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RELATIVELY COMPACT SETS IN VARIABLE-EXPONENT LEBESGUE SPACES

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Abstract. We study totally bounded sets in variable Lebesgue spaces. The full characterization of this kind of sets is given for the case of variable Lebesgue space on metric measure spaces. Furthermore, the sufficient conditions for compactness are shown without assuming log-Hölder continuity of the exponent.

Variable-exponent Lebesgue and Sobolev spaces are the natural extensions of the classical constant exponent L^p-spaces. This kind of theory finds many applications, for example in nonlinear elastic mechanics (see [28]), electrorheological fluids (see [27]), or image restoration (see [22]). During the last decade Lebesgue and Sobolev spaces with variable exponents have been studied intensively (see, e.g., the survey paper [6]). In particular, the Sobolev inequalities have been shown for variable-exponent spaces on Euclidean spaces (see [5], [7] and [4]) and on Riemannian manifolds (see [10]). Moreover, other types of spaces with variable exponent have been considered, such as Hardy spaces, Campanato spaces, and Besov spaces (see [24] and the references therein). Recently, the theory of variable-exponent spaces has been extended on metric measure spaces (in this context, we especially highlight [9], [16], [17] and [23]).

In our article we investigate relatively compact (precompact) sets in the variable Lebesgue space. In the classical L^p-spaces, the relatively compact sets are characterized by the celebrated Riesz–Kolmogorov theorem (see [20], [26]). The

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aim of this paper is to give a characterization of precompact sets in the variable Lebesgue space on arbitrary metric measure spaces equipped with doubling measures. Similar results have been shown recently (see [13]) for metric spaces with doubling measure satisfying some additional condition. In addition, we find sufficient conditions for compactness in these spaces, without assuming the log-Hölder continuity of the exponent. In the theory of variable-exponent spaces, the log-Hölder continuity is a commonly used assumption on the exponent; in particular, it guarantees the Sobolev-type embeddings for the variable Hajłasz–Sobolev spaces defined on metric spaces with a doubling measure (see [16]). Nevertheless, there are results in the theory of variable-exponent spaces achieved without log-Hölder continuity (see, e.g., [8]).

Let us mention some generalizations of the Riesz–Kolmogorov theorem (see [13], [14] for the complete bibliography). For instance, the papers [12], [15], [19], and [21] contain the characterizations of precompact sets in $L^p(X, \varrho, \mu)$, where (X, ϱ, μ) is a metric measure space. On the other hand, a sufficient condition for a set to be precompact in $L^p(\mathbb{R}^n, w\,dx)$, where w belongs to the A_p Muckenhoupt class, has been proved in [3]. More recently, the Riesz–Kolmogorov-type theorems have been extended to the variable-exponent Lebesgue spaces $L^{p(\cdot)}$ (see [25] for the Euclidean case and [13] for the case of metric measure spaces) and Banach function spaces (see [14]).

This paper is structured as follows. Section 1 presents fundamental details. In Section 2, we introduce the required norms, function spaces, and we recall standard results from the theory of variable-exponent spaces. We also recall basic facts about metric measure spaces. A characterization of relatively compact sets in the variable Lebesgue space on metric measure spaces is proved in Section 3.

1. Preliminaries

1.1. **Variable-exponent Lebesgue spaces.** We start by recalling some notation and basic facts about variable-exponent Lebesgue spaces. (Most of the properties for these spaces can be found in the book by Cruz-Uribe and Fiorenza [4] and in the monograph by Diening, Harjulehto, Hästö, and Růžička [5].)

Let (Ω, μ) be a σ-finite, complete measure space. By a *variable exponent*, we mean a bounded measurable function $p : \Omega \to (0, \infty]$. We denote the set of variable exponents on Ω by $\mathcal{P}(\Omega)$. For $U \subset \Omega$, we put

$$p_+(U) = \operatorname*{ess\,sup}_{x \in U} p(x), \qquad p_-(U) = \operatorname*{ess\,inf}_{x \in U} p(x).$$

If $U = \Omega$, we will write p_+, p_-. In the present paper, we assume that the variable-exponent functions are bounded (i.e., $0 < p_- \le p_+ < \infty$).

The variable-exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ consists of those μ-measurable functions $f : \Omega \to \mathbb{R}$, for which the expression

$$\rho_{p(\cdot)}(f) = \int_\Omega \left|f(x)\right|^{p(x)} d\mu(x)$$

is finite. This is a quasi-Banach space with respect to the expression

$$\|f\|_{p(\cdot)} = \inf\Big\{\lambda > 0 : \rho_{p(\cdot)}\Big(\frac{f}{\lambda}\Big) \leq 1\Big\},$$

where $f \in L^{p(\cdot)}(\Omega)$. In the case when $p_- \geq 1$, the variable-exponent Lebesgue space is a Banach function space. The variable Lebesgue space is a special case of the Musielak–Orlicz spaces. If the variable exponent p is constant, then $L^{p(\cdot)}(\Omega)$ is an ordinary Lebesgue space. It is needed to pass between norm and semimodular very often. In general, there are no functional relationships between norm and modular, but we have the following useful result.

Proposition 1.1. *Let $0 < p_- \leq p_+ < \infty$ and let $f \in L^{p(\cdot)}$. Then,*

$$\min\big\{\big(\rho_{p(\cdot)}(f)\big)^{1/p_-}, \big(\rho_{p(\cdot)}(f)\big)^{1/p_+}\big\} \leq \|f\|_{L^{p(\cdot)}} \leq \max\big\{\big(\rho_{p(\cdot)}(f)\big)^{1/p_-}, \big(\rho_{p(\cdot)}(f)\big)^{1/p_+}\big\}.$$

Moreover, if $p_- > 1$, then the Hölder inequality

$$\|fg\|_{L^1(\Omega)} \leq 2\|f\|_{L^{p(\cdot)}(\Omega)}\|g\|_{L^{p'(\cdot)}(\Omega)}$$

holds, where, as usual, $1 = \frac{1}{p(x)} + \frac{1}{p'(x)}$. Furthermore, we have the following lemma.

