Notre Dame Journal of Formal Logic
Volume 51, Number 3, 2010

Indestructible Strong Unfoldability

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Abstract Using the lottery preparation, we prove that any strongly unfoldable cardinal κ can be made indestructible by all $<\kappa$-closed κ^+-preserving forcing. This degree of indestructibility, we prove, is the best possible from this hypothesis within the class of $<\kappa$-closed forcing. From a stronger hypothesis, however, we prove that the strong unfoldability of κ can be made indestructible by all $<\kappa$-closed forcing. Such indestructibility, we prove, does not follow from indestructibility merely by $<\kappa$-directed closed forcing. Finally, we obtain global and universal forms of indestructibility for strong unfoldability, finding the exact consistency strength of universal indestructibility for strong unfoldability.

1 Introduction

The unfoldable cardinals were introduced by Villaveces in [20] along with their companion notion, the strongly unfoldable cardinals, which turn out to be the same as what Miyamoto [19] independently introduced as the H_{κ^+} reflecting cardinals. These cardinals lie relatively low in the large cardinal hierarchy, somewhat above the weakly compact cardinals, and they relativize to L in the sense that every unfoldable cardinal is unfoldable in L and in fact strongly unfoldable there, as in L the two notions coincide. For this reason, the notions of unfoldability and strong unfoldability, although not equivalent, have the same consistency strength, bounded below by the totally indescribable cardinals and above by the subtle cardinals.

Definition 1.1

(1) An inaccessible cardinal κ is *unfoldable* if for every ordinal θ it is θ-unfoldable, meaning that for every transitive set M of size κ with $\kappa \in M$ there is a transitive set N and an elementary embedding $j : M \to N$ with critical point κ and $\theta \leq j(\kappa)$.

Received November 5, 2008; accepted November 5, 2009; printed June 16, 2010
2010 Mathematics Subject Classification: Primary, 03E55, 03E40
Keywords: strongly unfoldable cardinal, forcing, indestructibility
 10.1215/00294527-2010-018

(2) An inaccessible cardinal κ is *strongly unfoldable* if for every ordinal θ it is θ-strongly unfoldable, meaning that for every transitive set M of size κ with $\kappa \in M \models \mathrm{ZFC}^-$ and $M^{<\kappa} \subseteq M$ there is a transitive set N and an elementary embedding $j : M \to N$ with critical point κ such that $\theta \leq j(\kappa)$ and $V_\theta \subseteq N$.

Unfoldability therefore generalizes the familiar weak compactness embedding property by insisting that the target models N be increasingly tall or close to V. In Definition 1.1, one can equivalently insist that $\theta < j(\kappa)$ by composing embeddings. After the introduction of unfoldability, it became gradually apparent that unfoldability embeddings were amenable to various techniques borrowed from much stronger large cardinal contexts. For example, Hamkins [9] adapted methods from strong cardinals to lift unfoldability and strong unfoldability embeddings through fast function forcing and the Easton support iterations that control the GCH and its failures. Džamonja and Hamkins [4] adapted methods from supercompact cardinals to show that $\Diamond_\kappa(\mathrm{REG})$ can fail at a strongly unfoldable cardinal κ. Miyamoto [19] used supercompactness methods with his H_{κ^+}-reflecting cardinals (equivalent to strong unfoldability) to obtain weak versions of PFA. Other PFA applications appeared in [14] and in [11]. The underlying indestructibility phenomenon hinted at in these arguments was verified in Johnstone's dissertation [14; 15], which established that every strongly unfoldable cardinal κ can be made indestructible by all $<\kappa$-closed κ-proper forcing. For a cardinal κ with $\kappa^{<\kappa} = \kappa$, the class of κ-proper forcing includes all κ^+-c.c. forcing and all $\leq\kappa$-closed forcing, as well as finite iterations of these, and is included within the class of all κ^+-preserving forcing. The main question left open in [15] was whether the indestructibility phenomenon could be extended to all $<\kappa$-closed κ^+-preserving forcing. In this article, we answer affirmatively.

Theorem 1.2 (Main Theorem) *If κ is strongly unfoldable, then after suitable preparatory forcing, the strong unfoldability of κ becomes indestructible by all $<\kappa$-closed κ^+-preserving set forcing.*

More specifically, the proof shows that if κ is strongly unfoldable, then after the lottery preparation of κ relative to a function with the strong unfoldability Menas property, the cardinal κ remains strongly unfoldable and becomes indestructible by all $<\kappa$-closed κ^+-preserving forcing. This class of forcing goes beyond the class of $<\kappa$-closed κ-proper forcing and includes, for example, the forcing to destroy certain stationary subsets of κ^+ (whereas classical arguments show that $<\kappa$-closed κ-proper forcing necessarily preserves such stationarity). We call particular attention to the fact that our theorem involves $<\kappa$-closed forcing and not merely $<\kappa$-directed closed forcing, a degree of indestructibility which is impossible for supercompact or even measurable cardinals. This issue is fully discussed in Section 5, where we prove among other things that indestructibility by $<\kappa$-closed forcing, for a strongly unfoldable cardinal κ, is not a consequence of indestructibility by $<\kappa$-directed closed forcing.

A key technical advance, highlighted in the proof of the Main Theorem, allows for the comparatively broad degree of indestructibility and seems to place this result beyond previously known methods. In broad strokes, to be sure, we perform an Easton support preparatory forcing and ultimately lift an embedding from the ground model to the forcing extensions—the same general outline of all similar indestructibility results—and more specifically we follow the method of [15] for obtaining indestructibility with strongly unfoldable cardinals. Nevertheless, at a key step of the argument we have a V-generic filter $g \subseteq \mathbb{Q}$ that we want to be X-generic for a suitable elementary substructure $X \prec H_\lambda$, and whereas in previous arguments this was accomplished by making restrictive hypotheses on $\mathbb{Q}$, such as requiring $\mathbb{Q}$ to have size κ, be κ^+-c.c. or, more generally, be κ-proper, we now obtain this situation by choosing X not in the ground model, but in the forcing extension, where we easily see that g is X-generic, and then applying critical facts about forcing extensions with the approximation and cover properties (see [10]) to conclude that X is actually in the ground model after all. This same new method also forms the basis of our related application in [11] to obtain strong fragments of the Proper Forcing Axiom (restricted to proper $\aleph_2$- or $\aleph_3$-preserving forcing) from the existence of an unfoldable cardinal. That argument appeals to the Jensen Covering Theorem in a context where the approximation and cover properties may fail. Our article [11] can be viewed as following up on this article.

After developing some preliminary material in Section 2, we prove the Main Theorem in Section 3. In Section 4, we prove that this degree of indestructibility is optimal from this hypothesis, within the class of $<\kappa$-closed forcing. In Section 5, we explore the issue of $<\kappa$-closed versus $<\kappa$-directed closed forcing. In Section 6, we explain how to make all strongly unfoldable cardinals simultaneously indestructible, as well as all partially strongly unfoldable cardinals, providing an exact equiconsistency result for such universal indestructibility.

2 Preliminaries and Background

Let us review some preliminary matters concerning strong unfoldability. Like their weaker cousins, the weakly compact cardinals, the unfoldable and strongly unfoldable cardinals admit numerous equivalent characterizations in terms of various embedding and extension properties. Villaveces [20] originally defined unfoldability and strong unfoldability by a certain extension property, which he then proved equivalent to the embedding characterization, which we take as the defining notion. (The equivalence between the extension and embedding formulations stated in [20], however, appears to break down when θ is a limit ordinal with cofinality ω; so we take the embedding definition as official.) In this article, it will be useful to have one very weak formulation of strong unfoldability, Lemma 2.1, to be used as a sufficient criterion when we want to verify that κ remains strongly unfoldable in the relevant forcing extension, and another very strong formulation of strong unfoldability, Lemma 2.2, to provide the most powerful embeddings in the ground model, with which we shall begin our main argument.

We use the notation ZFC$^-$ to mean the axiomatization of set theory consisting of the ZFC axioms except the Power Set axiom. For any regular uncountable cardinal δ, the collection H_δ of sets hereditarily of size less than δ is a model of ZFC$^-$. A *κ-model* is a transitive set M of size κ with $\kappa \in M \models \mathrm{ZFC}^-$ and $M^{<\kappa} \subseteq M$. A simple Löwenheim-Skolem argument shows that if $\kappa^{<\kappa} = \kappa$, then any $A \in H_{\kappa^+}$ can be placed into a κ-model $M \prec H_{\kappa^+}$. The notation $f : A \to B$ means that f is a partial function with $\mathrm{dom}(f) \subseteq A$ and $\mathrm{ran}(f) \subseteq B$. We use $\mathrm{Cof}_\delta\,\lambda$ to denote the set of ordinals $\alpha < \lambda$ with $\mathrm{cof}(\alpha) = \delta$, and V_α for the collection of sets with Lévy rank less than α, where $\mathrm{rank}(x) = \sup\{\,\mathrm{rank}(y) + 1 \mid y \in x\,\}$. We relativize this notation to other models of set theory by writing $\overline{V}_\theta$ and $V[G]_\theta$ for $(V_\theta)^{\overline{V}}$ and $(V_\theta)^{V[G]}$, respectively. A poset $\mathbb{P}$ is *$<\kappa$-strategically closed* if there is a strategy for the second player in the game of length κ allowing her to continue play, where the players alternate play to build a descending sequence $\langle p_\xi \mid \xi < \kappa\rangle$ of conditions in $\mathbb{P}$, with the second player playing at limit stages. The poset $\mathbb{P}$ is $\leq\kappa$-strategically closed if the second player has a winning strategy in the corresponding game of length $\kappa + 1$.

Lemma 2.1 *An inaccessible cardinal κ is θ-strongly unfoldable if and only if for every $A \subseteq \kappa$ there is a transitive set $M \models \mathrm{ZFC}^-$ of size κ with $\kappa, A \in M$ and a transitive set N with an embedding $j : M \to N$ with critical point κ such that $\theta \leq j(\kappa)$ and $V_\theta \subseteq N$.*

Lemma 2.1 can be easily deduced from Definition 1.1 by observing that any κ-model M' can be coded by a set $A \subseteq \kappa$, and so if there is $j : M \to N$ with $A \in M \models \mathrm{ZFC}^-$, then also $M' \in M$ and $j \restriction M' : M' \to j(M')$ is elementary. Since $V_\kappa \cap M' = V_\kappa \cap M$ and $\theta \leq j(\kappa)$, it follows that $V_\theta \cap j(M') = V_\theta \cap N$, and so this embedding witnesses the desired degree of strong unfoldability.

Lemma 2.2 ([4], Lemma 5) *An inaccessible cardinal κ with $\kappa \leq \theta$ is $\theta+1$-strongly unfoldable if and only if for every κ-model M there is a transitive set N and an embedding $j : M \to N$ with critical point κ such that $\theta < j(\kappa)$ and $N^{\beth_\theta} \subseteq N$ and $|N| = \beth_{\theta+1}$.*

Lemma 2.2 was proved by adapting the Hauser method [12] from the indescribable cardinal context. A similar observation was made in [19], although not in this local form. The supercompactness-like nature of the embedding provided by Lemma 2.2 allows us to borrow techniques for strong unfoldability from the supercompact cardinals.

The Main Theorem will be proved using the lottery preparation of [8], which works best when defined relative to a function $f : \kappa \to \kappa$ with high growth behavior. The specific desired property is that f should exhibit the *Menas* property for strong unfoldability: for every θ one should be able to find embeddings as in Lemmas 2.1 and 2.2 for which $f \in M$ and $j(f)(\kappa) \geq \theta$. Hamkins [9] proved that such a function can be added by Woodin's fast function forcing, but Johnstone [14] observed that there is no need for forcing, since every strongly unfoldable cardinal already has a function with the Menas property.

Lemma 2.3 ([14]) *Every strongly unfoldable cardinal has a function with the Menas property. Indeed, there is a class function $f : \mathrm{ORD} \to \mathrm{ORD}$ such that for every strongly unfoldable cardinal κ, the function $f \restriction \kappa : \kappa \to \kappa$ is definable in every κ-model and has the Menas property for κ.*

The desired function is simply the failure-of-strong-unfoldability function, defined for inaccessible δ so that $f(\delta)$ is the least θ such that δ is not θ-strongly unfoldable, if there is any such θ. If κ is θ-strongly unfoldable and $\theta > \kappa$, then there are θ-strong unfoldability embeddings $j : M \to N$ such that κ is not θ-unfoldable in N. Since $<\theta$-strong unfoldability is witnessed by objects in $V_\theta \subseteq N$, it follows that κ is $<\theta$-strongly unfoldable in N but not θ-strongly unfoldable, and so $j(f)(\kappa) = \theta$, as desired.

