}z,$$

as well as all of its powers, can be decomposed into a finite sum of operators, each of which satisfies conditions (A1) and (A2) when paired with any Dunford–Schwartz operator. Hence the conclusions of Theorem 1.1 apply whenever the operators $A_i$ are chosen to be powers of $V$.

## 2. Notation and tools

Before proceeding to the proof of our main result, we need to clarify some of the notions used and introduce notation that will simplify our arguments.

The proof works by iteratively splitting the functions into finitely many parts, so introducing vector indices will be very helpful. Given a vector $v \in \mathbb{N}^d$ $(d \ge 1)$, let $\overline{v} \in \mathbb{N}^{d-1}$ be the vector obtained by deleting its last coordinate, and let $v^*$ denote its last coordinate. Also, we will write $l(v) := d$ to denote the number of coordinates of the vector, $x \subset v$ if there exist vectors $w_0, w_1, \ldots, w_b$ $(b \ge 1)$ such that $w_0 = x$, $w_b = v$, and for each $1 \le i \le b$ we have $w_{i-1} = \overline{w_i}$, and finally, $x \subseteq v$ if $x = v$ or $x \subset v$.

Let $\mathcal{N}$ denote the set of all bounded sequences $\{a_n\} \subset \ell^\infty(\mathbb{C})$ satisfying

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^N |a_n| = 0.$$

By the Koopman–von Neumann lemma (see, e.g., Petersen [22, p. 65]), $(a_n) \in \mathcal{N}$ if and only if it lies in $\ell^\infty$ and converges to zero along a sequence of density 1.

*Definition* 2.1. Given a Banach space $E$ and an operator $T \in \mathcal{L}(E)$, the operator $T$ is said to have *relatively weakly compact orbits* if for each $f \in E$, the orbit set $\{T^n f | n \in \mathbb{N}\}$ is relatively weakly closed in $E$. For any such operator, there exists a corresponding *Jacobs–de Leeuw–Glicksberg* decomposition of the form (see [5, Theorem II.4.8])

$$E = E_r \oplus E_s,$$

where

$$E_r := \overline{\mathrm{lin}}\{f \in E : Tf = \lambda f \text{ for some } \lambda \in \mathbb{T}\},$$
$$E_s := \big\{f \in E : \big(\varphi(T^n f)\big) \in \mathcal{N} \text{ for every } \varphi \in E'\big\}.$$

Note that every power-bounded operator on a reflexive Banach space has relatively weakly compact orbits. Thus, for example, the above decomposition is valid for every contraction on $L^p(X,\mu)$ for $p \in (1,\infty)$. If $T$ is a *Dunford–Schwartz operator* on $(X,\mu)$, then $T$ clearly has relatively weakly compact orbits not only in $L^p(X,\mu)$ $(1 < p < \infty)$, but in $L^1(X,\mu)$ as well (see Lin, Olsen, and Tempelman [21, Proposition 2.6] and Kornfeld and Lin [14, pp. 226–227]). The Jacobs–de Leeuw–Glicksberg decomposition is therefore valid for Dunford–Schwartz operators on $L^p(X,\mu)$ for every $p \in [1,\infty)$.

Let $T$ be a Dunford–Schwartz operator on $(X,\mu)$. The *(linear) modulus* $|T|$ of $T$ is defined as the unique positive operator on $L^1(X,\mu)$ having the same $L^1$- and $L^\infty$-norm as $T$ such that $|T^n f| \le |T|^n |f|$ holds almost everywhere for every $f \in L^1(X,\mu)$ and every $n \in \mathbb{N}$. The modulus of a Dunford–Schwartz operator is again a Dunford–Schwartz operator. (For details, see Dunford and Schwartz [4, p. 672] and Krengel [15, pp. 159–160].) Also, it is easily seen that for $T$ Dunford–Schwartz, the operators $\lambda T$ $(\lambda \in \mathbb{T})$ are themselves Dunford–Schwartz and have the same modulus. For example, every Koopman operator (i.e., an operator induced by a $\mu$-preserving transformation on $X$) is a positive Dunford–Schwartz operator, and hence coincides with its modulus.

A key property of Dunford–Schwartz operators needed for this article is that the validity of pointwise ergodic theorems typically extends from Koopman operators to Dunford–Schwartz operators. For instance, for every $f \in L^1(X,\mu)$, the ergodic averages

$$\frac{1}{N}\sum_{n=1}^{N} T^n f$$

converge almost everywhere as $N \to \infty$ (see Dunford and Schwartz [4, p. 675]).

We will also need to define some classes of sequences that act as good weights for pointwise ergodic theorems. A sequence $(a_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ is called a *trigonometric polynomial* (see [12]) if it is of the form $a_n = \sum_{j=1}^{t} b_j \rho_j^n$, where the $b_j$'s are complex numbers and $\rho_j \in \mathbb{T}$ for all $1 \le j \le t$.

Let $\mathscr{P} \subset \ell^\infty$ denote the set of Bohr almost-periodic sequences, that is, the set of uniform limits of trigonometric polynomials. The following properties of the set $\mathscr{P}$ will be used: it is closed in $l^\infty$, closed under multiplication, and is a subclass of (Weyl) almost-periodic sequences $\mathrm{AP}(\mathbb{N})$, that is, sequences whose orbit under the left shift is relatively compact in $l^\infty$. Actually, $\mathrm{AP}(\mathbb{N}) = \mathscr{P} \oplus c_0$ (see Bellow and Losert [2, p. 316]), corresponding to the Jacobs–de Leeuw–Glicksberg decomposition of $\mathrm{AP}(\mathbb{N})$ induced by the left shift (see, e.g., [5, Theorem I.1.20]).

By Çömez, Lin, and Olsen [3, Theorem 2.5], every element $(a_n)_{n=1}^{\infty}$ of AP($\mathbb{N}$), and hence of $\mathscr{P}$, is a good weight for the pointwise ergodic theorem for Dunford–Schwartz operators. That is, for every Dunford–Schwartz operator $T$ on a probability space $(X,\mu)$ and every $f \in L^1(X,\mu)$, the weighted ergodic averages

$$\frac{1}{N}\sum_{n=1}^{N} a_n T^n f$$

converge almost everywhere as $N \to \infty$.