Lemma 1.2. *Let $0 < p_- \leq p_+ < \infty$ and let $f, g \in L^{p(\cdot)}(X)$. Then*

$$\|f+g\|_{L^{p(\cdot)}(\Omega)} \leq \max\{2^{\frac{1}{p_-}}, 2^{\frac{p_+}{p_-}}\}\big(\|f\|_{L^{p(\cdot)}(X)} + \|g\|_{L^{p(\cdot)}(X)}\big).$$

Proof. For brevity and the convenience of the reader, we give only a sketch of the proof. From the well-known inequality

$$(a+b)^\alpha \leq \max\{1, 2^{\alpha-1}\}(a^\alpha + b^\alpha),$$

where $a, b \geq 0$ and $\alpha > 0$, we obtain

$$\begin{aligned}
&\int_X \Big(\frac{|f(x)+g(x)|}{\max\{2^{\frac{1}{p_-}}, 2^{\frac{p_+}{p_-}}\}(\|f\|_{L^{p(\cdot)}(X,\mu)} + \|g\|_{L^{p(\cdot)}(X,\mu)})}\Big)^{p(x)} d\mu(x) \\
&\quad\leq \max\{1, 2^{p_+-1}\}\int_X \Big(\frac{|f(x)|}{\max\{2^{\frac{1}{p_-}}, 2^{\frac{p_+}{p_-}}\}(\|f\|_{L^{p(\cdot)}(X,\mu)} + \|g\|_{L^{p(\cdot)}(X,\mu)})}\Big)^{p(x)} d\mu(x) \\
&\qquad + \max\{1, 2^{p_+-1}\}\int_X \Big(\frac{|g(x)|}{\max\{2^{\frac{1}{p_-}}, 2^{\frac{p_+}{p_-}}\}(\|f\|_{L^{p(\cdot)}(X,\mu)} + \|g\|_{L^{p(\cdot)}(X,\mu)})}\Big)^{p(x)} d\mu(x) \\
&\quad\leq \max\{1, 2^{p_+-1}\}\int_X \big[2\max\{1, 2^{p_+-1}\}\big]^{-\frac{p(x)}{p_-}}\Big(\frac{|f(x)|}{\|f\|_{L^{p(\cdot)}(X,\mu)}}\Big)^{p(x)} d\mu(x) \\
&\qquad + \max\{1, 2^{p_+-1}\}\int_X \big[2\max\{1, 2^{p_+-1}\}\big]^{-\frac{p(x)}{p_-}}\Big(\frac{|g(x)|}{\|g\|_{L^{p(\cdot)}(X,\mu)}}\Big)^{p(x)} d\mu(x) \\
&\quad\leq \frac{1}{2}\Big(\int_X \Big(\frac{|f(x)|}{\|f\|_{L^{p(\cdot)}(X,\mu)}}\Big)^{p(x)} d\mu(x) + \int_X \Big(\frac{|g(x)|}{\|g\|_{L^{p(\cdot)}(X,\mu)}}\Big)^{p(x)} d\mu(x)\Big) = 1.
\end{aligned}$$

Thus,

$$\|f+g\|_{L^{p(\cdot)}(X,\mu)} \le \max\{2^{\frac{1}{p_-}}, 2^{\frac{p_+}{p_-}}\}\big(\|f\|_{L^{p(\cdot)}(X,\mu)} + \|g\|_{L^{p(\cdot)}(X,\mu)}\big). \qquad \square$$

Let (X,ρ) be a metric space $\Omega \subset X$. We say that a function $p:\Omega\to\mathbb{R}$ is *locally* log-*Hölder-continuous* on Ω if

$$\exists_{C_1>0}\ \forall_{x,y\in\Omega}\quad \big|p(x)-p(y)\big| \le \frac{C_1}{\log(e+\frac{1}{\rho(x,y)})}.$$

In addition, we say that the exponent function p satisfies the log-Hölder decay condition at infinity with a fixed point $x_0 \in X$ if

$$\exists_{p_\infty\in R}\ \exists_{C_2>0}\ \forall_{x\in\Omega}\quad \big|p(x)-p_\infty\big| \le \frac{C_2}{\log(e+\rho(x,x_0))}.$$

We also say that p is *globally* log-*Hölder-continuous* on Ω if it is locally log-Hölder-continuous on Ω and satisfies the log-Hölder decay condition at infinity. Then, the constant

$$C_{\log}(p) := \max\{C_1, C_2\}$$

is called the *log-Hölder constant* related to an exponent p. Subsequently, we define the set of log-Hölder continuous exponents by

$$\mathcal{P}_{\log}(\Omega) = \Big\{p\in\mathcal{P}(\Omega) : \frac{1}{p} \text{ is globally log-Hölder continuous}\Big\}.$$

1.2. **Metric measure spaces.** Let (X,ϱ,μ) be a metric measure space equipped with a metric ρ and the Borel regular measure μ. Denote by

$$B(x,r) = \big\{y\in X : \rho(x,y) < r\big\}$$

a ball of the radius $r>0$ with a center $x\in X$. We assume throughout the present article that the measure of every open nonempty set is positive and that the measure of every bounded set is finite. Additionally, we suppose that the measure μ satisfies a doubling condition. This means that there exists a constant $C_\mu > 0$ such that for every ball $B(x,r)$,

$$\mu\big(B(x,2r)\big) \le C_\mu \mu\big(B(x,r)\big).$$

It is well known (see [18]) that the doubling condition implies that there exists a positive constant D satisfying

$$\frac{\mu(B(x_2,r_2))}{\mu(B(x_1,r_1))} \le D\Big(\frac{r_2}{r_1}\Big)^s \quad \text{where } s=\log_2 C_\mu,$$

for all balls $B(x_2,r_2)$ and $B(x_1,r_1)$, with $r_2 \ge r_1 > 0$ and $x_1 \in B(x_2,r_2)$. It follows from the above inequality that if we fix a ball $B(x_0,R)$, then there exists $b>0$ such that the following inequality holds for $x\in B(x_0,R)$ and for $r\le R$:

$$\mu\big(B(x,r)\big) \ge br^s. \tag{1.1}$$

In particular, if X is bounded, then there exists $b>0$ such that the following inequality holds for $r < \operatorname{diam} X$:

$$\mu\big(B(x,r)\big) \ge br^s. \tag{1.2}$$

On the other hand, if the metric measure space equipped with a doubling measure is not bounded, then inequality (1.2) does not necessarily hold.