The key new method of this article, allowing us to enlarge the κ-properness argument of [14] to the class of κ^+-preserving forcing here, involves methods from [10], which we now discuss. Our typical situation will occur when $\overline{V}$ is a set forcing extension of V.

Definition 2.4 ([10]) Suppose that $V \subseteq \overline{V}$ is an extension of two transitive models of set theory and that δ is a cardinal in $\overline{V}$.

(1) The extension $V \subseteq \overline{V}$ satisfies the δ *approximation property* if whenever $A \subseteq V$ is a set in $\overline{V}$ and $A \cap a \in V$ for any $a \in V$ of size less than δ in V, then $A \in V$.
(2) The extension $V \subseteq \overline{V}$ satisfies the δ *cover property* if whenever $A \subseteq V$ is a set of size less than δ in $\overline{V}$, then there is a covering set $B \in V$ with $A \subseteq B$ and $|B|^V < \delta$.

We refer to the sets $A \cap a$ appearing in (1) as the δ-*approximations* to A, and so the approximation property asserts that if all the approximations to a subset of V are in V, then the set is in V. The following critical lemmas from [10] allow us to build the κ-models for which our main construction will succeed.

Lemma 2.5 ([10], Lemma 15) *Suppose that $V \subseteq \overline{V}$ satisfies the δ approximation and cover properties. If $\overline{X}^{<\delta} \subseteq \overline{X}$ in $\overline{V}$ and $\overline{X} \prec \overline{V}_\theta$ in the language with a predicate for V, so that $\langle \overline{X}, X, \in\rangle \prec \langle \overline{V}_\theta, V_\theta, \in\rangle$, where $X = \overline{X} \cap V$, then $X \in V$. Further, if $\overline{M}$ is the Mostowski collapse of $\overline{X}$, then the Mostowski collapse of X is the same as $\overline{M} \cap V$.*

The corresponding version of this lemma, using elementary substructures of H_θ in place of V_θ, can be proved in exactly the same way, and we will later employ this version of the lemma. Although it is often convenient in set theory to work with H_θ and its elementary substructures, because they model ZFC$^-$, the proof of Theorem 2.7 below (in [10]) uses the Power Set axiom and the V_α hierarchy, and it is not currently clear whether that proof can be adapted for ZFC$^-$ models. Therefore, following [10] for the applications of the approximation and cover properties and Theorem 2.7, we work instead with what we call the *models of set theory*, meaning that they satisfy a sufficiently rich fixed finite fragment of ZFC, taken for definiteness to be the Σ_{100} fragment of ZFC. Transitive such models are easily obtained by the Mostowski collapse of elementary substructures of a sufficiently rich V_θ.

Lemma 2.6 ([10], Lemma 16) *Suppose that $V \subseteq \overline{V}$ satisfies the δ approximation and cover properties and $\kappa \geq \delta$ is an inaccessible cardinal in $\overline{V}$. If $A \subseteq \kappa$ is any set in $\overline{V}$, then there is a transitive model of set theory $\overline{M}$ of size κ in $\overline{V}$ such that $A \in \overline{M}$, $\overline{M}^{<\kappa} \subseteq \overline{M}$ in $\overline{V}$ and $M = \overline{M} \cap V \in V$ is a model of set theory.*

Note that M may not necessarily have size κ in V, if κ^{+V} is collapsed in $\overline{V}$. The following theorem is a special case of the Main Theorem of [10], useful for our purposes here.

Theorem 2.7 *Suppose that $V \subseteq \overline{V}$ satisfies the δ approximation and cover properties, for some regular $\delta < \kappa$, and that $\overline{M}$ is a transitive model of set theory in $\overline{V}$ such that $\overline{M}^{<\kappa} \subseteq \overline{M}$ in $\overline{V}$ and $M = \overline{M} \cap V$ is an element of V and a model of set theory there. If $j : \overline{M} \to \overline{N}$ is a cofinal embedding with critical point κ, and $\overline{N}^\delta \subseteq \overline{N}$ in $\overline{V}$, then both $N = \bigcup j\, " \, M = \overline{N} \cap V$ and the restriction $j \upharpoonright M : M \to N$ are elements of V.*

By Lemmas 2.5 and 2.6, any subset $A \subseteq \kappa$ can be placed into such a model $\overline{M}$ to which the theorem applies. The corresponding restricted embedding $j \upharpoonright M : M \to N$ can witness the θ-strong unfoldability of κ in V, because if $V_\theta \subseteq \overline{N}$, then

$V_\theta \subseteq \overline{N} \cap V = N$. Note that strong unfoldability is indeed witnessed by cofinal embeddings, since if $j : M \to N$ is any embedding, then chopping N off at $\lambda = \sup j \text{ " ORD}^M$ leads to a cofinal embedding $j : M \to N_\lambda$, which is elementary by the Tarski-Vaught criterion for $N_\lambda \prec N$. Also, if $M^{<\kappa} \subseteq M$ and $N^{<\kappa} \subseteq N$, then the cofinality of λ is κ, and so $N_\lambda^{<\kappa} \subseteq N_\lambda$. So the theorem applies to these embeddings, and consequently, the corresponding extensions have no new strongly unfoldable cardinals above δ.

Corollary 2.8 *Suppose that $V \subseteq \overline{V}$ exhibits the δ approximation and cover properties. Then every strongly unfoldable cardinal above δ in $\overline{V}$ is strongly unfoldable in V.*

Extensions with the approximation and cover properties are very common in the large cardinal literature and include all those by small forcing, the Silver iteration (successively adding Cohen sets), the canonical forcing of the GCH, the Laver preparation and the lottery preparation. The following shows that any Easton support iteration of progressively closed forcing will exhibit suitable approximation and cover properties.

Definition 2.9 ([10]) A forcing notion has a *closure point* at δ when it factors as $\mathbb{P} * \dot{\mathbb{Q}}$, where $\mathbb{P}$ is nontrivial (in the sense that forcing with $\mathbb{P}$ necessarily adds a new set), $|\mathbb{P}| \leq \delta$ and $\mathbb{1} \Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is $\leq\delta$ strategically closed.

Lemma 2.10 ([10], Lemma 13) *Forcing with a closure point at δ satisfies the δ^+ approximation and cover properties.*

Proof We give an alternative proof here, following Mitchell [18], since this is simpler than the original argument of [10]. Suppose that $V \subseteq V[g][G]$ is a forcing extension by the forcing $g * G \subseteq \mathbb{P} * \dot{\mathbb{Q}}$, where $\mathbb{P}$ is nontrivial, $|\mathbb{P}| \leq \delta$, and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is $\leq\delta$-strategically closed. We aim to show that $V \subseteq V[g][G]$ has the δ^+ approximation and cover properties. The δ^+ cover property is easy here, since it holds for each step of the extension $V \subseteq V[g] \subseteq V[g][G]$. For the δ^+-approximation property, suppose that $A \in V[g][G]$ has $A \subseteq V$, but $A \notin V$. Let $\dot{A}$ be a $\mathbb{P} * \dot{\mathbb{Q}}$-name for A, forced by $\mathbb{1}$ to have those properties. Since $\dot{A}$ is forced not to be in V, it follows that for every condition $(p, \dot{q})$ there is $b \in V$ such that $(p, \dot{q})$ does not decide whether $\check{b} \in \dot{A}$. So there are $(p_0, \dot{q}_0)$ and $(p_1, \dot{q}_1)$ below $(p, \dot{q})$ such that $(p_0, \dot{q}_0) \Vdash \check{b} \notin \dot{A}$ and $(p_1, \dot{q}_1) \Vdash \check{b} \in \dot{A}$. Since $\mathbb{P}$ is nontrivial, we may assume without loss of generality that p_0 and p_1 are incompatible. Thus, by mixing, there is a name $\dot{q}'$ such that p_0 forces $\dot{q}' = \dot{q}_0$ and p_1 forces $\dot{q}' = \dot{q}_1$. Indeed, by further mixing we may assume that $\mathbb{1}$ forces $\dot{q}' \leq \dot{q}$. The point is that we could have used conditions $(p_0, \dot{q}')$ and $(p_1, \dot{q}')$ with the same second coordinate $\dot{q}'$ and which have $\mathbb{1} \Vdash_{\mathbb{P}} \dot{q}' \leq \dot{q}$. We now simply iterate this. Fix a $\mathbb{P}$-name τ for the strategy witnessing that $\mathbb{1} \Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is $\leq\delta$-strategically closed, and enumerate $\mathbb{P} = \{ p_\beta \mid \beta < \delta \}$. We build a sequence of $\mathbb{P}$-names $\dot{q}_\beta$ for elements of $\dot{\mathbb{Q}}$ such that $\alpha < \beta$ implies $(\mathbb{1}, \dot{q}_\beta) \leq (\mathbb{1}, \dot{q}_\alpha)$, and there are p^0_β and p^1_β below p_β and an element b_β such that $(p^0_\beta, \dot{q}_\beta) \Vdash \check{b}_\beta \notin \dot{A}$ and $(p^1_\beta, \dot{q}_\beta) \Vdash b_\beta \in \dot{A}$. Use the previous observation at successors, combined with an application of the strategy τ so that the resulting sequence accords with the strategy τ, allowing the construction to continue through limits. For any $(p_\beta, \dot{t}_\beta)$, there are as above p^0_β and p^1_β below p_β, a name $\dot{q}_\beta$ and an element b_β such that $(\mathbb{1}, \dot{q}_\beta) \leq (\mathbb{1}, \dot{t}_\beta)$ and $(p^0_\beta, \dot{q}_\beta) \Vdash b_\beta \notin \dot{A}$ and $(p^1_\beta, \dot{q}_\beta) \Vdash b_\beta \in \dot{A}$, as desired. Since

$\mathbb{1}$ forces that the $\dot{q}_\beta$ are descending and conform with the strategy, there is $\dot{q}_\delta$ such that $(\mathbb{1}, \dot{q}_\delta) \leq (\mathbb{1}, \dot{q}_\beta)$ for all $\beta < \delta$. But no condition $(p, \dot{r})$ stronger than $(1, \dot{q}_\delta)$ can decide $\dot{A}$ on $\{b_\beta \mid \beta < \delta\}$, since $p = p_\beta$ for some β. So not all approximations to A are in V. $\square$

The iterations we mentioned previously exhibit numerous closure points, often between any two nontrivial stages of forcing. In general, any forcing that first adds a Cohen real and then performs countably closed forcing will have a closure point at ω and consequently exhibit the ω_1 approximation and cover properties.

Our main construction will make use of the lottery preparation of [8], which we briefly review here. If κ is a cardinal and $f : \kappa \to \kappa$, then the *lottery preparation* of κ is the following Easton support forcing iteration $\mathbb{P}$ of length κ. Nontrivial forcing is performed only at inaccessible cardinal stages $\gamma \in \mathrm{dom}(f)$ such that $f\,"\,\gamma \subseteq \gamma$. At such a stage γ, the forcing is the lottery sum in $V^{\mathbb{P}_\gamma}$ of all posets $\mathbb{Q} \in H_{f(\gamma)^+}$ such that for every $\beta < \gamma$ there is a $<\beta$-strategically closed dense subset of $\mathbb{Q}$. For the purposes of this paper, it would be fine to use a simplified preparation, where the stage γ lottery includes just the $<\gamma$-closed $\mathbb{Q}$ in $H_{f(\gamma)^+}$. The *lottery sum* $\oplus\mathcal{A}$ of a collection of posets $\mathcal{A}$, also commonly called side-by-side forcing, is the poset $\{\langle \mathbb{Q}, p\rangle \mid p \in \mathbb{Q} \in \mathcal{A}\} \cup \{\mathbb{1}\}$, ordered with $\mathbb{1}$ above everything and $\langle \mathbb{Q}, p\rangle \leq \langle \mathbb{Q}', p'\rangle$ when $\mathbb{Q} = \mathbb{Q}'$ and $p \leq_{\mathbb{Q}} p'$. The generic filter in effect selects a "winning" poset from $\mathcal{A}$ and then forces with it. The thrust of [8] is that if j is an embedding with critical point κ and $j(f)(\kappa)$ is large, then the lottery sum at stage κ in $j(\mathbb{P})$ includes all the desired posets, and so by working below a condition opting for the correct poset in that lottery, one avoids the need for a Laver function. Since the lottery preparation exhibits a closure point between any two successive nontrivial stages of forcing, the resulting forcing extension $V \subseteq V[G]$ exhibits the δ approximation and cover property for numerous δ less than κ. This completes our review of the preliminary and background material.