A sequence $(a_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ is called *linear* (see [6]) if there exist a Banach space $E$, an operator $T \in \mathcal{L}(E)$ with relatively weakly compact orbits, and $y \in E$, $y' \in E'$ such that $a_n = y'(T^n y)$ for all $n \in \mathbb{N}$. Let us call a linear sequence *stable* if we can choose $y \in E_s$, and *reversible* if we can chose $y \in E_r$. It is easy to see that stable linear sequences all lie in $\mathcal{N}$, whereas reversible linear sequences all lie in $\mathscr{P}$.

We will later also need properties of polynomial subsequences of linear sequences, and thus a corresponding class of good weights for the pointwise polynomial ergodic theorem.

*Definition* 2.2. Given $1 \le p < \infty$ and a subsequence $(n_s)_{s\in\mathbb{N}}$ of $\mathbb{N}$, the class $B_{p,(n_s)_{s\in\mathbb{N}}}$ of $p,(n_s)_{s\in\mathbb{N}}$-Besicovitch sequences is the closure of the trigonometric polynomials in the $p,(n_s)_{s\in\mathbb{N}}$-seminorm defined by

$$\left\|(a_n)_{n\in\mathbb{N}}\right\|_{p,(n_s)_{s\in\mathbb{N}}}^{p} = \limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}|a_{n_s}|^p.$$

By [20, Theorem 2.1], the set of bounded sequences in these classes is independent of the choice of $p$; that is, $B_{1,(n_s)_{s\in\mathbb{N}}} \cap l^\infty = B_{p,(n_s)_{s\in\mathbb{N}}} \cap l^\infty$ for all $p \in (1,\infty)$. Note that the seminorm defined above is trivially dominated by the $l^\infty$-norm, and hence $\mathscr{P} \subset B_{1,(n_s)_{s\in\mathbb{N}}} \cap l^\infty$ for any subsequence $(n_s)_{s\in\mathbb{N}}$ of $\mathbb{N}$. The closedness of $\mathscr{P}$ under multiplication thus yields the following lemma.

**Lemma 2.3.** *Let $(a_{n;j})_{n\in\mathbb{N}}$ be a reversible linear sequence for each $1 \le j \le t$. Then $(b_n)_{n\in\mathbb{N}}$ defined by $b_n := \prod_{j=1}^{t} a_{n;j}$ lies in $B_{1,(n_s)_{s\in\mathbb{N}}} \cap l^\infty$ for any subsequence $(n_s)_{s\in\mathbb{N}}$ of $\mathbb{N}$.*

The essential property of elements of $B_{1,(n_s)_{s\in\mathbb{N}}} \cap l^\infty$ is given by the following theorem.

**Theorem 2.4** ([12, Theorem 2.1])**.** *Let $T$ be a Dunford–Schwartz operator on a standard probability space $(X,\mu)$, let $1 \le p < \infty$, and let $\mathbf{q}(x)$ be a polynomial with integer coefficients taking positive values on $\mathbb{N}$. Then for any $f \in L^p(X,\mu)$ and $(b_n)_{n\in\mathbb{N}} \in B_{1,(\mathbf{q}(n))_{n\in\mathbb{N}}} \cap l^\infty$, the limit*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} b_{\mathbf{q}(n)} T^{\mathbf{q}(n)} f$$

*exists almost surely.*

Finally, we need information about the sequences $(\lambda_{*;n})$ along polynomial indices. Recall that an operator is almost weakly stable if the stable part of the Jacobs–de Leeuw–Glicksberg decomposition is the whole space.

**Proposition 2.5** ([17, Theorem 1.1])**.** *Let $T$ be an almost weakly stable contraction on a Hilbert space $H$. Then $T$ is almost weakly polynomial stable; that is, for any $h \in H$ and nonconstant polynomial $\mathbf{q}$ with integer coefficients taking positive values on $\mathbb{N}$, the sequence $\{T^{\mathbf{q}(j)}h\}_{j=1}^{\infty}$ is almost weakly stable.*

As a consequence, we obtain the following result.

**Corollary 2.6.** *Let $T$ be a Dunford–Schwartz operator on the standard probability space $(X, \mu)$, let $\mathbf{q}$ be a nonconstant polynomial with integer coefficients taking positive values on $\mathbb{N}$, and let $A$ be an arbitrary operator on $L^2(X, \mu)$. Then for any $g, \varphi \in L^2(X, \mu)$ with $g$ in the stable part of $L^2(X, \mu)$ with respect to $T$, we have that the sequence $\langle AT^{\mathbf{q}(n)}g, \varphi\rangle$ is bounded and lies in $\mathcal{N}$.*

## 3. Proof of Theorem 1.1

We will proceed by successive splitting and reduction. For each operator $T_i$, starting from $T_2$, we split the functions it is applied to into several terms using condition (A1) of Theorem 1.1. Most of the obtained terms can be easily dealt with, but for the remaining "difficult" terms, we move on to $T_{i+1}$, up to and including $T_m$. We first prove part (1) of Theorem 1.1, and then we use this result to complete the proof for part (2) of Theorem 1.1.

In what follows, we will assume without loss of generality that for the constant in Theorem 1.1, we have $C \geq 1$. Given a function $f \in E_{1;s}$ and $\varepsilon \in (0, 1)$, do the following.

(I) First, set $d = 0$, $c := \varepsilon C^{-m}$, and let $\mathcal{I}_0$ consist of the empty index.