Furthermore, we define $(f)_A$ the integral average of the function f over the set A; that is,

$$(f)_A := \fint_A f\,d\mu = \frac{1}{\mu(A)} \int_A f\,d\mu.$$

1.3. **Space of measurable functions $L^0(X)$.** Let (X,μ) be a measure space such that $\mu(X) < \infty$. Then, by $L^0(X)$ we denote the space of measurable functions on X. This space is a complete metric space with respect to the metric

$$d_0(f,g) = \int_X \phi(f-g)\,d\mu,$$

where

$$\phi(t) = \frac{|t|}{1+|t|}.$$

It is well known that the convergence in this metric is equivalent to the convergence in measure. Using the method from [21], one can show the following theorem.

Theorem 1.3. *Let (X,ρ,μ) be a totally bounded metric measure space such that $\mu(X) < \infty$. A subset $\mathcal{F}$ of $L^0(X)$ is totally bounded if it is almost uniformly bounded and almost equicontinuous; that is, for any $\varepsilon > 0$, there exist $\delta > 0$ and $\lambda > 0$ such that, for any function $f \in \mathcal{F}$, there exists a measurable subset $E(f) \subset X$ satisfying the properties*

(1) $\mu(E(f)) < \varepsilon$;
(2) $|f(x) - f(y)| < \varepsilon$ *for* $x, y \in X \setminus E(f)$, $\rho(x,y) < \delta$;
(3) $|f(x)| \le \lambda$ *for* $x \in X \setminus E(f)$.

2. Totally bounded sets in $L^{p(\cdot)}(X,\varrho,\mu)$

In this section, we study totally bounded sets in $L^{p(\cdot)}(X,\varrho,\mu)$, where (X,ϱ,μ) is a metric space equipped with a doubling measure. We start with the following version of the Lebesgue–Vitali compactness-type theorem.

Theorem 2.1. *Suppose that (X,μ) is a measure space such that $\mu(X) < \infty$ and $p \in \mathcal{P}(X,\mu)$, $0 < p_- \le p_+ < \infty$. Then, a subset $\mathcal{F}$ of $L^{p(\cdot)}(X,\mu)$ is totally bounded if and only if the following conditions are satisfied:*

(i) *$\mathcal{F}$ is totally bounded in $L^0(X)$;*
(ii) *the family $\mathcal{F}$ is $p(\cdot)$-equi-integrable, that is,*

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{A\subset X} \quad \mu(A) < \delta \quad \Rightarrow \quad \sup_{f\in\mathcal{F}} \int_A \left|f(x)\right|^{p(x)} d\mu(x) < \varepsilon.$$

Proof. Assume that (i) and (ii) hold. Then, for a fixed $\epsilon > 0$, there exists $\delta > 0$ such that, for each set E satisfying $\mu(E) < \delta$, we have

$$\sup_{f\in\mathcal{F}} \int_E \left|f(x)\right|^{p(x)} d\mu(x) < \varepsilon.$$

Next, let us fix $\alpha \in (0,1)$ such that $\alpha^{p_-}\mu(X) < 2^{p_+ +1}\epsilon$, and suppose that $\{f_k\}_{k=1}^n \subset \mathcal{F}$ is a $\delta\phi(\alpha)$-net in $L^0(X)$. Then, for any $f \in \mathcal{F}$, there exists $k \in \{1, \dots, n\}$ such that $d_0(f, f_k) < \delta\phi(\alpha)$. Therefore, we get

$$\begin{aligned}
&\int_X \big|f(x) - f_k(x)\big|^{p(x)}\, d\mu(x) \\
&\quad = \int_X \big|f(x) - f_k(x)\big|^{p(x)}\, d\mu(x) \\
&\quad \le \int_{\{|f-f_k|\le\alpha\}} \big|f(x) - f_k(x)\big|^{p(x)}\, d\mu(x) + \int_{\{|f-f_k|>\alpha\}} \big|f(x) - f_k(x)\big|^{p(x)}\, d\mu(x) \\
&\quad \le \int_X \alpha^{p(x)}\, d\mu(x) + \int_{\{|f-f_k|>\alpha\}} \big|f(x) - f_k(x)\big|^{p(x)}\, d\mu(x) \\
&\quad \le \mu(X)\alpha^{p_-} + 2^{p_+ +1} \sup_{g\in\mathcal{F}} \int_{\{|f-f_k|>\alpha\}} \big|g(x)\big|^{p(x)}\, d\mu(x).
\end{aligned}$$

Moreover, in view of the monotonicity of ϕ and the Markov inequality, we get

$$\begin{aligned}
\mu\big(\{|f - f_k| > \alpha\}\big) &\le \mu\big(\{\phi\big(|f - f_k|\big) > \phi(\alpha)\}\big) \\
&\le \frac{1}{\phi(\alpha)} \int_X \phi\big(|f - f_k|\big)\, d\mu = \frac{d_0(f, f_k)}{\phi(\alpha)} < \delta.
\end{aligned}$$

Hence,

$$\int_X \big|f(x) - f_k(x)\big|^{p(x)}\, d\mu(x) \le 2^{p_+ +2}\varepsilon,$$

which proves that the set $\mathcal{F}$ is totally bounded in $L^{p(\cdot)}(X)$.