3 Indestructible Strong Unfoldability

We now proceed with the proof of our Main Theorem, restated here for convenience.

Main Theorem If κ is strongly unfoldable, then after suitable preparatory forcing, the strong unfoldability of κ becomes indestructible by all $<\kappa$-closed κ^+-preserving set forcing.

Proof Suppose that κ is strongly unfoldable. Let $f : \kappa \to \kappa$ be a function exhibiting the strong unfoldability Menas property, such as the function defined in Lemma 2.3. Let $\mathbb{P}$ be the lottery preparation of κ relative to f. Thus, $\mathbb{P}$ is the Easton support κ-iteration of forcing, which at every inaccessible stage $\gamma \in \mathrm{dom}(f)$, with $f\,"\,\gamma \subseteq \gamma$, forces with the lottery sum of all forcing notions in $H_{f(\gamma)^+}$ that for every $\beta < \gamma$ are $<\beta$-strategically closed. Suppose that $G \subseteq \mathbb{P}$ is V-generic and consider the model $V[G]$. We shall argue that κ is strongly unfoldable in $V[G]$ and the strong unfoldability of κ is indestructible over $V[G]$ by all further $<\kappa$-closed κ^+-preserving forcing. Because this includes trivial forcing, the latter property implies the former, and so it suffices to verify only that κ becomes indestructible. For this, suppose that $\mathbb{Q}$ is such a $<\kappa$-closed κ^+ preserving forcing in $V[G]$ and that $g \subseteq \mathbb{Q}$ is $V[G]$-generic. Fix a $\mathbb{P}$-name $\dot{\mathbb{Q}}$, forced to be $<\kappa$-closed and κ^+-preserving such that $\mathbb{Q} = \dot{\mathbb{Q}}_G$.

We will show that κ is strongly unfoldable in $V[G][g]$ by means of the embedding property. Fix any ordinal $\theta \geq \kappa$, large enough so that $\dot{\mathbb{Q}} \in H_\theta$, and consider any $A \subseteq \kappa$ in $V[G][g]$. Standard factoring arguments show that κ remains inaccessible in $V[G]$ (this is proved explicitly in [8, Theorem 3.3]) and consequently also in $V[G][g]$. Choose some large regular $\lambda > \beth_\theta$ and consider $H_\lambda^{V[G][g]} = H_\lambda[G * g]$. Let $\langle \overline{X}, X, \in \rangle \prec \langle H_\lambda[G * g], H_\lambda, \in \rangle$ be an elementary substructure chosen in $V[G][g]$, with $|\overline{X}| = \kappa$ and $\overline{X}^{<\kappa} \subseteq \overline{X}$ in $V[G * g]$, such that $\overline{X}$ contains all the objects in which we are interested: κ, $\mathbb{P}$, $\dot{\mathbb{Q}}$, G, g, f, and A. Although the elementary substructure $\overline{X}$ is in the extension $V[G * g]$ and the restriction $X = \overline{X} \cap V$ seems to be merely a subset of the ground model V, we make the critical observation, using the version of Lemma 2.5 adapted to H_λ, Lemma 2.10, and the subsequent observation that the lottery preparation admits numerous closure points below κ, that X is actually an *element* of the ground model V. It follows, of course, that the Mostowski collapse of X is also an element of V. By elementarity, every element of $\overline{X}$ has the form τ_{G*g} for some $\mathbb{P} * \dot{\mathbb{Q}}$ name $\tau \in X$, and so $\overline{X} = X[G][g]$. Since $X = \overline{X} \cap V = X[G * g] \cap V$, it follows that $G * g$ is X-generic. (To reiterate the remarks of the introduction, this is the key technical advance; we have obtained X-genericity here without any κ-properness hypothesis on $\mathbb{Q}$ by choosing X in the forcing extension, where it was easy to derive X-genericity, and then using Lemmas 2.5 and 2.10 to conclude that $X \in V$.) Continuing with the argument, let $\pi : X[G * g] \cong M[G * \pi(g)]$ be the Mostowski collapse of $\overline{X} = X[G * g]$. Since $\pi \restriction X : X \cong M$, it follows that M is the Mostowski collapse of X. Since $\mathbb{P} \subseteq X$, it follows that π is the identity on $\mathbb{P}$ and consequently on G. The poset $\mathbb{Q}$, however, may be larger than κ, so we let $\mathbb{Q}_0 = \pi(\mathbb{Q})$ and $g_0 = \pi(g)$, which is the same as π " g. Since $G * g$ was an X-generic filter, it follows that $G * g_0 \subseteq \mathbb{P} * \dot{\mathbb{Q}}_0$ is an M-generic filter. Since $\overline{X}^{<\kappa} \subseteq \overline{X}$ in $V[G * g]$, it follows easily that $X^{<\kappa} \subseteq X$ in V and consequently also $M^{<\kappa} \subseteq M$ in V. In summary, since the forcing $\mathbb{Q}$ was κ^+-preserving, what we have is a transitive set M of size κ in V with $\kappa \in M$ and $M^{<\kappa} \subseteq M \models \mathrm{ZFC}^-$. Since κ is strongly unfoldable in V, there is by Lemma 2.2 a $(\theta + 1)$-strong unfoldability embedding $j : M \to N$ in V, with $j(f)(\kappa) > \theta$ and $N^{\beth_\theta} \subseteq N$ and $|N| = \beth_{\theta+1}$.

We shall now lift the embedding $j : M \to N$ in two steps, first to the extension $j : M[G] \to N[j(G)]$, and then fully to $j : M[G][g_0] \to N[j(G)][j(g_0)]$. This final embedding, we shall argue, will witness the θ-strong unfoldability of κ with respect to A in $V[G][g]$. To begin the first step, consider the forcing $j(\mathbb{P})$, which is the lottery preparation of $j(\kappa)$ as computed in N relative to the function $j(f)$. Since $V_\kappa \subseteq N$ and $j(f) \restriction \kappa = f$, the forcing in $j(\mathbb{P})$ up to stage κ is the same as $\mathbb{P}$, the forcing we carried out in V. In particular, since $G \subseteq \mathbb{P} = j(\mathbb{P})_\kappa$ is V-generic for this forcing, it is also N-generic, and so $N[G]$ is a generic extension obtained by forcing over the first κ many stages of $j(\mathbb{P})$. Consider now the stage κ forcing in $j(\mathbb{P})$. This is the lottery sum of all sufficiently closed posets in $N[G]$ of hereditary size at most $j(f)(\kappa)$. Our assumptions on $\dot{\mathbb{Q}}$ and θ ensure that $\mathbb{Q}$ appears in this lottery. Below a condition opting for $\dot{\mathbb{Q}}$ in the stage κ lottery, therefore, we may factor $j(\mathbb{P})$ as $\mathbb{P} * \dot{\mathbb{Q}} * \mathbb{P}_{\text{tail}}$, where $\mathbb{P}_{\text{tail}}$ is the rest of the forcing, after stage κ up to $j(\kappa)$. Since $g \subseteq \mathbb{Q}$ is $V[G]$-generic, it is also $N[G]$-generic, and $N[G][g]$ is a generic extension arising from the first $\kappa+1$ many stages of forcing in $j(\mathbb{P})$. Because $j(f)(\kappa) > \theta$ and nontrivial forcing occurs in $j(\mathbb{P})$ only at inaccessible stages closed under $j(f)$, the next nontrivial stage of forcing is beyond $\beth_\theta$, and consequently $\mathbb{P}_{\text{tail}}$ is $\leq\beth_\theta$-closed in $N[G][g]$.

Let us suppose for a moment that the GCH holds at $\beth_\theta$ so that $\beth_{\theta+1} = \beth_\theta^+$ and consequently $|N| = \beth_\theta^+$. Since $N^{\beth_\theta} \subseteq N$ in V and $\mathbb{P} * \dot{\mathbb{Q}}$ is $\beth_\theta$-c.c., it follows that $N[G * g]^{\beth_\theta} \subseteq N[G * g]$ in $V[G * g]$. The forcing $\mathbb{P}_{\text{tail}}$, which is $\leq\beth_\theta$-closed in $N[G * g]$, is therefore also $\leq\beth_\theta$-closed in $V[G * g]$. By lining up the dense subsets of $\mathbb{P}_{\text{tail}}$ in $N[G * g]$ into a $\beth_\theta^+$-sequence in $V[G][g]$, we may construct by diagonalization an $N[G * g]$-generic filter $G_{\text{tail}} \subseteq \mathbb{P}_{\text{tail}}$ in $V[G][g]$. The combined filter $G * g * G_{\text{tail}} \subseteq \mathbb{P} * \dot{\mathbb{Q}} * \mathbb{P}_{\text{tail}}$ is therefore N-generic for $j(\mathbb{P})$, and we may lift the embedding to $j : M[G] \to N[j(G)]$ with $j(G) = G * g * G_{\text{tail}}$, completing the first step of the lifting process.

For the second step, still under the GCH assumption at $\beth_\theta$, consider now the forcing $j(\mathbb{Q}_0)$ in $N[j(G)]$. The usual closure arguments establish that $N[j(G)]^{\beth_\theta} \subseteq N[j(G)]$ in $V[G * g]$. These closure arguments also show that $M[G]$ is a κ-model in $V[G]$ and consequently that $M[G][g_0]$ is a κ-model in $V[G][g]$. Since $\mathbb{Q}_0$ is $<\kappa$-closed in $M[G]$, it follows by a density argument that the filter g_0 is a $<\kappa$-closed subset of $\mathbb{Q}_0$ in $M[G][g_0]$. Moreover, the model $V[G][g]$ agrees, but also sees that the filter g_0 has size κ. In $V[G][g]$, therefore, there is a descending κ-sequence of conditions $\langle p_\alpha \mid \alpha < \kappa\rangle$ that is downward cofinal in g_0 so that g_0 is the filter generated by this sequence. This sequence is not necessarily in $M[G][g_0]$, since this model doesn't necessarily agree that $\mathbb{Q}_0$ has size κ. Since $N[j(G)]$ is closed under κ-sequences, we conclude that $\langle j(p_\alpha) \mid \alpha < \kappa\rangle \in N[j(G)]$. This is a descending κ-sequence in $j(\mathbb{Q}_0)$, which is $<j(\kappa)$-closed in $N[j(G)]$, and so there is a master condition $p^* \in j(\mathbb{Q}_0)$ that is below every element of $j\,''\,g_0$. Because $j(\mathbb{Q}_0)$ is $\leq\beth_\theta$-closed in $N[j(G)]$ and $N[j(G)]^{\beth_\theta} \subseteq N[j(G)]$, it follows that $j(\mathbb{Q}_0)$ is $\leq\beth_\theta$-closed in $V[G][g]$. And since we have assumed that $N[j(G)]$ has size $\beth_\theta^+$, we may again construct by diagonalization an $N[j(G)]$-generic filter $g^* \subseteq j(\mathbb{Q}_0)$ below the master condition p^*. Because every element of $j\,''\,g_0$ lies above p^*, it follows that $j\,''\,g_0 \subseteq g^*$, and so we may lift the embedding to $j : M[G][g_0] \to N[j(G)][j(g_0)]$ with $j(g_0) = g^*$, completing the second step.

Let us argue that this lifted embedding witnesses the $(\theta + 1)$-strong unfoldability for A in $V[G][g]$. Because $N^{\beth_\theta} \subseteq N$ in V, it follows that $V_{\theta+1} \subseteq N$. Because $j(G) = G * g * G_{\text{tail}}$, it follows that G and g are both in $N[j(G)]$, and so $(V[G][g])_{\theta+1} = V_{\theta+1}[G][g] \subseteq N[j(G)]$. Since $\kappa \subseteq \overline{X}$, it follows that $\pi(A) = A$ and so $A \in M[G][g_0]$. Thus, the lifted embedding $j : M[G][g_0] \to N[j(G)][j(g_0)]$ is a $(\theta+1)$-strong unfoldability embedding for A in $V[G][g]$. Thus, under the GCH assumption at $\beth_\theta$, we have verified that κ is $(\theta+1)$-strongly unfoldable in $V[G][g]$, as desired.