(II) By assumption (A1), for each $f_v$ $(v \in \mathcal{I}_d)$ we may find a decomposition $E = \mathcal{U}_v \oplus \mathcal{R}_v$ with $\ell_v := \dim \mathcal{U}_v < \infty$ and

$$P_{\mathcal{R}_v} \mathscr{A}_{d+1,f_v} \subset B_{c_v}\big(0, L^\infty(X, \mu)\big).$$

For each $v \in \mathcal{I}_d$, choose a maximal linearly independent set $f_{v;1}, \ldots, f_{v;\ell_v}$ in $\mathcal{U}_v$. Then for each $n \in \mathbb{N}$, we can write the unique decomposition

$$A_{d+1}T_{d+1}^n f_v = \lambda_{v,1;n} f_{v,1} + \cdots + \lambda_{v,\ell_v;n} f_{v,\ell_v} + r_{v;n}$$

for appropriate coefficients $\lambda_{v,j;n} \in \mathbb{C}$ and some remainder term $r_{v,n} \in \mathcal{R}_v$ with $\|r_{v;n}\|_\infty < c_v$. Furthermore, choose elements $\varphi_{v;1}, \ldots, \varphi_{v;\ell_v} \in E'$ with the property

$$\varphi_{v;i}(f_{v,j}) = \delta_{i,j} \qquad \text{and} \qquad \varphi_{v;i}|_{\mathcal{R}_v} = 0 \quad \text{for every } i, j \in \{1, \ldots, \ell_v\}.$$

Set

$$u_v := \|f_v\| \cdot \|A_{d+1}^*\| \max_{1 \leq j \leq \ell_v} \|\varphi_{v;j}\|.$$

(III) Let

$$\mathcal{I}_{d+1} := \{w \in \mathbb{N}^{d+1} | \overline{w} \in \mathcal{I}_d, 1 \leq w^* \leq \ell_{\overline{w}}\}.$$

Also, for each $w \in \mathcal{I}^{d+1}$, let $c_w := c_{\overline{w}}/u_{\overline{w}}\ell_{\overline{w}}$.

(IV) Increase $d$ by 1, and unless $d = m-1$, restart from step (II).

(V) For each $w \in \mathcal{I}_{m-1}$, choose the function $\widetilde{f}_w \in L^\infty$ such that

$$\|f_w - \widetilde{f}_w\|_1 \le \|f_w - \widetilde{f}_w\|_p < c_w \cdot \varepsilon / |\mathcal{I}_{m-1}|.$$

*Proof of Theorem 1.1(1).* Applying the above splitting procedure to $f \in E_{1;s}$, we may bound our original Cesàro averages by a finite sum of averages. For almost every $z \in X$, we have

$$\begin{aligned}
&\frac{1}{N^k} \sum_{1\le n_1,\dots,n_k\le N} |T_m^{n_{\alpha(m)}} A_{m-1} T_{m-1}^{n_{\alpha(m-1)}} \cdots A_2 T_2^{n_{\alpha(2)}} A_1 T_1^{n_{\alpha(1)}} f|(z) \\
&\quad \le \sum_{v\in\mathcal{I}_{m-1}} \frac{1}{N^k} \sum_{1\le n_1,\dots,n_k\le N} |T_m^{n_{\alpha(m)}} f_v|(z) \prod_{l(x)>0, x\subseteq v} |\lambda_{x;n_{\alpha(l(x))}}| \\
&\qquad + \sum_{w\in\mathcal{I}_{l(w)}, 0\le l(w)<m-1} \frac{C^{m-2}}{N^k} \sum_{1\le n_1,\dots,n_k\le N} \left| r_{w;n_{\alpha(l(w)+1)}}(z) \right| \prod_{l(x)>0, x\subseteq w} |\lambda_{x;n_{\alpha(l(x))}}| \\
&\quad \le \sum_{v\in\mathcal{I}_{m-1}} \frac{1}{N^k} \sum_{1\le n_1,\dots,n_k\le N} |T_m^{n_{\alpha(m)}} \widetilde{f}_v|(z) \prod_{l(x)>0, x\subseteq v} |\lambda_{x;n_{\alpha(l(x))}}| \\
&\qquad + \sum_{v\in\mathcal{I}_{m-1}} \frac{1}{N^k} \sum_{1\le n_1,\dots,n_k\le N} \left| T_m^{n_{\alpha(m)}} (f_v - \widetilde{f}_v) \right|(z) \prod_{l(x)>0, x\subseteq v} |\lambda_{x;n_{\alpha(l(x))}}| \\
&\qquad + \sum_{w\in\mathcal{I}_{l(w)}, 0\le l(w)<m-1} \frac{C^{m}}{N^k} \sum_{1\le n_1,\dots,n_k\le N} c_w \prod_{l(x)>0, x\subseteq w} |\lambda_{x;n_{\alpha(l(x))}}|.
\end{aligned}$$

We will bound each of these three sums separately. Note that by the definition of the linear forms, we have for each $v \in \mathcal{I}_{l(v)}$ $(1 \le l(v) \le m-1)$

$$\lambda_{v;n} = \varphi_{\overline{v};v*}(A_{l(v)} T_{l(v)}^n f_{\overline{v}}) = (A_{l(v)}^* \varphi_{\overline{v};v*})(T_{l(v)}^n f_{\overline{v}}),$$

and hence

$$|\lambda_{v;n}| \le \|f_{\overline{v}}\| \cdot \|A_{l(v)}^*\| \max_{1\le j\le \ell_v} \|\varphi_{\overline{v};j}\| = u_{\overline{v}},$$

but also, since $f \in E_{1;s}$, we have $(\lambda_{j;n})_{n\in\mathbb{N}} \in \mathcal{N}$ for each $1 \le j \le \ell$.

Using that $\mathcal{N}$ is closed under multiplication by bounded sequences, on the one hand we obtain that

$$\begin{aligned}
&\lim_{N\to\infty} \sum_{w\in\mathcal{I}_{l(w)}, 0\le l(w)<m-1} \frac{C^m}{N^k} \sum_{1\le n_1,\dots,n_k\le N} c_w \prod_{l(x)>0, x\subseteq w} |\lambda_{x;n_{\alpha(l(x))}}| \\
&\quad = \sum_{w\in\mathcal{I}_{l(w)}, 0\le l(w)<m-1} C^m c_w \Big( \lim_{N\to\infty} \frac{1}{N^k} \sum_{1\le n_1,\dots,n_k\le N} \prod_{l(x)>0, x\subseteq w} |\lambda_{x;n_{\alpha(l(x))}}| \Big) \\
&\quad = C^m \sum_{w\in\mathcal{I}_0} c_w = C^m c = \varepsilon.
\end{aligned}$$

On the other hand, also using that $\widetilde{f}_v$ is essentially bounded for each $v \in \mathcal{I}_{m-1}$ and that $T_m$ as a Dunford–Schwartz operator is a contraction on $L^\infty$, we obtain that

$$\lim_{N\to\infty} \sum_{v\in\mathcal{I}_{m-1}} \frac{1}{N^k} \sum_{1\le n_1,\dots,n_k\le N} |T_m^{n_{\alpha(m)}} \widetilde{f}_v|(z) \prod_{l(x)>0, x\subseteq v} |\lambda_{x;n_{\alpha(l(x))}}| = 0$$

for almost every $z \in X$.