Now, assume that the family $\mathcal{F}$ is totally bounded. We aim to show that conditions (i) and (ii) are satisfied.

(i) Let us fix $\varepsilon > 0$ and take $\alpha \in (0,1)$ such that $\phi(\alpha)\mu(X) \le \varepsilon$. Additionally, let $\{f_k\}_{k=1}^n$ denote a $\varepsilon\alpha^{p_-}$-net in $L^{p(\cdot)}(X)$. Then, for any $f \in \mathcal{F}$, there exists $k \in \{1, \dots, n\}$ such that

$$\int_X \big|f(x) - f_k(x)\big|^{p(x)} d\mu(x) \le \varepsilon\alpha^{p_-}.$$

Therefore, we obtain

$$\begin{aligned}
d_0(f - f_k) &= \int_{|f-f_k|\le\alpha} \phi(f - f_k)\, d\mu + \int_{|f-f_k|>\alpha} \phi(f - f_k)\, d\mu \\
&\le \phi(\alpha)\mu(X) + \int_{|f-f_k|>\alpha} d\mu \le \varepsilon + \int_X \frac{|f(x) - f_k(x)|^{p(x)}}{\alpha^{p(x)}}\, d\mu(x) \le 2\varepsilon,
\end{aligned}$$

which proves (i).

(ii) We fix $\varepsilon > 0$ and assume that $\{f_k\}_{k=1}^n$ is a ε-net in $L^{p(\cdot)}(X)$. Since $f_k \in L^{p(\cdot)}(X, \mu)$, there exist $\delta_k > 0$, $k = 0, 1, \dots, n$, for which the following property holds:

$$\forall_{k=0,1,\dots,n}\ \forall_{A\subset X} \quad \mu(A) < \delta_k \quad \Rightarrow \quad \int_A \big|f_k(x)\big|^{p(x)}\, d\mu(x) < \varepsilon.$$

Then, for $\delta = \min\{\delta_k : k = 0, 1, \dots, n\}$ and $A \subset X$ satisfying $\mu(A) < \delta$, we have

$$\int_A \left|f_k(x)\right|^{p(x)} d\mu(x) < \varepsilon.$$

Since, for any $f \in \mathcal{F}$, there exists k such that $\int_X |f(x) - f_k(x)|^{p(x)} d\mu(x) \le \varepsilon$, we get

$$\begin{aligned}\int_A \left|f(x)\right|^{p(x)} d\mu(x) &\le 2^{p_+}\Big(\int_A \left|f_k(x) - f(x)\right|^{p(x)} d\mu(x) + \int_A \left|f_k(x)\right|^{p(x)} d\mu(x)\Big)\\ &\le 2^{p_+ + 1}\varepsilon.\end{aligned}$$

Consequently, $\mathcal{F}$ is $p(\cdot)$-equi-integrable. □

The following theorem will be needed in our further proofs.

Theorem 2.2 (see [1])**.** *Suppose that: (X, ρ, μ) is a metric measure space with a doubling measure, $p \in \mathcal{P}_{\log}(X, \mu)$, and suppose that $p_- > 1$. Then,*

$$\left\|\mathcal{M}(f)\right\|_{L^{p(\cdot)}(X,\mu)} \le \frac{Cp_-}{p_- - 1}\|f\|_{L^{p(\cdot)}(X,\mu)},$$

where a constant C depends on $\mu(B(x_0, 1))$, C_μ-doubling constant of measure μ, and where $C_{\log}(p)$ stands for a log*-Hölder constant of p.*

Now, for $q > 0$, let

$$\mathcal{M}_q f(x) = \sup_{r>0}\Big(\fint_{B(x,r)} \left|f(y)\right|^q d\mu(y)\Big)^{1/q}$$

denote the family of the Hardy–Littlewood maximal functions. We next state the following claim.

Corollary 2.3. *Let (X, ρ, μ) be a metric measure space with a doubling measure, $p \in \mathcal{P}_{\log}(X, \mu)$, and let $p_- > q$. Then,*

$$\left\|\mathcal{M}_q(f)\right\|_{L^{p(\cdot)}(X,\mu)} \le \Big(\frac{Cp_-}{p_- - q}\Big)^{1/q}\|f\|_{L^{p(\cdot)}(X,\mu)},$$

*where a constant C depends on $\mu(B(x_0, 1))$, C_μ-doubling constant of measure μ, and $C_{\log}(p)$-*log*-Hölder constant of p.*

Proof. Since $(p/q)_- > 1$, by Theorem 2.2 we get

$$\begin{aligned}\left\|\mathcal{M}_q(f)\right\|_{L^{p(\cdot)}(X,\mu)} &= \left\|\mathcal{M}(f^q)\right\|^{1/q}_{L^{\frac{p(\cdot)}{q}}(X,\mu)}\\ &\le \Big(\frac{C\frac{p_-}{q}}{\frac{p_-}{q} - 1}\Big)^{1/q}\|f^q\|^{1/q}_{L^{\frac{p(\cdot)}{q}}(X,\mu)} = \Big(\frac{Cp_-}{p_- - q}\Big)^{1/q}\|f\|_{L^{p(\cdot)}(X,\mu)}.\end{aligned}$$ □

We are in a position to state and prove our main result, which reads as follows.