It remains to consider the case that the GCH fails at $\beth_\theta$. In this case, let H be $V[G][g]$-generic for $\mathrm{Add}(\beth_\theta^+, 1)$, which naturally forces $2^{\beth_\theta} = \beth_\theta^+$. In $V[G][g][H]$, therefore, we do have the GCH hypothesis at $\beth_\theta$, and we can carry out the construction of the previous three paragraphs in $V[G][g][H]$. The result after the two steps of lifting is the embedding $j : M[G][g_0] \to N[j(G)][j(g_0)]$, which still has $A \in M[G][g_0]$ and $V_{\theta+1}[G][g] \subseteq N[j(G)]$. Since the H forcing is $\leq\beth_\theta$-closed, it adds no new subsets to $V_\theta[G][g]$, and so $(V_{\theta+1})^{V[G][g][H]} = (V_{\theta+1})^{V[G][g]} = V_{\theta+1}[G][g]$. So the embedding $j : M[G][g_0] \to N[j(G)][j(g_0)]$ is a $(\theta + 1)$-strong unfoldability embedding in $V[G][g][H]$. The final observation is that because θ-strong unfoldability is witnessed by extender embeddings having size at most $\beth_\theta$, the forcing H could not have added the θ-strong unfoldability extender arising from j, and so κ is θ-strongly unfoldable in $V[G][g]$, as desired. □

Since every unfoldable cardinal is strongly unfoldable in L and every strongly unfoldable cardinal is also weakly compact and totally indescribable, we obtain the following corollary.

Corollary 3.1 *If there is a model of* ZFC *with an unfoldable cardinal, then*

(1) *there is a model of* ZFC *with a weakly compact cardinal κ that is indestructible by all $<\kappa$-closed κ^+-preserving forcing, and*
(2) *there is a model of* ZFC *with a totally indescribable cardinal κ that is indestructible by all $<\kappa$-closed κ^+-preserving forcing.*

This provides a relatively low upper bound on the consistency strength of a weakly compact cardinal with this degree of indestructibility. But the exact strength of this situation remains open.

Question 3.2 *What is the exact consistency strength of a weakly compact cardinal κ that is indestructible by all $<\kappa$-closed κ^+-preserving forcing?*

The question is also open in the case of a weakly compact cardinal κ that is indestructible by all $<\kappa$-closed κ-proper forcing, or even only by all $<\kappa$-closed κ^+-c.c. forcing. The best upper bound currently known for any of these cases is the existence of an unfoldable cardinal, provided by our Main Theorem.

Let us turn now to local versions of the Main Theorem. The attentive reader will observe that our proof of the Main Theorem has actually established that if κ is $(\theta+1)$-strongly unfoldable and $2^{\beth_\theta} = \beth_\theta^+$, then after the lottery preparation relative to a function with the Menas property, the $(\theta+1)$-strong unfoldability of κ becomes indestructible by all $<\kappa$-closed κ^+-preserving forcing of size at most $\beth_\theta$. This GCH assumption of $2^{\beth_\theta} = \beth_\theta^+$ is relatively mild in the sense that it can be forced by adding a subset to $\beth_\theta^+$ while preserving $V_{\theta+1}$ and consequently the $(\theta+1)$-strong unfoldability of κ. But actually, we can prove a better local result, which omits both the GCH hypothesis and the restriction to successor ordinals by adapting lifting techniques from the strong cardinal context rather than from the supercompact cardinal context as we do above. Specifically,

Theorem 3.3 *If κ is θ-strongly unfoldable, $\kappa \le \theta$, then after the lottery preparation of κ relative to a function with the Menas property, the θ-strong unfoldability of κ becomes indestructible by all $<\kappa$-closed κ^+-preserving forcing of rank less than θ.*

Proof This theorem is obtained by combining the key method of our Main Theorem above with the proof of [14, Theorem 42], which obtained the corresponding result for $<\kappa$-closed κ-proper forcing. We sketch the argument. Suppose that $G \subseteq \mathbb{P}$ is V-generic for the lottery preparation $\mathbb{P}$ of κ relative to a function $f : \kappa \to \kappa$ with the θ-strong unfoldability Menas property for κ. Suppose that $g \subseteq \mathbb{Q}$ is V-generic for some $<\kappa$-closed κ^+-preserving forcing $\mathbb{Q}$ of rank less than θ. We want to show that κ remains θ-strongly unfoldable in $V[G][g]$. Fix any set $A \subseteq \kappa$, pick some large regular $\lambda > \theta$, and find $\overline{X} \prec H_\lambda^{V[G][g]}$ with $\overline{X}^{<\kappa} \subseteq \overline{X}$ as in Lemma 2.5, with $\{f, \mathbb{P}, \dot{\mathbb{Q}}, G, g, \kappa, A\} \subseteq \overline{X}$. It follows as before that $X = \overline{X} \cap V$ is in V, that $G * g$ is X-generic for $\mathbb{P} * \dot{\mathbb{Q}}$, and that $\overline{X} = X[G][g]$. If $\overline{M}$ is the Mostowski collapse of $\overline{X}$, then $\overline{M} = M[G][g_0]$, where M is the Mostowski collapse of X and

$g_0 \subseteq \mathbb{Q}_0$ is the Mostowski collapse of $g \subseteq \mathbb{Q}$. Now we make the key step, using Lemma 41 of [15], to obtain a θ-strong unfoldability embedding $j : M \to N$ in V with $j(f)(\kappa) < \theta$ and $V_\theta \subseteq N$ and $N = \{ j(h)(s) \mid h \in M, h : V_\kappa \to M, s \in S \}$, where $S = V_\theta \cup \{\theta, j \restriction \dot{\mathbb{Q}}_0\}$, and finally, crucially, $j \restriction \dot{\mathbb{Q}}_0 \in N$. Such an embedding is a very useful hybrid between the strongness-like extender embeddings, since it is generated by the seeds in S, and the supercompactness-like embeddings, since $j \restriction \dot{\mathbb{Q}}_0 \in N$. Let p be the condition opting for $\mathbb{Q}$ in the stage κ lottery of $j(\mathbb{P})$ so that $p = j(\vec{p})(s_0)$ for some function $\vec{p}$ and $s_0 \in S$. Let $Y = \{ j(h)(\kappa, s_0, \theta, j \restriction \dot{\mathbb{Q}}_0) \mid h \in M, h : \kappa \to M \} \prec N$ be the seed hull of $\langle \kappa, s_0, \theta, j \restriction \dot{\mathbb{Q}}_0 \rangle$, and consider $Y[G][g] \prec N[G][g]$. Since $Y^{<\kappa} \subseteq Y$ in V and $\mathbb{P} \subseteq Y$ is κ-c.c., it follows that $Y[G]^{<\kappa} \subseteq Y[G]$ in $V[G]$ and consequently that $Y[G][g]^{<\kappa} \subseteq Y[G][g]$ in $V[G][g]$. As $Y[G][g]$ has size κ and the tail forcing $\mathbb{P}_{\text{tail}}$, the part of $j(\mathbb{P})$ after stage κ, is $\leq\kappa$-closed, we may in $V[G][g]$ construct a $Y[G][g]$-generic $G^0_{\text{tail}} \subseteq \mathbb{P}_{\text{tail}}$. Since the next nontrivial stage of forcing in $\mathbb{P}_{\text{tail}}$ must be an inaccessible closure point of $j(f)$, it follows that $\mathbb{P}_{\text{tail}}$ is $\leq\beth_\theta$-closed in $N[G][g]$. It follows, we claim, that the filter G_{tail} generated by G^0_{tail} is actually $N[G][g]$-generic. To see this, suppose that $D \subseteq \mathbb{P}_{\text{tail}}$ is any open dense subset of $\mathbb{P}_{\text{tail}}$ in $N[G][g]$. So $D = j(\vec{D})(s)_{G*g}$ for some sequence $\vec{D}$ of names for open dense sets and some $s \in S$. In $Y[G][g]$, let $\bar{D}$ be the intersection of all $j(\vec{D})(t)_{G*g}$ for any $t \in S$ for which this gives an open dense subset of $\mathbb{P}_{\text{tail}}$. The set $\bar{D}$ is in $Y[G][g]$, since it was defined using only parameters in $Y[G][g]$. By the distributivity of the tail forcing, $\bar{D}$ is an open dense subset of $\mathbb{P}_{\text{tail}}$ in $Y[G][g]$. Consequently, G^0_{tail} contains elements from $\bar{D}$ and hence from D, since $\bar{D} \subseteq D$. Thus, G_{tail} is $N[G][g]$-generic, as we claimed. The combined filter $G * g * G_{\text{tail}}$ is therefore N-generic for $j(\mathbb{P})$ and we may lift the embedding to $j : M[G] \to N[j(G)]$ with $j(G) = G * g * G_{\text{tail}}$. Since $j \restriction \dot{\mathbb{Q}}_0 \in N$ and $g \in N[j(G)]$, we may build $j " g_0$ in $N[j(G)]$. Since $N[j(G)]$ can see that g_0 (and indeed all of $M[G][g_0]$) has size κ, it can build a descending κ-sequence cofinal in g_0, and consequently a descending κ-sequence cofinal in $j " g_0$. Since $j(\mathbb{Q}_0)$ is $<j(\kappa)$-closed, there is a master condition $p^* \in j(\mathbb{Q}_0)$ below every element of $j " g_0$. By elementarity, since $j \restriction \dot{\mathbb{Q}}_0 \in Y$, we may find such a condition p^* in $Y[j(G)]$, and consequently, we may again construct a $Y[j(G)]$-generic filter $g^* \subseteq j(\mathbb{Q}_0)$ containing p^*. It follows again by the distributivity of $j(\mathbb{Q}_0)$ that the filter generated by g^* is $N[j(G)]$-generic. Since $j " g_0 \subseteq g^*$, we may lift the embedding to $j : M[G][g_0] \to N[j(G)][j(g_0)]$, where $j(g_0) = g^*$. Since $V_\theta \subseteq N$, it follows that $V_\theta[G][g] \subseteq N[j(G)]$, and so this is a θ-strong unfoldability embedding. Since $A \in \overline{X}$, it follows that $A \in \overline{M} = M[G][g_0]$. So we have verified the θ-strong unfoldability of κ via Lemma 2.1. □

The following corollary is an immediate consequence of Theorem 3.3 using the fact that κ is Π^{m+1}_1-indescribable if and only if it is $(\kappa + m)$-strongly unfoldable and totally indescribable if and only if it is $(\kappa + m)$-strongly unfoldable for every $m < \omega$ (see [4] or [15]).

Corollary 3.4

(1) *If κ is Π^{m+1}_1-indescribable for some natural number m, then after the lottery preparation of κ relative to a function with the Menas property, the Π^{m+1}_1-indescribability of κ becomes indestructible by all $<\kappa$-closed, κ^+-preserving forcing of rank less than $\kappa + m$.*

(2) *If κ is totally indescribable, then after the lottery preparation of κ relative to a function with the Menas property, the total indescribability of κ is indestructible by all $<\kappa$-closed, κ^+-preserving forcing of size less than $\beth_{\kappa+\omega}$.*

We will argue in Section 4 that the degree of indestructibility provided by the Main Theorem, within the class of $<\kappa$-closed forcing, is optimal for its hypothesis, because having even the weak compactness of κ survive some $<\kappa$-closed forcing collapsing κ^+ has a far greater large cardinal consistency strength than unfoldability, implying the relative consistency of the failure of $\square_\kappa$. Nevertheless, we now show that if one begins with a much stronger hypothesis, such as a supercompact cardinal κ, then a greater degree of indestructibility is possible, dropping the κ^+-preserving requirement.