Thus only the middle sum remains to be bounded. To treat that term, we will make use of the pointwise ergodic theorem for Dunford–Schwartz operators. Since the modulus of a Dunford–Schwartz operator is itself Dunford–Schwartz, we may apply the pointwise ergodic theorem to $|T_m|$ and the functions $|f_v - \widetilde{f}_v|$ to obtain that for each $v \in \mathcal{I}_{m-1}$, there exists a function $0 \le \mathbf{f}_v \in L^1$ with $\|\mathbf{f}_v\|_1 \le |f_v - \widetilde{f}_v|_1$ and a set $S_v$ with $\mu(S_v) = 1$ such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |T_m|^n \big|(f_v - \widetilde{f}_v)\big|(z) = \mathbf{f}_v(z)$$

for all $z \in S_v$.

Note that by the norm bound in step (V), there then exists a set $\mathbf{S}_v \subset S_v$ with $\mu(\mathbf{S}_v) > 1 - \varepsilon/|\mathcal{I}_{m-1}|$ such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |T_m|^n \big|(f_v - \widetilde{f}_v)\big|(z) \le c_w$$

for all $z \in \mathbf{S}_v$. We obtain that for every $z \in \bigcap_{v\in\mathcal{I}_{m-1}} \mathbf{S}_v$, we have

$$\begin{aligned}
&\limsup_{N\to\infty} \sum_{v\in\mathcal{I}_{m-1}} \frac{1}{N^k} \sum_{1\le n_1,\dots,n_k\le N} \big|T_m^{n_{\alpha(m)}}(f_v - \widetilde{f}_v)\big|(z) \prod_{l(x)>0, x\subseteq v} |\lambda_{x;n_{\alpha(l(x))}}| \\
&\quad\le \lim_{N\to\infty} \sum_{v\in\mathcal{I}_{m-1}} \frac{1}{N^k} \sum_{1\le n_1,\dots,n_k\le N} \big(|T_m|^{n_{\alpha(m)}} |f_v - \widetilde{f}_v|\big)(z) \prod_{l(x)>0, x\subseteq v} u_{\overline{x}} \\
&\quad\le \sum_{v\in\mathcal{I}_{m-1}} c_v \prod_{l(x)>0, x\subseteq v} u_{\overline{x}} = \varepsilon C^{-m} \le \varepsilon.
\end{aligned}$$

In total, we obtain that for every $\varepsilon > 0$, we have

$$\limsup_{N\to\infty} \frac{1}{N^k} \sum_{1\le n_1,\dots,n_k\le N} |T_m^{n_{\alpha(m)}} A_{m-1} T_{m-1}^{n_{\alpha(m-1)}} \cdots A_2 T_2^{n_{\alpha(2)}} A_1 T_1^{n_{\alpha(1)}} f|(z) < \varepsilon$$

for all $z \in \bigcap_{v\in\mathcal{I}_{m-1}} \mathbf{S}_v$. Since $\mu(\bigcap_{v\in\mathcal{I}_{m-1}} \mathbf{S}_v) > 1 - |\mathcal{I}_{m-1}| \cdot \varepsilon/|\mathcal{I}_{m-1}| = 1 - \varepsilon$, letting $\varepsilon \to 0$ concludes our proof of part (1).

We now turn our attention to part (2), and we show almost-everywhere convergence of the averages on the reversible part $E_{1;r}$ with respect to the operator $T_1$. Again we will proceed by iterated splitting of the function but will also make use of part (1).

Given a function $f \in E_{1;r}$, and an $\varepsilon > 0$, do the following.

(i) First, set $d = 0$, $c := \varepsilon C^{-m}$, and let $\mathcal{I}_0$ consist of the empty index.

(ii) By assumption (A1), for each $f_v$ $(v \in \mathcal{I}_d)$ we may find a decomposition $E = \mathcal{U}_v \oplus \mathcal{R}_v$ with $\ell_v := \dim \mathcal{U}_v < \infty$ and

$$P_{\mathcal{R}_v} \mathscr{A}_{d+1,f_v} \subset B_{c_v}\big(0, L^\infty(X,\mu)\big).$$

For each $v \in \mathcal{I}_d$, choose a maximal linearly independent set $g_{v,1}, \ldots, g_{v,\ell_v}$ in $\mathcal{U}_v$. Then for each $n \in \mathbb{N}$, we can write the unique decomposition

$$A_{d+1} T_{d+1}^n f_v = \lambda_{v,1;n} g_{v,1} + \cdots + \lambda_{v,\ell_v;n} g_{v,\ell_v} + r_{v;n}$$

for appropriate coefficients $\lambda_{v,j;n} \in \mathbb{C}$ and some remainder term $r_{v,n} \in \mathcal{R}_v$ with $\|r_{v;n}\|_\infty < c_v$. Furthermore, choose elements $\varphi_{v;1}, \ldots, \varphi_{v;\ell_v} \in E'$ with the property

$$\varphi_{v;i}(g_{v,j}) = \delta_{i,j} \qquad \text{and} \qquad \varphi_{v;i}|_{\mathcal{R}_v} = 0 \quad \text{for every } i,j \in \{1, \ldots, \ell_v\}.$$

Set

$$u_v := \|f_v\| \cdot \|A_{d+1}^*\| \max_{1 \le j \le \ell_v} \|\varphi_{v;j}\|.$$

(iii) For each $v \in \mathcal{I}_d$ and $1 \le j \le \ell_v$, let $f_{v,j} := P_{E_{d+2;r}} g_{v,j}$ be the reversible part of $g_{v,j}$ with respect to $T_{d+2}$, and let $q_{v,j} := g_{v,j} - f_{v,j}$ be its stable part.

(iv) Let

$$\mathcal{I}_{d+1} := \{w \in \mathbb{N}^{d+1} | \overline{w} \in \mathcal{I}_d, 1 \le w^* \le \ell_{\overline{w}}\}.$$

Also, for each $w \in \mathcal{I}^{d+1}$, let $c_w := c_{\overline{w}} / u_{\overline{w}} \ell_{\overline{w}}$.