Theorem 2.4. *Let (X, ϱ, μ) be a metric measure space equipped with a doubling measure, let $p \in \mathcal{P}_{\log}(X, \mu)$, and let $0 < p_- \le p_+ < \infty$. Then, the family $\mathcal{F} \subset L^{p(\cdot)}(X, \mu)$ is totally bounded in $L^{p(\cdot)}(X, \mu)$ if and only if the following conditions are satisfied:*

(a) *$\mathcal{F}$ is bounded in $L^{p(\cdot)}(X,\mu)$; that is $\exists_{M>0}$,*

$$\sup_{f\in\mathcal{F}}\int_X \left|f(x)\right|^{p(x)}\,d\mu(x)\le M;$$

(b) $\exists_{0<q<p_-}$,

$$\lim_{r\to 0}\sup_{f\in\mathcal{F}}\int_X\Big(\fint_{B(x,r)}\left|f(x)-f(y)\right|^q\,d\mu(y)\Big)^{p(x)/q}\,d\mu(x)=0;$$

(c) *for some $x_0\in X$,*

$$\lim_{R\to\infty}\sup_{f\in\mathcal{F}}\int_{X\setminus B(x_0,R)}\left|f(x)\right|^{p(x)}\,d\mu(x)=0.$$

Proof. Assume that $\mathcal{F}$ is totally bounded in $L^{p(\cdot)}(X,\mu)$. We fix $0<\varepsilon<1$ and define $\widetilde{\varepsilon}:=(\frac{p_--q}{Cp_-})^{1-\frac{1}{q}}\varepsilon<1$, where C is a constant from Theorem 2.2. We also assume that $\{f_k\}_{k=1,\dots,N}$ is a $\widetilde{\varepsilon}$-net in $\mathcal{F}$. For $f\in\mathcal{F}$, we fix $k\in\{1,\dots,N\}$ satisfying

$$\|f_k-f\|_{L^{p(\cdot)}(X,\mu)}<\widetilde{\varepsilon}.$$

By virtue of Proposition 1.1, we have

$$\int_X\left|f_k(x)-f(x)\right|^{p(x)}\,d\mu(x)<\widetilde{\varepsilon}^{p_-}.$$

Put $M_1=\sup_{k=1,\dots,N}\|f_k\|_{L^{p(\cdot)}(X,\mu)}$. Our goal is to show that conditions (a)–(c) are satisfied.

(a) Let $f\in\mathcal{F}$. It follows from Lemma 1.2 that

$$\begin{aligned}\|f\|_{L^{p(\cdot)}(X,\mu)}&\le\max\{2^{\frac{1}{p_-}},2^{\frac{p_+}{p_-}}\}\big(\|f_k\|_{L^{p(\cdot)}(X,\mu)}+\|f-f_k\|_{L^{p(\cdot)}(X,\mu)}\big)\\&\le\max\{2^{\frac{1}{p_-}},2^{\frac{p_+}{p_-}}\}(M_1+\varepsilon)=M.\end{aligned}$$

(b) Direct calculations lead us to the following estimates

$$\begin{aligned}&\int_X\Big(\fint_{B(x,r)}\left|f(x)-f(y)\right|^q\,d\mu(y)\Big)^{\frac{p(x)}{q}}\,d\mu(x)\\&\le 2^{p_+}\Big(\int_X\Big(\fint_{B(x,r)}\left|f(x)-f_k(x)\right|^q\,d\mu(y)\Big)^{\frac{p(x)}{q}}\,d\mu(x)\\&\quad+\int_X\Big(\fint_{B(x,r)}\left|f_k(x)-f(y)\right|^q\,d\mu(y)\Big)^{\frac{p(x)}{q}}\,d\mu(x)\Big)\\&\le 2^{p_+}\int_X\left|f(x)-f_k(x)\right|^{p(x)}\,d\mu(x)\\&\quad+4^{p_+}\int_X\Big(\fint_{B(x,r)}\left|f_k(x)-f_k(y)\right|^q\,d\mu(y)\Big)^{\frac{p(x)}{q}}\,d\mu(x)\\&\quad+4^{p_+}\int_X\Big(\fint_{B(x,r)}\left|f_k(y)-f(y)\right|^q\,d\mu(y)\Big)^{\frac{p(x)}{q}}\,d\mu(x)\\&=2^{p_+}I_1+4^{p_+}I_2+4^{p_+}I_3.\end{aligned}$$

Now, we give the bounds for integrals $I_1 - I_3$.

It is obvious that $I_1 < \tilde{\varepsilon}^{p_-}$. Moreover, by Corollary 2.3, we have

$$\begin{aligned}
&\Big(\fint_{B(x,r)} \big|f_k(x) - f_k(y)\big|^q \, d\mu(y)\Big)^{\frac{p(x)}{q}} \\
&\quad \le 2^{p_+}\Big(\big|f_k(x)\big|^q + \fint_{B(x,r)} \big|f_k(y)\big|^q \, d\mu(y)\Big)^{\frac{p(x)}{q}} \\
&\quad \le 2^{p_+(1+\frac{1}{q})-1}\Big(\big|f_k(x)\big|^{p(x)} + \Big(\fint_{B(x,r)} \big|f_k(y)\big|^q \, d\mu(y)\Big)^{\frac{p(x)}{q}}\Big) \\
&\quad \le 2^{p_+(1+\frac{1}{q})-1}\big(\big|f_k(x)\big|^{p(x)} + \big[\mathcal{M}_q f_k(x)\big]^{p(x)}\big) \in L_1(X,\mu).
\end{aligned}$$

Since $|f_k|^q \in L^1_{\mathrm{loc}}(X,\mu)$, it follows from the Lebesgue differentiation theorem that

$$\fint_{B(x,r)} \big|f_k(x) - f_k(y)\big|^q \, d\mu(y) \to 0,$$

for $r \to +0$ and almost all $x \in X$. Hence, by virtue of the Lebesgue dominated convergence theorem, $I_2 < \varepsilon$ for sufficiently small $r > 0$.