Theorem 3.5 *If κ is supercompact, then after suitable preparatory forcing, the strong unfoldability of κ becomes indestructible by all $<\kappa$-closed forcing (whether or not this forcing collapses κ^+).*

Proof We follow the proof of the Main Theorem, but in the event that the additional forcing collapses κ^+, we make use of the stronger hypothesis we have made on κ in V. Suppose that κ is supercompact in V. Clearly, κ is also strongly unfoldable in V. Let $f : \kappa \to \kappa$ be any function with the strong unfoldability Menas property; by pushing it somewhat higher, if necessary, we may also assume that f has the supercompactness Menas property as well. Let $\mathbb{P}$ be the lottery preparation of κ relative to f, and suppose that $G \subseteq \mathbb{P}$ is V-generic. The proof of the Main Theorem shows that κ remains strongly unfoldable in $V[G]$ and the strong unfoldability of κ is indestructible by further $<\kappa$-closed κ^+-preserving forcing over $V[G]$. In fact, [8, Corollary 4.6] shows that κ remains supercompact in $V[G]$ and the supercompactness of κ becomes indestructible over $V[G]$ by all $<\kappa$-directed closed forcing. We want to show that the strong unfoldability of κ is indestructible over $V[G]$ by $<\kappa$-closed forcing $\mathbb{Q}$, which may happen to collapse κ^+ and perhaps other cardinals. Suppose $g \subseteq \mathbb{Q}$ is $V[G]$-generic, and as in the Main Theorem, suppose θ is large enough above κ so that $\dot{\mathbb{Q}} \in H_\theta$, where $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for $\mathbb{Q}$, forced to be $<\kappa$-closed. Fix any $A \subseteq \kappa$ in $V[G][g]$. Choose any regular $\lambda > \theta, \kappa$ and let $\overline{X} \prec H_\lambda^{V[G][g]}$ be an elementary substructure in $V[G][g]$ of size κ in $V[G][g]$ in a language with a predicate for V so that $\langle \overline{X}, X, \in \rangle \prec \langle H_\lambda^{V[G][g]}, H_\lambda^V, \in \rangle$, where $X = \overline{X} \cap V$, and such that $\overline{X}^{<\kappa} \subseteq \overline{X}$ in $V[G][g]$ and $\mathbb{P}, \dot{\mathbb{Q}}, G, g, A, f, \kappa \in \overline{X}$. The Mostowski collapse of $\overline{X} = X[G][g]$ has the form $M[G][g_0]$, where M is the Mostowski collapse of X. Since the combined forcing $\mathbb{P} * \dot{\mathbb{Q}}$ has closure points below κ, we know by Lemmas 2.5 and 2.10 that X and hence M is in V. But unlike the situation in the Main Theorem, we don't necessarily know here that X has size κ in V, since κ^+ may have been collapsed, although we do know that X has size less than θ in V, since it has size κ in $V[G][g]$ and θ remains a cardinal there. Now is precisely where we shall make use of our greater assumption on κ in V. Since κ was supercompact in V, there is a $\beth_\theta$-supercompactness embedding $j : V \to N$ having critical point κ and $N^{\beth_\theta} \subseteq N$ in V, with $j(f)(\kappa) > \theta$. Consider the elementary embedding $j \upharpoonright M : M \to j(M)$ obtained by restricting to M. Since $M^{<\kappa} \subseteq M$ in V, it follows that $j(M)^{<j(\kappa)} \subseteq j(M)$ in N, and therefore $j(M)^{\beth_\theta} \subseteq j(M)$ in V. In particular, $V_{\theta+1} \subseteq j(M)$. In V, we may construct $Y \prec j(M)$ with $\mathrm{ran}(j) \subseteq Y$ and $Y^{\beth_\theta} \subseteq Y$ and of size $|Y| = 2^{\beth_\theta}$. If $\pi : Y \cong N_0$ is the Mostowski collapse,

then the induced factor embedding $j_0 = \pi \circ j : M \to N_0$ has critical point κ with $N_0^{\beth_\theta} \subseteq N_0$ and $|N_0| = 2^{\beth_\theta}$. In particular, $j_0 \in N_0$. This is something like a $(\theta + 1)$-unfoldability embedding, except that the domain M may have size larger than κ, although it does have size κ in $V[G][g]$. The point now is that the proof of the Main Theorem can simply proceed with this embedding j_0, since the argument never used that the domain had size κ in V except to get the embedding initially. So first we consider the case when the GCH holds at $\beth_\theta$. We lift the embedding in the first step to $j_0 : M[G] \to N_0[j_0(G)]$, where $j_0(G) = G * g * G_{\text{tail}}$, using the forcing $g \subseteq \mathbb{Q}$ in the lottery at stage κ in $j(\mathbb{P})$ and constructing G_{tail} by diagonalization against the $\beth_\theta^+$ enumeration of the dense sets in $N_0[G][g]$. Using the fact that $j_0 \in N_0$ and $g \in N_0[j_0(G)]$, we can build $j_0 \text{ "} g_0$, and since $N_0[G][g]$ knows that g_0 has size κ, as in the Main Theorem, we build a descending κ-sequence cofinal in $j_0 \text{ "} g_0$ and appeal to the fact that $j_0(\mathbb{Q}_0)$ is $<j_0(\kappa)$-closed to obtain a master condition $p^* \in j(\mathbb{Q}_0)$ below every element of $j_0 \text{ "} g_0$. Another diagonalization argument builds an $N_0[j_0(G)]$-generic filter $g^* \subseteq j_0(\mathbb{Q}_0)$ containing p^*, and we may therefore lift the embedding in $V[G][g]$ to $j_0 : M[G][g_0] \to N_0[j_0(G)][j_0(g_0)]$, where $j_0(g_0) = g^*$. This lifted embedding witnesses the $(\theta + 1)$-strongness of κ for A in $V[G][g]$, as desired. As in the Main Theorem, if the GCH happens to fail at $\beth_\theta$ in V, then we may simply force it over $V[G][g]$, use the previous argument to show that κ is θ-strongly unfoldable in the resulting extension, and then argue that the extra forcing could not have created the θ-strong unfoldability extender embeddings, so again κ is θ-strongly unfoldable in $V[G][g]$, as desired. $\square$

The Laver preparation of a supercompact cardinal κ can be easily modified, by allowing $<\gamma$-closed forcing at each stage γ rather than merely $<\gamma$-directed closed forcing, in order to attain the indestructibility identified in Theorem 3.5 (in addition to the supercompactness indestructibility by $<\kappa$-directed closed forcing). Our proof of Theorem 5.2 will show, however, that the unmodified Laver preparation, as defined in [17], definitely does not create this degree of indestructibility.

4 The Degree of Indestructibility Is Optimal from Our Hypothesis

We now briefly explain the sense in which the indestructibility result of the Main Theorem is optimal. For this, we make use of Jensen's principle $\square_\kappa$, which asserts of an uncountable cardinal κ that there is a sequence $\langle C_\xi \mid \xi < \kappa^+, \xi \in \text{Lim}\rangle$ such that for any limit ordinal $\xi < \kappa^+$, the set C_ξ is a closed unbounded subset of ξ, having order type less than κ if $\text{cof}(\xi) < \kappa$ and such that $C_\beta = C_\xi \cap \beta$ whenever β is a limit point of C_ξ. It follows that if $\beta < \kappa^+$ has cofinality κ, then C_β has order type exactly κ. The significance of the following theorem flows principally from the fact that the failure of $\square_\kappa$ when κ is weakly compact is a very strong hypothesis, known to imply $\text{AD}^{L(\mathbb{R})}$, for example, which has the strength of infinitely many Woodin cardinals, far above the existence of an unfoldable cardinal in the large cardinal hierarchy.

Theorem 4.1 *If a cardinal κ is weakly compact in a $<\kappa$-closed forcing extension $V[G]$ collapsing κ^{+V}, then $\square_\kappa$ fails in V.*

Proof This theorem is widely known, but we outline the proof. We begin with Jensen's fact that $\square_\kappa$ implies that there is a stationary nonreflecting subset $S \subseteq \kappa^+$ such that $S \subseteq \text{Cof}_\omega \kappa^+$. To see this, suppose $\vec{C} = \langle C_\xi \mid \xi < \kappa^+, \xi \in \text{Lim}\rangle$ is a $\square_\kappa$ sequence. We define the partial function $F : \kappa^+ \to \kappa^+$ on the limit ordinals below

κ^+ so that $F(\xi)$ equals the order type of C_ξ. By Fodor's lemma there is a stationary subset S of κ^+ with $S \subseteq \mathrm{Cof}_\omega\,\kappa^+$ on which F is constant. No initial segment $S \cap \beta$ can be stationary, for any $\beta < \kappa^+$ of uncountable cofinality, because the restriction of F to the limit points of C_β is injective, and the club C_β can thus have at most one of its limit points in S. The set $S \subseteq \mathrm{Cof}_\omega\,\kappa^+$ is therefore a stationary nonreflecting subset of κ^+.

Next, we argue that S remains stationary and nonreflecting in the forcing extension $V[G]$, obtained by $<\kappa$-closed κ^+-collapsing forcing $G \subseteq \mathbb{P}$. The set S is nonreflecting in $V[G]$, of course, since the initial segments of S were nonstationary in V and consequently remain nonstationary in $V[G]$. To see that S remains stationary in $V[G]$, suppose that $\dot{C}$ is a $\mathbb{P}$-name in V forced by the condition p to be club in κ^{+V}. Since S is stationary in V, there exists in V an elementary substructure $X \prec V_\theta$ of size κ, for some very large regular θ, with $\gamma = X \cap \kappa^{+V} \in S$ and $\kappa, S, p, \mathbb{P}, \dot{C} \in X$. Since γ has cofinality ω, we may construct a countable $Y \prec X$ such that $\sup(Y \cap \kappa^{+V}) = \gamma$ and $\kappa, S, p, \mathbb{P}, \dot{C} \in Y$. Thus also $Y \prec V_\theta$. Since $\mathbb{P}$ is countably closed, a simple descending sequence argument produces a condition p^* inside every open dense subset of $\mathbb{P}$ in Y. Since $p \in Y$ forces that $\dot{C}$ is unbounded in κ^{+V}, the condition p^* will force that $\dot{C}$ is unbounded in γ, and consequently force that γ is in $\dot{C}$. Since $\gamma \in S$, we have therefore found a condition below p forcing that $\dot{C} \cap \check{S}$ is nonempty, and so S remains stationary in $V[G]$, as desired.

Suppose now toward contradiction that κ is weakly compact in $V[G]$. Since $\mathbb{P}$ is $<\kappa$-closed and collapses κ^+, it follows that the cofinality of κ^{+V} in $V[G]$ is κ, so there is a club set $C \subseteq \kappa^{+V}$ of order type exactly κ. Since κ^{+V} now has size κ, we may find in $V[G]$ a κ-model M with $S, C \in M$ such that M agrees that S is stationary and nonreflecting, and a weak compactness embedding $j : M \to N$ having critical point κ. We may assume by the Hauser [12] embedding characterization that M and j are elements of N and that they have size κ there. Since $j(C)$ has order type $j(\kappa)$, it follows that $\delta = \sup j\,``\,\kappa^{+V} < j(\kappa^{+V})$. Observe that $j\,``\,C$ is a closed unbounded subset of δ in N, since C has order type κ and j has critical point κ and M is a κ-model in $V[G]$. If $D \subseteq j\,``\,C$ is any smaller club set in $N$, then $D = j\,``\,C_0$ for some smaller club $C_0 \subseteq C$ in N using the fact that $j \in N$. Since S is stationary in $V[G]$, it is stationary in N, and so $S \cap C_0$ is not empty. It follows that $j\,``\,S \cap j\,``\,C_0$ is also not empty, and so $j(S) \cap D$ is not empty. This shows that $j(S) \cap \delta$, a proper initial segment of $j(S)$, is stationary in N, contradicting the fact obtained by elementarity that $j(S)$ is stationary and nonreflecting in N. □

We note that by Kunen's theorem, if as in Theorem 4.1 the cardinal κ is weakly compact in a $<\kappa$-closed extension $V[G]$, then it is weakly compact in V (see also the proof of Theorem 5.1). Using Theorem 4.1, we now deduce that within the class of $<\kappa$-closed forcing, the conclusion of the Main Theorem is optimal.

