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Independence of the Dual Axiom in Modal K with Primitive ◇

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Abstract Explicit axioms relating $\Diamond\phi$ and $\Box\phi$ appear to be needed if $\Diamond$ is taken to be primitive. We prove that such axioms are in fact indispensable.

1 Introduction

Blackburn, de Rijke, and Venema [1] formulated systems of propositional modal logic with $\Diamond$ as primitive, and with $\Box p$ defined as $\neg\Diamond\neg p$. In axiomatizing these logics, the authors resorted to an axiom that is not needed when $\Box$ is the modal primitive. This is the *dual axiom*:

$$\Diamond p \leftrightarrow \neg\Box\neg p.$$

The purpose of this article is to show that such an axiom is indispensable: in fact, both $\Diamond p \to \Diamond\neg\neg p$ and $\Diamond\neg\neg p \to \Diamond p$ can be invalidated in a modal logic with $\Diamond$ as primitive and with the usual Boolean axioms, the necessitation rule, and the **K** axiom. Of course, these axioms cannot be invalidated in Kripke frames, or even in Boolean propositional logic. So the models used in this article are somewhat exotic.

2 Eight-Valued Models for Modality

We will use many-valued models with eight values. It is best to think of these values as made up out of two 4-element Boolean algebras $\mathbf{B}$ and $\mathbf{B}'$. See Figure 1 for a picture.

The units of the two Boolean algebras, $\vee$ and $\vee'$, are the only designated values: a formula is valid if it only receives values in $\{\vee, \vee'\}$. Negation is nonstandard. Within $\mathbf{B}$, it is as expected, but the "complement" of an element of $\mathbf{B}'$ is the complement

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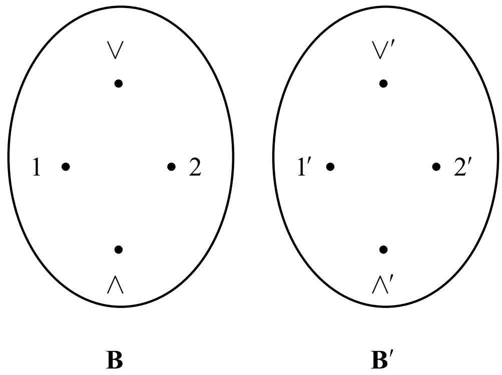

Figure 1 An 8-valued model.

of its twin in **B**. The conditional $\rightarrow$ is the "union" or least upper bound of the complement of the antecedent with the twin of the consequent.

We now spell these ideas out explicitly in the following definition of the functions $f_\neg$ and $f_\rightarrow$ that serve to interpret negation and the conditional.

Definition 1 $f_\neg$, $f_\rightarrow$

$$f_\neg(\wedge) = \vee, \qquad f_\neg(\vee) = \wedge, \qquad f_\neg(1) = 2, \qquad f_\neg(2) = 1,$$
$$f_\neg(\wedge') = \vee, \qquad f_\neg(\vee') = \wedge, \qquad f_\neg(1') = 2, \qquad f_\neg(2') = 1,$$
$$\mathrm{Twin}(x) = f_\neg\big(f_\neg(x)\big) = x \quad \text{if } x \in \mathbf{B}, \qquad f_\neg\big(f_\neg(x)\big) \quad \text{if } x \in \mathbf{B}',$$
$$\text{For } x, y \in \mathbf{B}: \quad f_\rightarrow(\wedge, x) = \vee, \qquad f_\rightarrow(x, \wedge) = f_\neg(x), \qquad f_\rightarrow(\vee, x) = x,$$
$$f_\rightarrow(1, 2) = 2, \qquad f_\rightarrow(2, 1) = 1,$$
$$\text{For } x, y \in \mathbf{B}': \quad f_\rightarrow(x, y) = f_\rightarrow\big(\mathrm{Twin}(x), \mathrm{Twin}(y)\big).$$

These conditions overlap in places, but the overlaps are consistent.

We will postpone the definition of $f_\diamond$.