(v) Increase $d$ by 1, and unless $d = m-1$, restart from step (II).

□

*Proof of Theorem 1.1(2).* Let us apply the iterated decomposition (i)–(vi) detailed above to the function $f \in E_{1;r}$. We obtain that

$$\begin{aligned}
&\frac{1}{N^k} \sum_{1 \le n_1, \ldots, n_k \le N} T_m^{n_{\alpha(m)}} A_{m-1} T_{m-1}^{n_{\alpha(m-1)}} \cdots A_2 T_2^{n_{\alpha(2)}} A_1 T_1^{n_{\alpha(1)}} f \\
&\quad = \sum_{v \in \mathcal{I}_{m-1}} \frac{1}{N^k} \sum_{1 \le n_1, \ldots, n_k \le N} \Big( \prod_{l(x) > 0, x \subseteq v} \lambda_{x; n_{\alpha(l(x))}} \Big) T_m^{n_{\alpha(m)}} g_v \\
&\qquad + \sum_{w \in \mathcal{I}_{l(w)}, 0 < l(w) < m-1} \frac{1}{N^k} \sum_{1 \le n_1, \ldots, n_k \le N} T_m^{n_{\alpha(m)}} A_{m-1} T_{m-1}^{n_{\alpha(m-1)}} \cdots \\
&\qquad A_{\ell(w)+1} T_{\ell(w)+1}^{n_{\alpha(\ell(w)+1)}} q_w \prod_{l(x) > 0, x \subseteq w} \lambda_{x; n_{\alpha(l(x))}} \\
&\qquad + \sum_{w \in \mathcal{I}_{l(w)}, 0 \le l(w) < m-1} \frac{1}{N^k} \sum_{1 \le n_1, \ldots, n_k \le N} T_m^{n_{\alpha(m)}} A_{m-1} T_{m-1}^{n_{\alpha(m-1)}} \cdots \\
&\qquad A_{\ell(w)+2} T_{\ell(w)+2}^{n_{\alpha(\ell(w)+2)}} r_{w; n_{\alpha(l(w)+1)}} \prod_{l(x) > 0, x \subseteq w} \lambda_{x; n_{\alpha(l(x))}}.
\end{aligned}$$

First, let us look at the terms

$$T_m^{n_{\alpha(m)}} A_{m-1} T_{m-1}^{n_{\alpha(m-1)}} \cdots A_{\ell(w)+1} T_{\ell(w)+1}^{n_{\alpha(\ell(w)+1)}} q_w \prod_{l(x)>0, x\subseteq w} \lambda_{x;n_{\alpha(l(x))}}$$

involving the $q_w$'s. For each $w \in \mathcal{I}_{l(w)}$ with $0 < l(w) < m-1$, we note that the products

$$\prod_{l(x)>0, x\subseteq w} \lambda_{x;n_{\alpha(l(x))}}$$

are bounded in absolute value by the constant

$$\prod_{l(x)>0, x\subseteq w} u_{\overline{x}},$$

and by using part (1) with the new value $m' := m - l(w) > 1$, we obtain for each $w$ that

$$\begin{aligned}
&\limsup_{N\to\infty} \Big| \frac{1}{N^k} \sum_{1\le n_1,\dots,n_k\le N} T_m^{n_{\alpha(m)}} A_{m-1} T_{m-1}^{n_{\alpha(m-1)}} \cdots \\
&\qquad A_{\ell(w)+1} T_{\ell(w)+1}^{n_{\alpha(\ell(w)+1)}} q_w \prod_{l(x)>0, x\subseteq w} \lambda_{x;n_{\alpha(l(x))}} \Big| (z) \\
&\quad \le \Big( \prod_{l(x)>0, x\subseteq w} u_{\overline{x}} \Big) \limsup_{N\to\infty} \frac{1}{N^k} \sum_{1\le n_1,\dots,n_k\le N} |T_m^{n_{\alpha(m)}} A_{m-1} T_{m-1}^{n_{\alpha(m-1)}} \cdots \\
&\qquad A_{\ell(w)+1} T_{\ell(w)+1}^{n_{\alpha(\ell(w)+1)}} q_w | (z) = 0
\end{aligned}$$

for almost every $z \in X$. Since there are finitely many different such terms involving the $q_w$'s, they contribute a total of 0 to the Cesàro means on a set of full measure.

Second, let us look at the terms

$$T_m^{n_{\alpha(m)}} A_{m-1} T_{m-1}^{n_{\alpha(m-1)}} \cdots A_{\ell(w)+2} T_{\ell(w)+2}^{n_{\alpha(\ell(w)+2)}} r_{w;n_{\alpha(l(w)+1)}} \prod_{l(x)>0, x\subseteq w} \lambda_{x;n_{\alpha(l(x))}}$$

involving the $r_{w;*}$'s. Note that since we work on the reversible part and lack a coefficient sequence $\lambda_*$ in $\mathcal{N}$, we cannot conclude the same way as in part (1). Let us therefore fix $w \in \mathcal{I}_{l(w)}$ with $0 \le l(w) < m-1$. Then, using (A2), we have that

$$\begin{aligned}
&\Big\| T_m^{n_{\alpha(m)}} A_{m-1} T_{m-1}^{n_{\alpha(m-1)}} \cdots A_{\ell(w)+2} T_{\ell(w)+2}^{n_{\alpha(\ell(w)+2)}} r_{w;n_{\alpha(l(w)+1)}} \prod_{l(x)>0, x\subseteq w} \lambda_{x;n_{\alpha(l(x))}} \Big\|_\infty \\
&\quad \le C^{m-l(w)-2} \| r_{w;n_{\alpha(l(w)+1)}} \| \prod_{l(x)>0, x\subseteq w} u_{\overline{x}} < C^m c_w \prod_{l(x)>0, x\subseteq w} u_{\overline{x}} \\
&\quad = \varepsilon \prod_{x\subset w} \frac{1}{\ell_x}.
\end{aligned}$$