Corollary 4.2 *Using the same hypothesis as in the Main Theorem, if consistent, one cannot provably improve the conclusion to indestructibility by any strictly larger class of $<\kappa$-closed forcing.*

The point is that any such improvement would lead to the situation of Theorem 4.1 and consequently to the failure of $\square_\kappa$ when κ is weakly compact, a conclusion having a much stronger consistency strength than the hypothesis, which would violate the Incompleteness Theorem. Indeed, what Theorem 4.1 shows is that one cannot improve the Main Theorem in this way from any consistent large cardinal hypothesis below the existence of infinitely many Woodin cardinals. From the hypothesis that κ is supercompact, however, we have already seen in Theorem 3.5 that one can improve the conclusion to provide indestructibility by all $<\kappa$-closed forcing.

The hypotheses on the forcing in Theorem 4.1 can be considerably weakened; the proof shows that it is sufficient, for example, that $\mathbb{P}$ preserves stationary subsets of $\mathrm{Cof}_\omega\, \kappa^{+V}$ and forces $\mathrm{cof}(\kappa^{+V}) = \kappa$, a hypothesis true of any proper, $<\kappa$-distributive forcing collapsing κ^+. The conclusion of Theorem 4.1 has reportedly been improved (see Jensen, Schimmerling, Schindler, Steel [13]) to the fact that if κ is weakly compact after collapsing κ^+, then there is nondomestic pre-mouse, which implies the consistency of AD and even $\mathrm{AD}_{\mathbb{R}}$.

Let us now turn to the possibility of weakening the $<\kappa$-closed requirement in the Main Theorem. We begin by showing that it cannot be weakened to $<\kappa$-strategically closed forcing.

Theorem 4.3 *It is not possible to weaken "$<\kappa$-closed" to "$<\kappa$-strategically closed" in the Main Theorem, because for any uncountable, regular cardinal κ with $2^{<\kappa} = \kappa$, there is a $<\kappa$-strategically closed cardinal-preserving notion of forcing that destroys the weak compactness of κ.*

Proof Suppose κ is an uncountable regular cardinal with $2^{<\kappa} = \kappa$. The standard forcing $\mathbb{P}$ to add a κ-Souslin tree by initial segment is the desired forcing notion. Because weakly compact cardinals have the tree property, they cannot admit such a Souslin tree. Conditions in $\mathbb{P}$ are the empty tree $\mathbb{1}$ and subtrees p of $2^{<\kappa}$ such that

(1) $\mathrm{ht}(p) = \lambda + 1$ for some limit ordinal $\lambda < \kappa$,
(2) each level of p has size less than κ,
(3) p has binary splitting at each level except its top level, and
(4) every $s \in p$ is extended by some $t \supseteq s$ which lies in the top level of p.

For conditions $p, q \in \mathbb{P}$, we have that $p \le q$ exactly if p end-extends the tree q. It is well known that $\mathbb{P}$ preserves all cardinals and adds a κ-Souslin tree. Here, we show that $\mathbb{P}$ is $<\kappa$-strategically closed. Note that not every descending chain $\langle p_\xi : \xi < \omega_1\rangle$ in $\mathbb{P}$ has a lower bound in $\mathbb{P}$, since the trees p_ξ may even be converging to an Aronszajn tree $p = \bigcup_\xi p_\xi$ with no possibility of extending further. Thus, $\mathbb{P}$ is not $\le\omega_1$-closed. But it is $<\kappa$-strategically closed, because as the sequence of conditions $\langle p_\xi \mid \xi < \kappa\rangle$ is revealed during play, Player II can assign to each node s in p_ξ a branch through p_ξ which she promises to extend in subsequent play. Player I is required by condition (4) to extend these branches at successor stages of play, and at limit stages of play, Player II can play the unions of her promised branches as the desired limit tree. So Player II has a winning strategy allowing her to continue play indefinitely up to κ. $\square$

We conclude this section by explaining another sense in which the indestructibility provided by the Main Theorem, even in the case of indestructible weak compactness or indescribability, is optimal. Namely, if one attains such indestructibility via preparatory forcing that resembles the Laver or lottery preparations—an Easton support iteration of progressively closed forcing—then such an extension will have numerous closure points below κ and consequently by Lemma 2.10 will exhibit the δ approximation and cover properties. The following theorem then shows that κ must have been either strongly unfoldable or supercompact in the ground model, depending on whether the degree of indestructibility allows for κ^+ to be collapsed or not. This conforms exactly with the hypotheses of the Main Theorem and of Theorem 3.5. We conclude that for the method of proof employed, consequently, the hypotheses of the Main Theorem and of Theorem 3.5 are optimal. Recall that κ is $(\kappa+1)$-strongly unfoldable if and only if it is Π^2_1-indescribable.

Theorem 4.4 *Suppose that κ is a cardinal in a forcing extension $V[G]$ having a closure point below κ. Then*

(1) *if the $(\kappa+2)$-strong unfoldability of κ is indestructible over $V[G]$ by all forcing of the form $\mathrm{Add}(\kappa,\theta)$, then κ was strongly unfoldable in V;*
(2) *if the $(\kappa+1)$-strong unfoldability of κ is indestructible over V by all forcing of the form $\mathrm{Add}(\kappa,\theta)$, then κ is strongly unfoldable in L;*
(3) *if the weak compactness of κ is indestructible over $V[G]$ by all forcing of the form $\mathrm{Coll}(\kappa,\theta)$, then κ was supercompact in V.*

Proof For statement (1), suppose $A \subseteq \kappa$ in V and fix any ordinal θ. Force over $V[G]$ to add a $V[G]$-generic filter $H \subseteq \mathrm{Add}(\kappa, \beth^V_\theta)$, and suppose that κ remains $(\kappa+2)$-strongly unfoldable in $V[G][H]$. Notice that the combined forcing $V \subseteq V[G*H]$ still admits the same closure point. By Lemma 2.6, we may find a κ-model $\overline{M}$ in $V[G][H]$ for which $M = \overline{M} \cap V$ is an element of V and $A \in M$. By $(\kappa+2)$-strong unfoldability, there is an elementary embedding $j : \overline{M} \to \overline{N}$ with critical point κ in $V[G][H]$, having $\overline{N}^{\beth_{\kappa+1}} \subseteq \overline{N}$ in $V[G][H]$. In $V[G][H]$, since $\beth_{\kappa+1} = 2^\kappa \geq \beth^V_\theta$ and V^V_θ has size $\beth^V_\theta$, it follows by coding with subsets of κ that $V^V_\theta \subseteq \overline{N}$. The embedding j is not cofinal in $\overline{N}$, but if we let $\overline{N}_0 = \overline{N}_\lambda$, where $\lambda = \sup j\,"\,\overline{M} \cap \mathrm{ORD}$, then $j : \overline{M} \to \overline{N}_0$ remains elementary, and $V^V_\theta \subseteq \overline{N}_0$, and also $\overline{N}_0^{<\kappa} \subseteq \overline{N}_0$ in $V[G][H]$. By Theorem 2.7, it follows that $j \restriction M : M \to N_0$, where $N_0 = \bigcup j\,"\,M = \overline{N}_0 \cap V$ is an element of V. Thus, the restriction $j \restriction M : M \to N_0$ with $V^V_\theta \subseteq N_0$ witnesses θ-strong unfoldability for A in V, as desired (note: in the case that $V[G]$ collapses κ^{+V}, then M may have size κ^{+V} or more in V, but this is no obstacle, since we may simply restrict j further to a κ-model inside M containing A). The same argument works with all subsets of V_θ, and so this is actually a $(\theta+1)$-strong unfoldability embedding.

For statement (2), suppose that θ is any cardinal and that κ remains $(\kappa+1)$-strongly unfoldable in $V[H]$, a forcing extension obtained by forcing with $H \subseteq \mathrm{Add}(\kappa,\theta)$. Thus, $2^\kappa \geq \theta$ in $V[H]$, and consequently the $(\kappa+1)$-strong unfoldability embeddings $j : M \to N$ that exist in $V[G]$ have $P(\kappa)^{V[H]} \subseteq N$ and consequently $j(\kappa) > \theta$. Thus, κ is θ-unfoldable in $V[H]$ and consequently θ-unfoldable in L. Since θ was arbitrary, it follows that κ is unfoldable in L, and hence also strongly unfoldable there, as desired. Statement (3) is essentially the Main Theorem of [2], and we omit the proof. □

The more specific facts are that if after closure point forcing the weak compactness of κ becomes indestructible by $\mathrm{Coll}(\kappa, 2^{\theta^{<\kappa}})$, then κ is θ-supercompact in V (see [2, Theorem 3]). Similarly, the argument above shows that if after closure point forcing the $(\kappa+2)$-strong unfoldability of κ becomes indestructible by $\mathrm{Add}(\kappa,\beth_\theta)$, then κ is $(\theta+1)$-strongly unfoldable in V. This is a converse of sorts to Theorem 3.3 and Corollary 3.4 and shows that the size limitations appearing in those results cannot be omitted without a totally different proof method, lacking closure points.

5 Directed Closed Versus Closed Forcing

We have called attention to the fact that our theorem concerns $<\kappa$-closed forcing as opposed to $<\kappa$-directed closed forcing. Let us now discuss this issue in more detail. Laver's landmark theorem [17] showed that any supercompact cardinal κ can be made indestructible by further $<\kappa$-directed closed forcing, and this result was subsequently generalized to a number of other cardinals. In the supercompactness argument, one uses the directed closure of the forcing when constructing the master condition in the final step of the lifting argument. Although this use of directed closure can be weakened in some ways, it cannot be weakened to provide indestructibility by all $<\kappa$-closed forcing, because such a degree of indestructibility is impossible for supercompact or even measurable cardinals. It was observed in [16] that the class of $<\kappa$-closed forcing includes the forcing to add a slim κ-Kurepa tree (the forcing is $<\kappa$-closed and κ^+-c.c.), which necessarily destroys measurability and more. Indeed, [16] shows that no amount of $<\theta$-closure, for arbitrarily large θ, is sufficient to overcome the lack of directed closure, and there are such posets destroying the supercompactness of κ. Gitman, Reitz, and Johnstone had observed that, with a suitable preparation, the particular forcing to add a slim Kurepa tree is not a problem for weakly compact, strongly unfoldable and even strongly Ramsey cardinals (see [5, Theorem 2.56]). For strong unfoldability this fact is of course generalized by the main theorem of [14; 15] and further generalized by the Main Theorem of this article.

In the case of weak compactness, it turns out by Theorem 5.1 that indestructibility by $<\kappa$-directed closed separative forcing is equivalent to indestructibility by $<\kappa$-closed separative forcing. This is special for weak compactness and is primarily a consequence of the fact that weak compactness is downward absolute through $<\kappa$-closed forcing. Theorem 5.2 will show, in an extreme way, that the corresponding fact is not true for strong unfoldability.

Theorem 5.1 *The following are equivalent:*

1. *The weak compactness of κ is indestructible by $<\kappa$-closed separative forcing;*
2. *The weak compactness of κ is indestructible by $<\kappa$-directed closed separative forcing;*
3. *The weak compactness of κ is indestructible by all forcing of the form* $\mathrm{Coll}(\kappa, \theta)$.