In order to give the bound for I_3, observe that in view of Corollary 2.3, we have

$$\big\|\mathcal{M}_q(f_k - f)\big\|_{L^{p(\cdot)}(X,\mu)} \le \Big(\frac{Cp_-}{p_- - q}\Big)^{\frac{1}{q}} \|f_k - f\|_{L^{p(\cdot)}(X,\mu)} \le \Big(\frac{p_- - q}{Cp_-}\Big)^{1-\frac{1}{q}} \varepsilon < 1.$$

Thus, by Proposition 1.1,

$$\begin{aligned}
I_3 &= \int_X \Big(\fint_{B(x,r)} \big|f_k(y) - f(y)\big|^q \, d\mu(y)\Big)^{\frac{p(x)}{q}} \, d\mu(x) \\
&\le \int_X \big(\mathcal{M}_q\big(|f_k - f|\big)(x)\big)^{p(x)} \, d\mu(x) \\
&\le \big\|\mathcal{M}_q\big(|f_k - f|\big)\big\|^{p_-}_{L^{p(\cdot)}(X,\mu)} \\
&\le \Big(\frac{p_- - q}{Cp_-}\Big)^{p_- - \frac{p_-}{q}} \varepsilon^{p_-}.
\end{aligned}$$

(c) For $k = 1, \dots, N$, we fix R_k satisfying

$$\int_{X \setminus B(x_0,R_k)} \big|f_k(x)\big|^{p(x)} \, d\mu(x) < \varepsilon.$$

Then, for $R = \max\{R_k : k = 1, \dots, N\}$, we have

$$\begin{aligned}
&\int_{X \setminus B(x_0,R)} \big|f(x)\big|^{p(x)} \, d\mu(x) \\
&\quad \le 2^{p_+}\Big(\int_X \big|f(x) - f_k(x)\big|^{p(x)} \, d\mu(x) + \int_{X \setminus B(x_0,R_k)} \big|f_k(x)\big|^{p(x)} \, d\mu(x)\Big) \\
&\quad < 2^{p_+}(\tilde{\varepsilon}^{p_-} + \varepsilon).
\end{aligned}$$

Now, let us show the converse (i.e., the sufficiency of conditions (a)–(c)). Let us fix $\epsilon > 0$. Then, there exists R such that

$$\sup_{f\in\mathcal{F}} \int_{X\setminus B(x_0,R)} \left|f(x)\right|^{p(x)}\, d\mu(x) \le \epsilon. \tag{2.1}$$

Now, let us introduce the following Lipschitz cut-off function with the Lipschitz constant $\frac{1}{R}$:

$$\phi_R(x) := \begin{cases} \frac{1}{R}(2R-\rho(x,x_0)) & \text{if } x \in B(x_0,2R)\setminus B(x_0,R),\\ 1 & \text{if } x\in B(x_0,R),\\ 0 & \text{if } x \in X\setminus B(x_0,2R), \end{cases}$$

and define the set $\mathcal{F}_R$ by

$$\mathcal{F}_R = \{f\phi_R : f\in\mathcal{F}\}.$$

Our objective is to show that $\mathcal{F}_R$ is totally bounded in $L^{p(\cdot)}(B(x_0,R))$. We start with the following lemma.

Lemma 2.5. *The family $\mathcal{F}_R$ satisfies the following conditions:*

(i) *$\mathcal{F}_R$ is bounded in $L^{p(\cdot)}(X)$;*

(ii)

$$\lim_{r\to 0}\sup_{g\in\mathcal{F}_R}\int_X \Big(⨍_{B(x,r)} \left|g(x)-g(y)\right|^q d\mu(y)\Big)^{p(x)/q} d\mu(x) = 0.$$

Proof. The proof of (i) is straightforward and we omit it. Thus, we turn to the proof of condition (ii). Let $g\in\mathcal{F}_R$. Then, $g=\phi_R f$, where $f\in\mathcal{F}$. Applying the properties of the cut-off function ϕ_R, we have

$$\begin{aligned} &⨍_{B(x,r)} \left|g(x)-g(y)\right|^q d\mu(y)\\ &\quad\le 2^q\Big(⨍_{B(x,r)} \left|f(x)-f(y)\right|^q \phi_R^q(y)\, d\mu(y)\\ &\qquad + ⨍_{B(x,r)} \left|\phi_R(x)-\phi_R(y)\right|^q \left|f(x)\right|^q d\mu(y)\Big)\\ &\quad\le 2^q\Big(⨍_{B(x,r)} \left|f(x)-f(y)\right|^q d\mu(y) + r^q\left|f(x)\right|^q\Big). \end{aligned}$$

Hence, we conclude that

$$\begin{aligned} &\int_X\Big(⨍_{B(x,r)} \left|g(x)-g(y)\right|^q d\mu(y)\Big)^{p(x)/q} d\mu(x)\\ &\quad\le 2^{p_+ + \frac{p_+}{q}}\Big(\int_X\Big(⨍_{B(x,r)} \left|f(x)-f(y)\right|^q d\mu(y)\Big)^{p(x)/q} d\mu(x)\\ &\qquad + r^{p_-}\int_X \left|f(x)\right|^{p(x)} d\mu(x)\Big), \end{aligned}$$

which completes the proof of the lemma. □

Lemma 2.6. *The family $\mathcal{F}_R$ is $p(\cdot)$-equi-integrable.*

Proof. It is obvious that

$$|f(x)|^q \le 2^q\big(|f(x)-f(y)|^q + |f(y)|^q\big).$$

Averaging it with respect to $y \in B(x,r)$, we get

$$\begin{aligned}|f(x)|^q \le 2^q\Big(&\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|f(x)-f(y)|^q\,d\mu(y)\\ &+\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|f(y)|^q\,d\mu(y)\Big).\end{aligned}$$