This in turn implies that for every $N$,

$$\Big\| \sum_{w\in\mathcal{I}_{l(w)},0\le l(w)<m-1} \frac{1}{N^k} \sum_{1\le n_1,\dots,n_k\le N} T_m^{n_{\alpha(m)}} A_{m-1} T_{m-1}^{n_{\alpha(m-1)}} \cdots$$

$$A_{\ell(w)+2} T_{\ell(w)+2}^{n_{\alpha(\ell(w)+2)}} r_{w;n_{\alpha(l(w)+1)}} \prod_{l(x)>0,x\subseteq w} \lambda_{x;n_{\alpha(l(x))}} \Big\|_\infty$$

$$\le \sum_{w\in\mathcal{I}_{l(w)},0\le l(w)<m-1} \frac{1}{N^k} \sum_{1\le n_1,\dots,n_k\le N}$$

$$\times \Big\| T_m^{n_{\alpha(m)}} A_{m-1} T_{m-1}^{n_{\alpha(m-1)}} \cdots A_{\ell(w)+2} T_{\ell(w)+2}^{n_{\alpha(\ell(w)+2)}} r_{w;n_{\alpha(l(w)+1)}} \prod_{l(x)>0,x\subseteq w} \lambda_{x;n_{\alpha(l(x))}} \Big\|_\infty$$

$$< \sum_{w\in\mathcal{I}_{l(w)},0\le l(w)<m-1} \varepsilon \prod_{x\subset w} \frac{1}{\ell_x} = \varepsilon \sum_{d=1}^{m-2} \sum_{w\in\mathcal{I}_d} \prod_{x\subset w} \frac{1}{\ell_x}$$

$$= \varepsilon \sum_{d=1}^{m-2} 1 = \varepsilon(m-2).$$

It only remains to estimate the terms

$$\Big( \prod_{l(x)>0,x\subseteq v} \lambda_{x;n_{\alpha(l(x))}} \Big) T_m^{n_{\alpha(m)}} g_v$$

involving the functions $g_v$ ($v\in\mathcal{I}_{m-1}$). We have

$$\frac{1}{N^k} \sum_{1\le n_1,\dots,n_k\le N} \Big( \prod_{l(x)>0,x\subseteq v} \lambda_{x;n_{\alpha(l(x))}} \Big) T_m^{n_{\alpha(m)}} g_v$$

$$= \Big( \frac{1}{N^{k-1}} \sum_{1\le n_j\le N(1\le j\le k, j\ne\alpha(m))} \Big( \prod_{l(x)>0,\alpha(l(x))\ne\alpha(m),x\subseteq v} \lambda_{x;n_{\alpha(l(x))}} \Big) \Big)$$

$$\times \Big( \frac{1}{N} \sum_{n=1}^{N} \Big( \prod_{l(x)>0,\alpha(l(x))=\alpha(m),x\subseteq v} \lambda_{x;n} \Big) T_m^n g_v \Big).$$

We will show that as $N$ tends to infinity, the first, complex-valued factor is convergent, whereas the second, function-valued factor converges almost everywhere. This will then imply that the product also converges almost everywhere.

Let us fix $v\in\mathcal{I}_{m-1}$. We obtain for each $x\subseteq v$ with $l(x)>0$ that

$$\lambda_{x;n} = \varphi_{\overline{x};x*}(A_{l(x)} T_{l(x)}^n f_{\overline{x}}) = \langle A_{l(x)}^* \varphi_{\overline{x};x*}, T_{l(x)}^n f_{\overline{x}} \rangle,$$

and since $f_{\overline{x}}$ is in the reversible part of $E$ with respect to $T_{l(x)}$, the sequence $(\lambda_{x;n})_{n\in\mathbb{N}}$ is a reversible linear sequence. Using that $\mathscr{P}$ is closed under multiplication, we have that for each $1\le j\le m$

$$\Big( \prod_{l(x)>0,\alpha(l(x))=j,x\subseteq v} \lambda_{x;n} \Big)_{n\in\mathbb{N}} \in \mathscr{P}.$$

In particular, for each $v \in \mathcal{I}_{m-1}$, the Cesàro means

$$\Big(\frac{1}{N^{k-1}} \sum_{1\le n_j\le N(1\le j\le k, j\ne\alpha(m))} \Big(\prod_{l(x)>0,\alpha(l(x))\ne\alpha(m),x\subseteq v} \lambda_{x;n_{\alpha(l(x))}}\Big)\Big)$$

converge.

Finally, let us turn our attention to the factor

$$\frac{1}{N}\sum_{n=1}^{N}\Big(\prod_{l(x)>0,\alpha(l(x))=\alpha(m),x\subseteq v} \lambda_{x;n}\Big) T_m^n g_v.$$

Since elements of $\mathscr{P}$ are good weights for the pointwise ergodic theorem for Dunford–Schwartz operators, this converges pointwise almost everywhere.