3 Axioms

The system in which we are interested has the following four axioms, together with the rules of modus ponens, necessitation, and substitution:

(1) $p \rightarrow_\bullet q \rightarrow p$,
(2) $(p \rightarrow_\bullet q \rightarrow r) \rightarrow_\bullet (p \rightarrow q) \rightarrow_\bullet p \rightarrow r$,
(3) $(\neg p \rightarrow \neg q) \rightarrow_\bullet q \rightarrow p$,
(4) $\neg\Diamond\neg(p \rightarrow q) \rightarrow_\bullet \neg\Diamond\neg p \rightarrow \neg\Diamond\neg q$.

Axioms (1)–(3) are complete for Boolean propositional logic. Axiom (4) is the modal axiom **K**, with $\Diamond$ primitive. Together, these axioms partially axiomatize the modal system **K**, including all the usual axioms, but not the dual axiom.

4 Preliminaries

The relation $x \preceq y$ over $\mathbf{B} \cup \mathbf{B}'$ is given by the least upper bound in Figure 1. It is the transitive closure of

$$\big\{\langle\wedge, 1\rangle, \langle\wedge, 2\rangle, \langle 1, \vee\rangle, \langle 2, \vee\rangle, \langle\wedge', 1'\rangle, \langle\wedge', 2'\rangle, \langle 1', \vee'\rangle, \langle 2', \vee'\rangle\big\}.$$

We begin with some easily verifiable claims, stated without proof.

Claim 1 *We have* $f_{\to}(x, y) \in \{\vee, \vee'\}$ *iff* $f_{\to}(x, y) = \vee$.

Claim 2 *We have* $f_{\to}(x, y) = \vee$ *iff* $x \preceq y$ *iff* $\mathrm{Twin}(x) \preceq \mathrm{Twin}(y)$.

Claim 3 *We have* $f_{\neg}(x) \preceq f_{\neg}(y)$ *iff* $y \preceq x$ *iff* $\mathrm{Twin}(y) \preceq \mathrm{Twin}(x)$.

Claim 4 *Where* lub *is the least-upper-bound operator in* $\mathbf{B} \cup \mathbf{B}'$,

$$f_{\to}(x, y) = \mathrm{lub}\big(f_{\neg}(x), \mathrm{Twin}(y)\big).$$

Claim 5 *Where* glb *is the greatest-lower-bound (GLB) operator in* $\mathbf{B} \cup \mathbf{B}'$, $f_{\to}(x, f_{\to}(y, z)) = \wedge$ *iff*

$$\mathrm{glb}\big(\mathrm{Twin}(x), \mathrm{Twin}(y)\big) \preceq \mathrm{Twin}(z).$$

Claim 6 *We have* $f_{\to}(w, f_{\to}(x, f_{\to}(y, z))) = \vee$ *iff* $\mathrm{glb}(\mathrm{Twin}(w), \mathrm{Twin}(x), \mathrm{Twin}(y)) \preceq \mathrm{Twin}(z)$.

5 Nonmodal Soundness

Let V be a mapping of propositional variables to values in $\mathbf{B} \cup \mathbf{B}'$. The mapping V is extended to nonmodal formulas in the usual way, interpreting $\neg$ with $f_{\neg}$ and $\to$ with $f_{\to}$.

The validity of Axioms (1)–(3) and the rules of substitution and modus ponens follows from the fact that $V(\phi) = V'(\phi)$ if $V'(\phi) = V(\mathrm{Twin}(\phi))$. But we will also provide direct arguments.

Validity of the substitution rule. Substitution is valid in any many-valued matrix.

Validity of modus ponens. Suppose that $V(\phi \to \psi), V(\phi) \in \{\vee, \vee'\}$, and let $\mathrm{Twin}(V(\psi)) = x$. By Claims 1 and 2, $\vee \preceq x$, so $x = \vee$, so $V(\psi) \in \{\vee, \vee'\}$.

Validity of Axiom (1). Suppose that $V(p \to_{\bullet} q \to p) \notin \{\vee, \vee'\}$. By Claims 1 and 5, $\mathrm{glb}(x, y) \not\preceq x$, where $x = \mathrm{Twin}(V(p))$, $y = \mathrm{Twin}(V(q))$. But this is impossible.

Validity of Axiom (2). Consider $(p \to_{\bullet} q \to r) \to_{\bullet} (p \to q) \to_{\bullet} p \to r$, and let $x = \mathrm{Twin}(V(p))$, $y = \mathrm{Twin}(V(q))$, $z = \mathrm{Twin}(V(r))$.