Proof (Thanks to James Cummings for this suggestion and a helpful discussion.) Clearly (1) implies (2) and (2) implies (3). So suppose that the weak compactness of κ is indestructible by the collapse forcing $\mathrm{Coll}(\kappa, \theta)$. The point will be that any $<\kappa$-closed separative forcing is absorbed by collapse forcing $\mathrm{Coll}(\kappa, \theta)$ for sufficiently large θ, and this collapse forcing is $<\kappa$-directed closed and separative. Specifically, suppose that $\mathbb{Q}$ is any $<\kappa$-closed separative forcing and $G \subseteq \mathbb{Q}$ is any V-generic filter. Consider the two-step iterated forcing $\mathbb{Q} * \mathrm{Coll}(\kappa, \theta)$, where $\theta = |\mathbb{Q}|^{<\kappa}$. This forcing is $<\kappa$-closed and separative, necessarily collapses θ to κ, and has size $\theta = \theta^{<\kappa}$. It is a well-known result that all such posets are forcing equivalent to $\mathrm{Coll}(\kappa, \theta)$. This result is proved by building a dense copy of the tree $\theta^{<\kappa}$ inside the poset via an argument that breaks down if the poset is not $<\kappa$-closed, or if it is not separative. If $H \subseteq \mathrm{Coll}(\kappa, \theta)$ is any $V[G]$-generic filter, we may thus view $V[G][H]$ as one-step forcing extension of V, obtained by simply forcing with $\mathrm{Coll}(\kappa, \theta)$ over

V. Since $\mathrm{Coll}(\kappa, \theta)$ is $<\kappa$-directed closed, we therefore know that κ remains weakly compact in $V[G][H]$. We complete the proof by proving that the weak compactness of κ in $V[G][H]$ is downward absolute through the $<\kappa$-closed forcing to $V[G]$. Certainly, κ is inaccessible in $V[G]$. Suppose that T is a κ-tree in $V[G]$. By the tree property in $V[G][H]$, there is a κ-branch through T in $V[G][H]$. But since the forcing from $V[G]$ to $V[G][H]$ was $<\kappa$-closed, we can in $V[G]$ build a pseudogeneric sequence of conditions that decide more and more of this branch, thereby producing a branch through T in $V[G]$. Thus, κ is inaccessible and has the tree property in $V[G]$ and is consequently weakly compact there. Thus, $\mathbb{Q}$ preserved the weak compactness of κ, as desired. □

Unfortunately, the fact expressed in Theorem 5.1 does not generalize to strong unfoldability. The key element of the proof of Theorem 5.1 is the fact that weak compactness is downward absolute, via the tree property, through $<\kappa$-closed forcing. This is simply not true of strong unfoldability. The easiest way to see this is to carry out the Easton support iteration of length κ, adding a subset to γ at every inaccessible cardinal $\gamma < \kappa$ (but do nothing at stage κ). If G is a V-generic filter for this iteration, then κ will not be $(\kappa+1)$-strongly unfoldable in $V[G]$, essentially because the unfoldability embeddings $j : M[G] \to N[j(G)]$ would have to have done forcing at stage κ in $j(G)$, and by strong unfoldability, the $N[G]$-generic that would be used would have to be actually $V[G]$-generic, since by Theorem 2.7 we would have $V_{\kappa+1} \subseteq N \subseteq V$, contradicting the assumption that the embedding was in $V[G]$. Nevertheless, forcing over $V[G]$ to add a $V[G]$-generic subset $g \subseteq \kappa$ will now resurrect the strong unfoldability of κ by the usual lifting arguments. Thus, κ is strongly unfoldable in $V[G][g]$, but not in $V[G]$, and so there is no general downward absoluteness of strong unfoldability even through the forcing $\mathrm{Add}(\kappa, 1)$. This argument can be modified to use $\mathrm{Add}(\kappa^+, 1)$ or $\mathrm{Add}(\kappa^{++}, 1)$ or others, producing a model $\overline{V}$, in which κ is not strongly unfoldable, but becomes strongly unfoldable by $\leq\kappa$-closed or $\leq\kappa^+$-closed forcing, respectively.

Indeed, the model provided by Theorem 5.2 is an extreme opposite case, where the strong unfoldability of κ is fully indestructible by $<\kappa$-directed closed forcing, but definitely destroyed by any $<\kappa$-closed forcing not forcing equivalent to $<\kappa$-directed closed forcing. This contrasts sharply with Theorem 5.1 and shows that the improvement from $<\kappa$-directed closed to $<\kappa$-closed forcing, introduced by Johnstone in [14], is a genuine improvement.

Theorem 5.2 *If κ is strongly unfoldable in V, then there is a forcing extension $V[G]$ such that*

(1) *the strong unfoldability of κ is indestructible over $V[G]$ by all $<\kappa$-directed closed κ^+-preserving forcing, but*
(2) *the strong unfoldability of κ is destroyed over $V[G]$ by all $<\kappa$-closed forcing that is not forcing equivalent below a condition to $<\kappa$-directed closed forcing.*

If κ is supercompact in V, then there is a forcing extension $V[G]$ such that

(3) *the supercompactness of κ is indestructible over $V[G]$ by all $<\kappa$-directed closed forcing, but*
(4) *the strong unfoldability of κ is destroyed over $V[G]$ by all $<\kappa$-closed forcing that is not forcing equivalent below a condition to $<\kappa$-directed closed forcing.*

Proof This argument follows the main idea and theme of [6]. Suppose that κ is strongly unfoldable. Let f be the failure-of-strong unfoldability function as defined after Lemma 2.3, which has a particularly strong form of the Menas property. Let $\mathbb{P}$ be the modified lottery preparation, for which at every inaccessible stage γ having f " $\gamma \subseteq \gamma$, we force with the lottery sum of all $<\gamma$-directed closed forcing $\mathbb{Q} \in H_{f(\gamma)^+}$. That is, we limit the lottery preparation to use only directed closed as opposed to closed forcing. Suppose that $G \subseteq \mathbb{P}$ is V-generic for κ. It is straightforward to follow through the proof of the Main Theorem and check that this modification does not cause any problems in verifying that κ is strongly unfoldable in $V[G]$ and that the strong unfoldability of κ is indestructible by further $<\kappa$-directed closed κ^+-preserving forcing over $V[G]$. So (1) holds. Similarly, we may follow through the proof of Theorem 3.5 and check that if κ is initially supercompact, then in $V[G]$ the supercompactness of κ becomes indestructible by all $<\kappa$-directed closed forcing. So (3) holds as well.

We now prove (2) and (4) at the same time. Suppose that $\mathbb{Q}$ is $<\kappa$-closed forcing in $V[G]$ and κ remains strongly unfoldable in $V[G][H]$, where $H \subseteq \mathbb{Q}$ is $V[G]$-generic. Choose $\theta > \kappa$ large enough so that $\mathbb{Q} \in V[G]_\theta$, and consider in $V[G][H]$ a $(\theta + 1)$-strong unfoldability embedding $j : M[G][H_0] \to N[j(G)][j(H_0)]$, where the κ-model $M[G][H_0]$ arises as the Mostowski collapse of $\overline{X} = X[G][H]$ as in Lemma 2.5 and the Main Theorem. (As in the Main Theorem, $H_0 \subseteq \mathbb{Q}_0$ is an $M[G]$-generic filter, because it is the pointwise image under the collapse of the $X[G]$-generic filter $H \subseteq \mathbb{Q}$.) Without loss of generality, we may assume that κ is not $(\theta + 1)$-strongly unfoldable in $N[j(G)][j[(H_0)]$, that j is cofinal, and that the target model $N[j(G)][j[(H_0)]$ is closed under $<\kappa$-sequences. It follows that $j(f)(\kappa) = \theta + 1$, which is the strong form of the Menas property we mentioned. The N-generic filter $j(G)$ selected some poset $\mathbb{Q}_1$ in the stage κ lottery of $j(\mathbb{P})$ with hereditary size at most the size of $j(f)(\kappa) = \theta + 1$. The filter $j(G)$ also added an $N[G]$-generic filter $H_1 \subseteq \mathbb{Q}_1$. Without loss of generality, let us thus assume that $\mathbb{Q}_1$ is a subset of θ. We may factor the forcing $j(\mathbb{P})$ below a condition in $j(G)$ as $\mathbb{P} * \mathbb{Q}_1 * \mathbb{P}_{\text{tail}}$, where $\mathbb{P}_{\text{tail}}$ is the forcing after stage κ up to $j(\kappa)$. The generic filter $j(G)$ similarly factors as $G * H_1 * G_{\text{tail}}$. Since $H \in V[G][H]_\theta$, it follows that $H \in N[j(G)]$. By the main result of [10] as in Theorem 2.7, we know that the restriction $j : M \to N$ is in V and that $N = N[j(G)][j(H_0)] \cap V$. Thus, $V_{\theta+1} \subseteq N$ and $V[G]_{\theta+1} = N[G]_{\theta+1}$. It follows that $V[G]$ and $N[G]$ have all the same subsets of $\mathbb{Q}_1$ and so $H_1 \subseteq \mathbb{Q}_1$ is fully $V[G]$-generic. Since $j(f)(\kappa) = \theta + 1$, it follows that the next stage of forcing after stage κ in $j(\mathbb{P})$ is beyond θ. Consequently, the forcing $G_{\text{tail}} \subseteq \mathbb{P}_{\text{tail}}$ is highly closed in $N[G][H_1]$, and so $N[G][H_1]_{\theta+1} = N[j(G)]_{\theta+1} = V[G][H]_{\theta+1}$. In particular, $H \in N[G][H_1] \subseteq V[G][H_1] \subseteq V[G][H]$, and consequently $V[G][H_1] = V[G][H]$. Since the respective forcing to add $H \subseteq \mathbb{Q}$ and $H_1 \subseteq \mathbb{Q}_1$ over $V[G]$ gave rise to the same forcing extension, it follows that $\mathbb{Q}$ and $\mathbb{Q}_1$ are forcing equivalent below respective conditions $h \in H$ and $h_1 \in H_1$. That is, forcing with $\mathbb{Q} \restriction h$ is forcing equivalent to $\mathbb{Q}_1 \restriction h_1$. Since $\mathbb{Q}_1$ was allowed in the stage κ lottery of $j(\mathbb{P})$, it was $<\kappa$-directed closed in $N[G]$. Since $N[G]$ has all the same subsets of $\mathbb{Q}_1$ as $V[G]$, it follows that both $\mathbb{Q}_1$ and its restriction $\mathbb{Q}_1 \restriction h_1$ are $<\kappa$-directed closed in $V[G]$. So below a condition, $\mathbb{Q}$ is forcing equivalent to $<\kappa$-directed closed forcing, as claimed in (2) and (4). □

Our expectation is that the method of the previous theorem would generalize to a full "As you like it" style theorem in the sense of [6], attaining indestructibility and destructibility of strong unfoldability by any locally definable precise class of $<\kappa$-closed κ^+-preserving forcing notions (e.g., the ones that collapse κ^{++}, or the ones that don't, etc.).

6 Global and Universal Indestructiblity

We begin with *global* indestructibility, where all strongly unfoldable cardinals become simultaneously indestructible.

Theorem 6.1 *If V satisfies* ZFC*, then there is a class forcing extension $V[G]$ satisfying* ZFC *such that*

(1) *every strongly unfoldable cardinal in V remains strongly unfoldable in $V[G]$,*
(2) *every strongly unfoldable cardinal κ in $V[G]$ is indestructible by $<\kappa$-closed, κ^+-preserving set forcing,*
(3) *no new strongly unfoldable cardinals are created.*

Proof Let $f : \mathrm{ORD} \to \mathrm{ORD}$ be the global Menas function for strong unfoldability provided by Lemma 2.3. Let $\mathbb{P}$ be the (possibly proper class) lottery preparation forcing relative to f, and suppose that $G \subseteq \mathbb{P}$ is V-generic. Since the forcing in $\mathbb{P}$ becomes increasingly closed, the usual arguments establish that $V[G] \models \mathrm{ZFC}$, even when $\mathbb{P}$ is a proper class. If κ is strongly unfoldable in V, then the forcing $\mathbb{P}_\kappa$ up to κ is precisely the lottery preparation of κ relative to $f \restriction \kappa$, which has the strong unfoldability Menas property for κ. The Main Theorem therefore establishes that κ remains strongly unfoldable in $V[G_\kappa]$ and becomes indestructible there by any further $<\kappa$-closed κ^+-preserving forcing. In particular, since the forcing at stage κ itself is trivial, as strongly unfoldable cardinals are not in the domain of f, the cardinal κ remains strongly unfoldable after further forcing with the forcing $G_{\kappa,\theta} \subseteq \mathbb{P}_{\kappa,\theta}$ for the stages from κ up to any larger ordinal θ. That is, κ remains strongly unfoldable in every $V[G_\theta]$. Since the subsequent forcing $G_{\theta,\infty} \subseteq \mathbb{P}_{\theta,\infty}$ is $<\theta$-closed, it doesn't destroy unfoldability down low, and so by taking θ arbitrarily large we conclude that κ remains strongly unfoldable in $V[G]$, establishing (1). For (2), suppose that $\mathbb{Q}$ is any $<\kappa$-closed κ^+-preserving forcing in $V[G]$ and that $H \subseteq \mathbb{Q}$ is a $V[G]$-generic filter. Since $\mathbb{Q}$ is a set, we know $\mathbb{Q} \in V[G_\theta]$ for some large enough θ. We may therefore factor $\mathbb{P} * \mathbb{Q}$ as $\mathbb{P}_\kappa * (\mathbb{P}_{\kappa,\theta} * \mathbb{Q}) * \mathbb{P}_{\theta,\infty}$. The middle forcing $\mathbb{P}_{\kappa,\theta} * \mathbb{Q}$ is $<\kappa$-closed and κ^+-preserving over $V[G_\kappa]$, and so it preserves the strong unfoldability of κ because κ was indestructible in $V[G_\kappa]$. The tail forcing $\mathbb{P}_{\theta,\infty}$ is highly distributive (but no longer closed if $\mathbb{Q}$ is nontrivial), and so does not affect the strong unfoldability of κ down low. Once again, by taking θ arbitrarily large, we see that κ remains strongly unfoldable in $V[G][H]$, establishing the desired indestructibility as in (2), but only for those strongly unfoldable cardinals in $V[G]$ that were also strongly unfoldable in V. But, since the lottery preparation $\mathbb{P}$ admits a closure point between any two nontrivial stages of forcing, if we simply insist that nontrivial forcing occurs at stage ω, or at the least inaccessible cardinal, then by Corollary 2.8, the forcing $\mathbb{P}$ has created no new strongly unfoldable cardinals, thereby completing the argument for (2) and also establishing (3). $\square$