In conclusion, for almost every $z \in X$, we have

$$\begin{aligned}
&\Big(\limsup_{N\to\infty} - \liminf_{N\to\infty}\Big)\frac{1}{N^k}\sum_{1\le n_1,\dots,n_k\le N} (T_m^{n_{\alpha(m)}} A_{m-1} T_{m-1}^{n_{\alpha(m-1)}} \cdots A_2 T_2^{n_{\alpha(2)}} A_1 T_1^{n_{\alpha(1)}} f)(z)\\
&\quad \le \sum_{v\in\mathcal{I}_{m-1}} \Big(\limsup_{N\to\infty} - \liminf_{N\to\infty}\Big)\frac{1}{N^k}\sum_{1\le n_1,\dots,n_k\le N}\Big(\prod_{l(x)>0,x\subseteq v}\lambda_{x;n_{\alpha(l(x))}}\Big)(T_m^{n_{\alpha(m)}} g_v)(z)\\
&\qquad + \sum_{w\in\mathcal{I}_{l(w)},0<l(w)<m-1}\Big(\limsup_{N\to\infty} - \liminf_{N\to\infty}\Big)\frac{1}{N^k}\\
&\qquad \times \sum_{1\le n_1,\dots,n_k\le N} (T_m^{n_{\alpha(m)}} A_{m-1} T_{m-1}^{n_{\alpha(m-1)}} \cdots\\
&\qquad A_{\ell(w)+1} T_{\ell(w)+1}^{n_{\alpha(\ell(w)+1)}} q_w)(z) \prod_{l(x)>0,x\subseteq w}\lambda_{x;n_{\alpha(l(x))}}\\
&\qquad + \sum_{w\in\mathcal{I}_{l(w)},0\le l(w)<m-1}\Big(\limsup_{N\to\infty} - \liminf_{N\to\infty}\Big)\frac{1}{N^k}\\
&\qquad \times \sum_{1\le n_1,\dots,n_k\le N} (T_m^{n_{\alpha(m)}} A_{m-1} T_{m-1}^{n_{\alpha(m-1)}} \cdots\\
&\qquad A_{\ell(w)+2} T_{\ell(w)+2}^{n_{\alpha(\ell(w)+2)}} r_{w;n_{\alpha(l(w)+1)}})(z) \prod_{l(x)>0,x\subseteq w}\lambda_{x;n_{\alpha(l(x))}}\\
&\quad = 0 + 0 + \sum_{w\in\mathcal{I}_{l(w)},0\le l(w)<m-1}\Big(\limsup_{N\to\infty} - \liminf_{N\to\infty}\Big)\\
&\qquad \times \frac{1}{N^k}\sum_{1\le n_1,\dots,n_k\le N} (T_m^{n_{\alpha(m)}} A_{m-1} T_{m-1}^{n_{\alpha(m-1)}} \cdots\\
&\qquad A_{\ell(w)+2} T_{\ell(w)+2}^{n_{\alpha(\ell(w)+2)}} r_{w;n_{\alpha(l(w)+1)}})(z) \prod_{l(x)>0,x\subseteq w}\lambda_{x;n_{\alpha(l(x))}}\\
&\quad \le 2\sup_{N\in\mathbb{N}}\Big\| \sum_{w\in\mathcal{I}_{l(w)},0\le l(w)<m-1}\frac{1}{N^k}
\end{aligned}$$

$$\times \sum_{1\le n_1,\dots,n_k\le N} T_m^{n_{\alpha(m)}} A_{m-1} T_{m-1}^{n_{\alpha(m-1)}} \cdots$$
$$A_{\ell(w)+2} T_{\ell(w)+2}^{n_{\alpha(\ell(w)+2)}} r_{w;n_{\alpha(l(w)+1)}} \prod_{l(x)>0, x\subseteq w} \lambda_{x;n_{\alpha(l(x))}} \Big\|_\infty$$
$$\le 2\varepsilon(m-2).$$

Since this holds for every $\varepsilon > 0$, we have thus concluded the proof of Theorem 1.1(2). ☐

*Remark.* The pointwise limit—if it exists—is clearly the same as the strong limit, and takes the form given in [7, Theorem 3].

## 4. Pointwise polynomial ergodic version

In this section, our goal is to prove a polynomial version of Theorem 1.1. With these tools in hand, we can now state and prove almost everywhere pointwise convergence of entangled means on Hilbert spaces.

**Theorem 4.1.** *Let $m > 1$ and $k$ be positive integers, let $\alpha : \{1,\dots,m\} \to \{1,\dots,k\}$ be a not-necessarily surjective map, and let $T_1, T_2, \dots, T_m$ be Dunford–Schwartz operators on a standard probability space $(X,\mu)$. Let $E := L^2(X,\mu)$, and let $E = E_{j,r} \oplus E_{j,s}$ be the Jacobs–de Leeuw–Glicksberg decomposition corresponding to $T_j$ $(1 \le j \le m)$. Furthermore, let $A_j \in \mathcal{L}(E)$ $(1 \le j < m)$ be bounded operators. Suppose that conditions (A1) and (A2) of Theorem 1.1 hold. Furthermore, let $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_k$ be nonconstant polynomials with integer coefficients taking positive values on $\mathbb{N}$. Then we have the following:*

(1) *for each $f \in E_{1,s}$,*
$$\frac{1}{N^k} \sum_{1\le n_1,\dots,n_k\le N} |T_m^{\mathbf{q}_{\alpha(m)}(n_{\alpha(m)})} \cdots A_2 T_2^{\mathbf{q}_{\alpha(2)}(n_{\alpha(2)})} A_1 T_1^{\mathbf{q}_{\alpha(1)}(n_{\alpha(1)})} f| \to 0$$
*pointwise almost everywhere;*

(2) *for each $f \in E_{1,r}$, the averages*
$$\frac{1}{N^k} \sum_{1\le n_1,\dots,n_k\le N} T_m^{\mathbf{q}_{\alpha(m)}(n_{\alpha(m)})} \cdots A_2 T_2^{\mathbf{q}_{\alpha(2)}(n_{\alpha(2)})} A_1 T_1^{\mathbf{q}_{\alpha(1)}(n_{\alpha(1)})} f$$
*converge pointwise almost everywhere.*

*Proof.* We will follow the proof of Theorem 1.1, using the same recursive splitting. The question is then why the convergences still hold when averaging along polynomial subsequences.

For part (1), we have three terms to bound: those involving the remainder functions $r_{*;n}$, the ones involving the essentially bounded functions $\widetilde{f}_*$, and finally the ones with the small approximation errors $f_* - \widetilde{f}_*$. Using Corollary 2.6, we obtain that the subsequences $\lambda_{j;q(n)}$ involved $(1 \le j \le \ell)$ also lie in $\mathcal{N}$, leading to the same bounds as in the linear case for the first two types of terms. For the terms involving the functions $f_* - \widetilde{f}_*$, we use the polynomial version of the pointwise ergodic theorem for Dunford–Schwartz operators, Theorem 2.4, to obtain that for each $v \in \mathcal{I}_{m-1}$, there exists a function $0 \leq \mathbf{f}_v \in L^1$ and a set $S_v$ with $\mu(S_v) = 1$ such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |T_m|^{\mathbf{q}(n)} \big|(f_v - \widetilde{f}_v)\big|(z) = \mathbf{f}_v(z)$$

for all $z \in S_v$. Since the polynomial Cesàro means are also contractive in $L^1$ for Dunford–Schwartz operators, the rest of the arguments remain unchanged, and this concludes the proof of part (1).