Suppose now that $V((p \to_{\bullet} q \to r) \to_{\bullet} (p \to q) \to_{\bullet} p \to r) \notin \{\vee, \vee'\}$. Then, by Claims 1 and 6 and the definition of $f_{\to}$, $\mathrm{glb}(f_{\to}(x, f_{\to}(y, z)), f_{\to}(x, y), x) \not\preceq z$. Let GLB be $\mathrm{glb}(f_{\to}(x, f_{\to}(y, z)), f_{\to}(x, y), x)$. Then either

(i) $z = \wedge$ and GLB $\in \{\vee, 1, 2\}$, or
(ii) $z = 1$ and GLB $\in \{\vee, 2\}$, or
(iii) $z = 2$ and GLB $\in \{\vee, 1\}$.

In case (i), GLB $= \mathrm{glb}(f_{\to}(x, f_{\neg}(y)), f_{\to}(x, y), x) = \wedge$, and we have a contradiction, because $\wedge \preceq z$ and, by hypothesis, GLB $\not\preceq z$.

In case (ii), in view of the definition of $f_{\to}$, either

(ii.1) $f_{\to}(y, 1) = 1$, or
(ii.2) $f_{\to}(y, 1) = \wedge$.

In case (ii.1), GLB $= \mathrm{glb}(f_{\to}(x, 1), f_{\to}(x, y), x)$ and $y \in \{\wedge, 2\}$. If $y = \wedge$, then GLB $= \mathrm{glb}(f_{\to}(x, 1), f_{\neg}(x), x) \in \{\wedge, 1\}$, and we have a contradiction. If $y = 2$, then GLB $= \mathrm{glb}(f_{\to}(x, 1), f_{\to}(x, 2), x) = \wedge$, and again we have a contradiction.

In case (ii.2), GLB $= \mathrm{glb}(f_{\to}(x, \wedge), f_{\to}(x, y), x) = \mathrm{glb}(f_{\to}(x, y), x)$ and $y \in \{1, \wedge\}$. If $y = 1$, then GLB $= \mathrm{glb}(f_{\to}(x, 1), x)$ and $y \in \{1, \wedge\}$. If $y = 1$, then

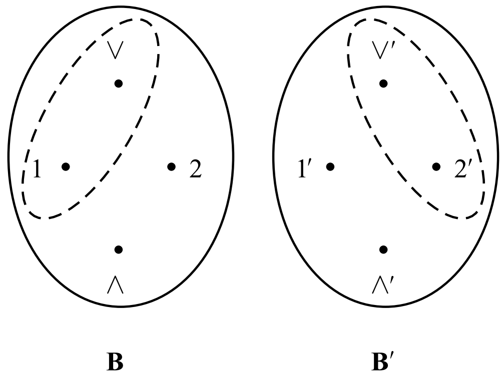

Figure 2 Regions of the 8-valued model.

$\mathrm{GLB} = \mathrm{glb}(f_{\rightarrow}(x, 1), x) \in \{1, \wedge\}$, and we have a contradiction. If $y = \wedge$, then $\mathrm{GLB} = \mathrm{glb}(f_{\neg}(x), x) = \wedge$, and again we have a contradiction.

The reasoning in case (iii) is like that in case (ii).

Validity of Axiom (3). By Claim 4, $f_{\rightarrow}(f_{\neg}(x), f_{\neg}(y)) = \mathrm{lub}(f_{\neg}(f_{\neg}(x)), \mathrm{Twin}(f_{\neg}(y))) = \mathrm{lub}(\mathrm{Twin}(x), f_{\neg}(y))$. But $f_{\rightarrow}(y, x) = \mathrm{lub}(f_{\neg}(y), \mathrm{Twin}(x))$. Therefore $V(\neg p \rightarrow \neg q) = V(q \rightarrow p)$, so that $V(\neg p \rightarrow \neg q) \rightarrow (q \rightarrow p) = \vee$.

This completes the detailed proof of soundness for the Boolean axioms.

6 Interpreting $\Diamond$

To interpret $\Diamond$, we revert to the picture of our model in Figure 2 and elaborate it by including two regions of **B** and **B**′. These are shown by the dashed lines in the elaborated picture.