Subsequently, we raise the above inequality to the $\frac{p(x)}{q}$th power and integrate the newly obtained expression over $E \subset X$. By the Hölder inequality, we have

$$\begin{aligned}&\int_E |f(x)|^{p(x)}\,d\mu(x)\\
&\le 2^{p_+ + \frac{p_+}{q}}\Big(\int_E\Big(⨍_{B(x,r)}|f(x)-f(y)|^q\,d\mu(y)\Big)^{p(x)/q}\,d\mu(x)\\
&\quad+\int_E\Big(\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|f(y)|^q\,d\mu(y)\Big)^{\frac{p(x)}{q}}\,d\mu(x)\Big)\\
&\le 2^{p_+ + \frac{p_+}{q}}\int_E\Big(⨍_{B(x,r)}|f(x)-f(y)|^q\,d\mu(y)\Big)^{p(x)/q}\,d\mu(x)\\
&\quad+2^{p_+ + 2\frac{p_+}{q}}\int_E\Big(\frac{1}{\mu(B(x,r))}\|f^q\|_{L_{\frac{p(\cdot)}{q}}(B(x,r))}\|1\|_{L_{\frac{p(\cdot)}{p(\cdot)-q}}(B(x,r))}\Big)^{\frac{p(x)}{q}}\,d\mu(x).\end{aligned}$$

Due to Proposition 1.1, we get

$$\|1\|_{L_{\frac{p(\cdot)}{p(\cdot)-q}}(B(x,r))} \le \max\Big\{\big[\mu\big(B(x,r)\big)\big]^{\frac{p_+-q}{p_+}},\big[\mu\big(B(x,r)\big)\big]^{\frac{p_--q}{p_-}}\Big\}.$$

Therefore, by (1.1), there exists a constant C such that

$$\begin{aligned}\frac{1}{\mu(B(x,r))}\cdot\|1\|_{L_{\frac{p(\cdot)}{p(\cdot)-q}}(B(x,r))} &\le \max\Big\{\big[\mu\big(B(x,r)\big)\big]^{-\frac{q}{p_+}},\big[\mu\big(B(x,r)\big)\big]^{-\frac{q}{p_-}}\Big\}\\
&\le C^q\max\{r^{-\frac{sq}{p_+}},r^{-\frac{sq}{p_-}}\} = C^q r^{-\frac{sq}{p_-}},\end{aligned}$$

provided that $r < 1$.

It is clear that

$$\|f^q\|_{L_{\frac{p(\cdot)}{q}}(B(x,r))} = \|f\|^q_{L_{p(\cdot)}(B(x,r))} \le M^q.$$

Therefore, we get

$$\begin{aligned}
&\int_E \left|f(x)\right|^{p(x)}\,d\mu(x)\\
&\quad\le 2^{p_+ + \frac{p_+}{q}}\Big(\int_E\Big(\fint_{B(x,r)}\left|f(x)-f(y)\right|^q\,d\mu(y)\Big)^{p(x)/q}\,d\mu(x)\\
&\qquad + 2^{\frac{p_+}{q}}(MC+1)^{p_+}r^{-\frac{sp_+ + q}{p_-}}\mu(E)\Big),
\end{aligned}$$

and we obtain that the family $\mathcal{F}_R$ is indeed $p(\cdot)$-equi-integrable. $\square$

Lemma 2.7. *$\mathcal{F}_R$ is totally bounded in $L^0(B(x_0,R))$.*

Proof. Let us note that, since the measure μ is doubling, the metric space (X,ρ) is doubling (see [18]). Thus, $(B(x_0,R),\rho)$ is a doubling metric space and hence, $(B(x_0,R),\rho)$ is totally bounded.

Let us fix $\varepsilon > 0$. We take $\delta > 0$ such that, for all $f \in \mathcal{F}_R$, the following condition holds:

$$\int_X\Big(\fint_{B(x,2\delta)}\left|f(x)-f(y)\right|^q\,d\mu(y)\Big)^{p(x)/q}\,d\mu(x) < \varepsilon^{p_+ + 1}.$$

Hence, in view of the Markov inequality, we get

$$\begin{aligned}
&\mu\Big\{x : \Big(\fint_{B(x,2\delta)}\left|f(x)-f(y)\right|^q\,d\mu(y)\Big)^{1/q} > \varepsilon\Big\}\\
&\quad\le \frac{1}{\varepsilon^{p_+}}\int_X\Big(\fint_{B(x,2\delta)}\left|f(x)-f(y)\right|^q\,d\mu(y)\Big)^{p(x)/q}\,d\mu(x) < \varepsilon.
\end{aligned}$$

Next, for any $x, y \in B(x_0,R)$ satisfying $\rho(x,y) < \delta$, we have

$$\begin{aligned}
\left|f(x)-f(y)\right| \le 2^{1+\frac{1}{q}}\Big(&\Big(\fint_{B(x,\delta)}\left|f(x)-f(y)\right|^q\,d\mu(y)\Big)^{1/q}\\
&+\Big(\fint_{B(x,\delta)}\left|f(y)-f(z)\right|^q\,d\mu(z)\Big)^{1/q}\Big).
\end{aligned}$$

Since $B(x,\delta) \subset B(y,2\delta)$ and the measure is doubling, the last inequality implies that, for any $x, y \in X$ satisfying $\rho(x,y) < \delta$,

$$\begin{aligned}
\left|f(x)-f(y)\right| \le 2^{1+\frac{1}{q}}C_d^{-\frac{1}{q}}\Big(&\Big(\fint_{B(x,2\delta)}\left|f(x)-f(z)\right|^q\,d\mu(z)\Big)^{1/q}\\
&+\Big(\fint_{B(y,2\delta)}\left|f(z)-f(y)\right|^q\,d\mu(z)\Big)^{1/q}\Big).
\end{aligned}$$

By virtue of the Markov inequality and boundedness of $\mathcal{F}_R$ in $L_{p(\cdot)}(X)$, we have

$$\mu\big\{|f| > \lambda\big\} = \mu\big\{|f|^{p(x)} > \lambda^{p(x)}\big\} \le \frac{1}{\lambda^{p_-}}\int_X \left|f(x)\right|^{p(x)}\,d\mu(x) \le \frac{1}{\lambda^{p_-}}M.$$