For part (2), we again have three types of terms. The terms involving the functions $q_*$ can again be treated using part (1) and shown to have a zero contribution almost everywhere, and the terms with the $r_{*;n}$'s also do not require any change in the arguments used. Only the terms involving the functions $g_v$ ($v \in \mathcal{I}_{m-1}$) remain. For these, we use Lemma 2.3 combined with Theorem 2.4 to obtain the almost-everywhere convergence needed. $\square$

## 5. The continuous case

In this section, we finally turn our attention to a variant of the above results, where we replace the discrete action of the Dunford–Schwartz operators with the continuous-action $C_0$-semigroups. In other words, the semigroups $\{T_i^n | n \in \mathbb{N}^+\}$ are replaced by strongly continuous semigroups $\{T_i(t) | t \in [0, \infty)\}$.

Let $T(\cdot) := (T(t))_{t\in[0,\infty)}$ be a $C_0$-semigroup of Dunford–Schwartz operators on $L^1(X, \mu)$. Then, by the standard approximation argument, using that the unit ball in $L^\infty(X, \mu)$ is invariant under the semigroup, $T(\cdot)$ is automatically a $C_0$-semigroup (of contractions) on $L^p(X, \mu)$ for every $1 \leq p < \infty$. In addition, by Fubini's theorem (see, e.g., Satō [23, p. 3]), for every $f \in L^1(X, \mu)$ the function $(T(\cdot)f)(x)$ is Lebesgue integrable over finite intervals in $[0, \infty)$ for almost every $x \in X$. Similarly, for $C_0$-semigroups $T_0(\cdot), \ldots, T_a(\cdot)$ on $E := L^p(X, \mu)$, operators $A_0, \ldots, A_{a-1} \in \mathcal{L}(E)$, and $f \in E$, the product

$$\big(T_a(\cdot)A_{a-1}T_{a-1}(\cdot) \cdots A_1T_1(\cdot)A_0T_0(\cdot)f\big)(x)$$

is Lebesgue integrable over finite intervals in $[0, \infty)$ for almost every $x \in X$.

By Dunford and Schwartz [4, pp. 694, 708], the pointwise ergodic theorem extends to every strongly measurable semigroup $T(\cdot)$ of Dunford–Schwartz operators. In addition, it can be shown through a simple adaptation of the arguments in Lin, Olsen, and Tempelman [21, Proof of Proposition 2.6] that every $C_0$-semigroup of Dunford–Schwartz operators has relatively weakly compact orbits in $L^1(X, \mu)$. Thus, the continuous version of the Jacobs–de Leeuw–Glicksberg decomposition (see, e.g., [5, Theorem III.5.7]) is valid for such semigroups.

In the discrete case, the modulus $|T|$ of the operator $T$ was used to obtain a discrete semigroup of positive operators that dominates $(T^n)_{n\in\mathbb{N}}$ while keeping the Dunford–Schwartz property. The time-continuous case turns out to be more involved, as there is no "first" operator whose modulus can be used to generate the dominating semigroup. Just as in the discrete case, we usually have $|T^2| \neq |T|^2$; in the $C_0$ setting, $(|T(t)|)_{t\geq 0}$ will generally not be a strongly continuous semigroup. For example, by Kipnis [13] or Kubokawa [16], for a $C_0$-semigroup $T(\cdot)$ of contractions there exists a minimal $C_0$-semigroup of positive operators dominating $T(\cdot)$, which we will denote by $|T|(\cdot)$. Of course, $|T|(\cdot) = T(\cdot)$ for positive semigroups. Moreover, the construction in [13, pp. 372–373] implies that if $T(\cdot)$ consists of Dunford–Schwartz operators, then so does $|T|(\cdot)$.

With the above, the proof of Theorem 1.1 can be extended to the time-continuous setting to obtain the following $C_0$ version of our main theorem.

**Theorem 5.1.** *Let $m > 1$ and $k$ be positive integers, let $\alpha : \{1,\dots,m\} \to \{1,\dots,k\}$ be a not-necessarily surjective map, and let $(T_1(t))_{t\geq 0},\dots,(T_m(t))_{t\geq 0}$ be $C_0$-semigroups of Dunford–Schwartz operators on a standard probability space $(X,\mu)$. Let $p \in [1,\infty)$, let $E := L^p(X,\mu)$, and let $E = E_{j,r} \oplus E_{j,s}$ be the Jacobs–de Leeuw–Glicksberg decomposition corresponding to $T_j(\cdot)$ $(1 \leq j \leq m)$. Furthermore, let $A_j \in \mathcal{L}(E)$ $(1 \leq j < m-1)$ be bounded operators. For a function $f \in E$ and an index $1 \leq j \leq m-1$, write $\mathscr{A}_{j,f} := \{A_jT_j(t)f | t \in [0,\infty)\}$. Suppose that the following conditions hold.*

(A1c) *(Twisted compactness) For any function $f \in E$, index $1 \leq j \leq m-1$, and $\varepsilon > 0$, there exists a decomposition $E = \mathcal{U} \oplus \mathcal{R}$ with $\dim \mathcal{U} < \infty$ such that*

$$P_{\mathcal{R}}\mathscr{A}_{j,f} \subset B_\varepsilon\big(0, L^\infty(X,\mu)\big),$$

*with $P_{\mathcal{R}}$ denoting the projection onto $\mathcal{R}$ along $\mathcal{U}$.*

(A2c) *(Joint $\mathcal{L}^\infty$-boundedness) There exists a constant $C > 0$ such that we have*

$$\big\{A_jT_j(t) | t \in [0,\infty), 1 \leq j \leq m-1\big\} \subset B_C\big(0, \mathcal{L}\big(L^\infty(X,\mu)\big)\big).$$

*Then we have the following:*

(1) *for each $f \in E_{1,s}$,*

$$\lim_{\mathcal{T}\to\infty} \frac{1}{\mathcal{T}^k} \int_{\{t_1,\dots,t_k\}\in[0,\mathcal{T}]^k} \big|T_m(t_{\alpha(m)}) \cdots A_2T_2(t_{\alpha(2)})A_1T_1(t_{\alpha(1)})f\big| \to 0$$

*pointwise almost everywhere;*

(2) *for each $f \in E_{1,r}$,*

$$\frac{1}{\mathcal{T}^k} \int_{\{t_1,\dots,t_k\}\in[0,\mathcal{T}]^k} T_m(t_{\alpha(m)})A_{m-1}T_{m-1}(t_{\alpha(m-1)}) \cdots A_2T_2(t_{\alpha(2)})A_1T_1(t_{\alpha(1)})f$$

*converges pointwise almost everywhere.*

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