The interpretation of $\Diamond$ is sensitive to whether you are working in **B** or in **B**′. In the former case, the value is $\vee$ for arguments in the area $\{1, \vee\}$. Otherwise it is $\wedge$. In the latter case, the value is $\vee$ for arguments in the circled area $\{2', \vee'\}$. Otherwise it is $\wedge$. Here is the official definition.

For $x \in \mathbf{B}$, $f_{\Diamond}(x) = \vee$ if $x \in \{1, \vee\}$ and $f_{\Diamond}(x) = \wedge$ if $x \notin \{1, \vee\}$.
For $x \in \mathbf{B}'$, $f_{\Diamond}(x) = \vee$ if $x \in \{2', \vee'\}$ and $f_{\Diamond}(x) = \wedge$ if $x \notin \{2', \vee'\}$.

7 Modal Soundness

Let $f_{\Box}(x) = f_{\neg}(f_{\Diamond}(f_{\neg}(x)))$. We state two more easily verified claims.

Claim 7 *For all x, $f_{\Box}(x) \in \{\vee, \vee'\}$.*

Claim 8 *For all x, $f_{\Box}(x) = \vee$ iff* $\mathrm{Twin}(x) \in \{\vee, 1\}$.

We now check the validity of the necessitation rule. Suppose that, for all V, $V(\phi) \in \{\vee, \vee'\}$. Then, in view of Claim 8, $f_{\Box}(x) = \vee$, where $x = \mathrm{Twin}(V(\phi))$. So $V(\neg\Diamond\neg\phi) = \vee$, for all V.

Now consider Axiom (4): $\neg\Diamond\neg(p \rightarrow q) \rightarrow_{\bullet} \neg\Diamond\neg p \rightarrow \neg\Diamond\neg q$. Let $\mathrm{Twin}(V(p)) = x$ and $\mathrm{Twin}(V(q)) = y$, and suppose that $\mathrm{lub}(f_{\Box}(f_{\rightarrow}(x, y)), f_{\Box}(x)) \not\leq f_{\Box}(y)$. Then, in view of Claim 7, $f_{\Box}(f_{\rightarrow}(x, y)) = \vee$, $f_{\Box}(x) = \vee$,

$f_{\Box}(y) = \wedge$. By Claim 8, $f_{\to}(x, y) \in \{\vee, 1\}$, $f_{\Box}(x) = \{\vee, 1\}$, and $f_{\Box}(y) = \{\wedge, 2\}$. But this is impossible.

This completes the proof of the soundness of the Boolean and modal axioms and rules for this interpretation. It remains to show that the dual axiom is invalid.

8 Invalidity of the Dual Axiom

Recall that the dual axiom is $\Diamond p \to \neg\Box\neg p$. Since with $\Diamond$ primitive, $\Box p$ is defined as $\neg\Diamond\neg p$, and $f_{\neg}(f_{\neg}(x)) = x$, this axiom amounts to $\Diamond p \to \Diamond\neg\neg p$.

To invalidate $\Diamond p \to \Diamond\neg\neg p$ (and hence, the dual axiom), let $V(p) = 2'$. Then $V(\Diamond p) = \vee$. But $V(\neg\neg p) = 1$, so $V(\Diamond\neg\neg p) = \wedge$. So $V(\Diamond p \to \Diamond\neg\neg p) = \wedge$.

To invalidate the converse formula $\Diamond\neg\neg p \to \Diamond p$, let $V(p) = 1'$. Then $V(\Diamond p) = \wedge$ and $V(\neg\neg p) = 2$, so $V(\Diamond\neg\neg p \to \Diamond p) = \wedge$.

This completes the proof of the independence of the dual axiom from the other axioms.

9 Conclusion

It would be nice if the models used in this proof were useful for some other purpose, but none has yet occurred to me.

Acknowledgment

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Reference

[1] Blackburn, P., M. de Rijke, and Y. Venema, *Modal Logic*, vol. 53 of *Cambridge Tracts in Theoretical Computer Science*, Cambridge University Press, Cambridge, 2001. Zbl 1108.03309. MR 1837791. DOI 10.1023/A:1010358017657. 381

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