Therefore, there exists λ such that

$$\sup_{f\in\mathcal{F}}\mu\big\{|f| > \lambda\big\} < \varepsilon.$$

We set

$$E(f) = \{x : |f(x)| > \lambda\} \cup \left\{x : \left(\fint_{B(x,2\delta)} |f(x) - f(y)|^q \, d\mu(y)\right)^{1/q} > \varepsilon\right\}.$$

Then, $\mu(E(f)) < 2\varepsilon$. Moreover,

$$|f(x)| \le \lambda \quad \text{for } x \in B(x_0, R) \setminus E(f),$$
$$|f(x) - f(y)| \le 2^{2+\frac{1}{q}} C_d^{-\frac{1}{q}} \varepsilon \quad \text{for } x, y \in B(x_0, R) \setminus E(f), \rho(x, y) < \delta.$$

Thus, by Theorem 1.3, we get that $\mathcal{F}_R$ is totally bounded in $L^0(B(x_0, R))$. □

Thus, by Lemmas 2.6 and 2.7, conditions (i) and (ii) of Theorem 2.1 are satisfied. Hence, we proved that $\mathcal{F}_R$ is totally bounded in $L^{p(\cdot)}(B(x_0, R))$. Finally, combining the total boundedness with inequality (2.1), we obtain that $\mathcal{F}$ is totally bounded in $L^{p(\cdot)}(X, \mu)$, which completes the proof of Theorem 2.4. □

3. Remarks

We summarize the paper with some remarks. Following the proof of Theorem 2.4, we get the result below.

Theorem 3.1. *Let (X, ϱ, μ) be a metric measure space equipped with doubling measure, let $p \in \mathcal{P}_{\log}(X, \mu)$, and let $0 < p_- \le p_+ < \infty$. Then, the family $\mathcal{F} \subset L^{p(\cdot)}(X, \mu)$ is totally bounded in $L^{p(\cdot)}(X, \mu)$ if and only if the following conditions are satisfied:*

(a) *$\mathcal{F}$ is bounded in $L^{p(\cdot)}(X, \mu)$; that is, $\exists_{M>0}$,*

$$\sup_{f \in \mathcal{F}} \int_X |f(x)|^{p(x)} \, d\mu(x) \le M;$$

(b) $\forall_{0<q<p_-}$,

$$\lim_{r \to 0} \sup_{f \in \mathcal{F}} \int_X \left(\fint_{B(x,r)} |f(x) - f(y)|^q \, d\mu(y)\right)^{p(x)/q} d\mu(x) = 0;$$

(c) *for some $x_0 \in X$,*

$$\lim_{R \to \infty} \sup_{f \in \mathcal{F}} \int_{X \setminus B(x_0,R)} |f(x)|^{p(x)} \, d\mu(x) = 0.$$

Moreover, combining Theorem 2.4 with Theorem 3.1, we get the following characterization of precompact sets in $L^{p(\cdot)}(X, \varrho, \mu)$.

Theorem 3.2. *Let (X, ϱ, μ) be a metric measure space equipped with doubling measure, let $p \in \mathcal{P}_{\log}(X, \mu)$, and let $1 < p_- \le p_+ < \infty$. Then, the family $\mathcal{F} \subset L^{p(\cdot)}(X, \mu)$ is totally bounded in $L^{p(\cdot)}(X, \mu)$ if and only if the following conditions are satisfied:*

(a) *$\mathcal{F}$ is bounded in $L^{p(\cdot)}(X, \mu)$; that is, $\exists_{M>0}$,*

$$\sup_{f \in \mathcal{F}} \int_X |f(x)|^{p(x)} \, d\mu(x) \le M;$$

(b)

$$\lim_{r\to 0}\sup_{f\in\mathcal{F}}\int_X\Big(\fint_{B(x,r)}\big|f(x)-f(y)\big|\,d\mu(y)\Big)^{p(x)}\,d\mu(x)=0;$$

(c) *for some* $x_0\in X$,

$$\lim_{R\to\infty}\sup_{f\in\mathcal{F}}\int_{X\setminus B(x_0,R)}\big|f(x)\big|^{p(x)}\,d\mu(x)=0.$$

The above assertion is very similar to the result stated in [13]. Let us note that in [13], the authors assumed additionally that

$$\inf\big\{\mu\big(B(x,1)\big):x\in X\big\}>0.$$

Let us stress now that an arbitrary doubling measure does not necessarily satisfy this assumption (see [13] for details). More recently, it was shown that if the metric measure space (X,ρ,μ) with doubling measure μ admits the Sobolev inequality (see [11]) or the Trudinger–Moser embedding (see [2]), then the lower bound for the measure holds.

Finally, following the proof of Theorem 2.4, we get the sufficient conditions for compactness in the form of the claim below.

Theorem 3.3. *Let* (X,ϱ,μ) *be a metric measure space equipped with doubling measure, and let* p *be a measurable map such that* $0<p_-\le p_+<\infty$. *If the family* $\mathcal{F}\subset L^{p(\cdot)}(X,\mu)$ *satisfies the conditions*

(a) $\mathcal{F}$ *is bounded in* $L^{p(\cdot)}(X,\mu)$; *that is,* $\exists_{M>0}$,

$$\sup_{f\in\mathcal{F}}\int_X\big|f(x)\big|^{p(x)}\,d\mu(x)\le M;$$

(b) $\exists_{0<q<p_-}$,

$$\lim_{r\to 0}\sup_{f\in\mathcal{F}}\int_X\Big(\fint_{B(x,r)}\big|f(x)-f(y)\big|^q\,d\mu(y)\Big)^{p(x)/q}\,d\mu(x)=0;$$

(c) *for some* $x_0\in X$,

$$\lim_{R\to\infty}\sup_{f\in\mathcal{F}}\int_{X\setminus B(x_0,R)}\big|f(x)\big|^{p(x)}\,d\mu(x)=0;$$

then the family $\mathcal{F}$ *is totally bounded in* $L^{p(\cdot)}(X,\mu)$.

It is worth mentioning that there are no assumptions on the regularity of exponent p in Theorem 3.3.

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