In contrast, we turn now to *universal* indestructibility, for which any degree of strong unfoldability exhibited by any cardinal is made indestructible. The difference is whether or not the cardinals that are only partially strongly unfoldable, such as the weakly compact and indescribable cardinals, are also indestructible. Specifically, we say that there is *universal indestructibility for strong unfoldability* if whenever any cardinal κ is θ-strongly unfoldable, then it remains θ-strongly unfoldable after any $<\kappa$-closed κ^+-preserving forcing. (From a stronger hypothesis in the ground model, we will also be able to omit the limitation to κ^+-preserving forcing.) The universal indestructibility theme was introduced by Apter and Hamkins [1], who treated the case of universal indestructibility for supercompactness and partial supercompactness. In order to do so, they introduced the method of *trial-by-fire* forcing, which we adapt here to the case of strong unfoldability. The results here are parallel to [1] via an analogy between strong unfoldability and supercompactness, beginning with the following basic limitation.

Theorem 6.2 *Universal indestructibility for strong unfoldability is inconsistent with the existence of two strongly unfoldable cardinals, and even with a cardinal κ that is $(\delta + 1)$-strongly unfoldable for a weakly compact cardinal δ above κ.*

Proof This is the strong unfoldability analogue of [1, Theorem 10]. Suppose that universal indestructibility for strong unfoldability holds, but κ is $(\delta + 1)$-strongly unfoldable for some weakly compact cardinal δ above κ. Force to add a V-generic subset $A \subseteq \kappa$ using $\mathrm{Add}(\kappa, 1)$. By assumption, κ remains $(\delta+1)$-strongly unfoldable in $V[A]$. Also, since this is small forcing relative to δ, we know δ remains weakly compact in $V[A]$. But also, because this is small forcing, the Main Theorem of [7] shows that the weak compactness of δ is now destructible in $V[A]$; further forcing with $\mathrm{Add}(\delta, 1)^{V[A]}$ will destroy the weak compactness of δ. Let M be any κ-model in $V[A]$, and fix $j : M \to N$ a $(\delta + 1)$-strong unfoldability embedding. Since $V[A]_{\delta+1} \subseteq N$, the model N agrees that δ is weakly compact and that adding a subset to δ will destroy the weak compactness of δ. This means that N thinks that there is a destructible weakly compact cardinal below $j(\kappa)$, namely, δ, and consequently by elementarity there will be such a destructible weakly compact cardinal γ below κ in M. Since M and V agree on V_κ, this means that γ was a destructible weakly compact cardinal in V, violating our assumption that universal indestructibility holds. □

The argument shows that if the strong unfoldability of κ is indestructible by $\mathrm{Add}(\kappa, 1)$ and all weakly compact cardinals $\gamma < \kappa$ are indestructible by $\mathrm{Add}(\gamma, 1)$, then there are no weakly compact cardinals above κ. So if we aim to produce a model of universal indestructibility for strong unfoldability, then the most we can hope for is one strongly unfoldable cardinal, with essentially no large cardinals above it. Furthermore, our method will require us, we claim, to begin with many strongly unfoldable cardinals in the ground model. The reason is that in order to force universal indestructibility for strong unfoldability, we will of course carry out a kind of universal preparatory forcing, which will have numerous closure points. It follows by Theorem 4.4 that any cardinals that survive this preparation with any nontrivial degree of strong unfoldability intact will have begun as fully strongly unfoldable cardinals in the ground model. That is, Theorem 4.4 shows that if γ is $(\gamma + 2)$-strongly unfoldable in our final model $V[G]$ and indestructible by all $\mathrm{Add}(\gamma, \theta)$, then it must have been strongly unfoldable in V. Since we aim to produce universal indestructibility with one strongly unfoldable cardinal, there will be many such indestructible $(\gamma + 2)$-strongly unfoldable cardinals γ lower down, and we must consequently begin with many strongly unfoldable cardinals in V. This phenomenon exactly parallels the situation in [1] with supercompactness.

The hypothesis we use is a strong unfoldability analogue of the Mitchell rank, inspired by the similar hypothesis in the case of supercompactness and strongness introduced by Sargsyan (see [3]) to improve on that in [1]. This definition proceeds inductively on α, simultaneously in all transitive ZFC$^-$ models containing the cardinal κ and the ordinal α.

Definition 6.3 An inaccessible cardinal κ is *strongly unfoldable of degree* α, for an ordinal α, if for every ordinal θ it is θ*-strongly unfoldable of degree* α, meaning that for every $A \subseteq \kappa$ there is a transitive set $M \models$ ZFC$^-$ of size κ with $\kappa, A \in M$ and a transitive set N with $\alpha \in N$ and an embedding $j : M \to N$ having critical point κ with $j(\kappa) > \max\{\theta, \alpha\}$ and $V_\theta \subseteq N$ such that κ is strongly unfoldable in N of every degree $\beta < \alpha$. We say that κ is $<\theta$-strongly unfoldable of degree α, if κ is θ'-strongly unfoldable of degree α for every $\theta' < \theta$.

By using an induced factor embedding, if necessary, we may assume that the witnessing embedding $j : M \to N$ in Definition 6.3 has hereditary size $\max\{\beth_\theta, \alpha, \kappa\}$. It follows that whenever $\kappa, \alpha < \theta$ and κ is $<\theta$-strongly unfoldable of degree α, then any transitive set containing $V_\theta \cup \{\theta\}$ will see this. Moreover, induction on α shows that if N is any transitive set such that κ is θ-strongly unfoldable of degree α in N and $P(\kappa) \cup V_\theta \subseteq N$, then κ really is θ-strongly unfoldable of degree α. Note that κ is θ-strongly unfoldable of degree 0 if and only if it is θ-strongly unfoldable. Thus, a cardinal κ is κ-strongly unfoldable of degree 0 exactly if it is weakly compact, and κ is strongly unfoldable of degree 0 exactly if it is strongly unfoldable. If κ is θ-strongly unfoldable of degree α, then it is also θ-strongly unfoldable of every degree α' less than α. Lastly, note that we could have equivalently replaced in Definition 6.3 the requirement that κ is inaccessible by merely requiring that κ is a beth fixed point, since even if $\theta = \alpha = 0$ it is not difficult to show that the embedding property itself already implies that κ is an uncountable regular cardinal; combined with $\beth_\kappa = \kappa$ it thus follows that κ is inaccessible.

An inaccessible cardinal κ is Σ_2*-reflecting* if $V_\kappa \prec_{\Sigma_2} V$. Since the Σ_2 statements of set theory are characterized up to equivalence as those of the form "$\exists\alpha\ V_\alpha \models \psi$," for any first-order assertion ψ (not necessarily Σ_2), it follows that any such ψ true in any V_α, with parameters from V_κ, is true in some V_α for $\alpha < \kappa$. A standard reflection argument, just like that for strong cardinals, shows that every strongly unfoldable cardinal is Σ_2-reflecting.

Lemma 6.4 *Suppose that κ is Σ_2-reflecting and $\gamma < \kappa$.*

(1) *If γ is θ-strongly unfoldable of every degree $\beta < \kappa$, then γ is θ-strongly unfoldable of every degree $\beta \in$* ORD.
(2) *If γ is $<\kappa$-strongly unfoldable of degree α, then γ is strongly unfoldable of degree α.*

Consequently, if γ is $<\kappa$-strongly unfoldable of degree β for every $\beta < \kappa$, then γ is strongly unfoldable of degree β for all $\beta \in$ ORD.

Proof The question whether γ is θ-strongly unfoldable of degree β can be answered in V_α, for any α above θ, γ and β. Consequently, any failure of this for either θ or β above κ reflects below κ by Σ_2-reflection. □

Lemma 6.5 *The following are equivalent, for any weakly compact cardinal κ and any ordinals α, θ.*

(1) *The cardinal κ is θ-strongly unfoldable of degree α.*

(2) *For every κ-model M in which κ is Σ_2-reflecting, there is a transitive set N and an embedding $j : M \to N$ having critical point κ, with $j(\kappa) > \max\{\theta, \alpha\}$ and $V_\theta \subseteq N$ such that κ is strongly unfoldable in N of every degree $\beta < \alpha$.*

(3) *For every $A \in H_{\kappa^+}$ there is a κ-model M with $A \in M$, a transitive set N, and an embedding $j : M \to N$ having critical point κ, with $j(\kappa) > \max\{\theta, \alpha\}$ and $V_\theta \subseteq N$ such that κ is strongly unfoldable in N of every degree $\beta < \alpha$.*

Proof (1 implies 2) Fix any κ-model M in which κ is Σ_2-reflecting. Code M with a set $A \subseteq \kappa$ and fix by (1) a transitive set $\overline{M} \models \mathrm{ZFC}^-$ with $\kappa, A \in \overline{M}$ and an embedding $j : \overline{M} \to \overline{N}$ having critical point κ, with $\theta, \alpha < j(\kappa)$ and $V_\theta \subseteq \overline{N}$ such that κ is strongly unfoldable in $\overline{N}$ of every degree $\beta < \alpha$. In particular, $\overline{N}$ thinks that κ is ${<}j(\kappa)$-strongly unfoldable of every degree $\beta < \alpha$. We claim that $j \restriction M : M \to j(M)$ will witness (2). Since M is a κ-model in $\overline{M}$, it follows by elementarity that $j(M)^{<j(\kappa)} \subseteq j(M)$ in $\overline{N}$, and consequently $j(M)$ and $\overline{N}$ agree up to rank $j(\kappa)$. Since $\alpha < j(\kappa)$, it follows that $j(M)$ agrees that κ is ${<}j(\kappa)$-strongly unfoldable of every degree $\beta < \alpha$, since this is verifiable in $\overline{N}_{j(\kappa)} = j(M)_{j(\kappa)}$. Since $j(\kappa)$ is Σ_2-reflecting in $j(M)$, it follows by Lemma 6.4 that κ is strongly unfoldable in $j(M)$ of every degree $\beta < \alpha$, and so $j \restriction M$ witnesses (2).

(2 implies 3) Fix any $A \in H_{\kappa^+}$, and suppose that $A' \subseteq \kappa$ codes the set A. It is easy to find a κ-model $\overline{M}$ such that $A' \in \overline{M}$. Since κ is weakly compact, there is an embedding $j : \overline{M} \to \overline{N}$ with critical point κ. By using an induced factor embedding, if necessary, we may assume that $\overline{N}$ is a κ-model also. The rank initial segment $M = \overline{N}_{j(\kappa)}$ is thus a κ-model with $A' \in M$ and $V_\kappa \prec M$. It follows that $A \in M$ and that κ is Σ_2-reflecting in M. Statement (2) now provides the desired embedding.

(3 implies 1) Immediate. □

Assertion (2) can be strengthened without loss of generality to assert also that $j : M \to N$ is a Hauser embedding, meaning that $M, j \in N$ and, moreover, have size κ in N. This is because the Hauser argument (see [4, Lemma 5]) shows that the restricted embedding $j \restriction M$ of the proof necessarily has this Hauser property. Assertion (3) can be strengthened to assert that M is a κ-model in which κ is Σ_2-reflecting, since this is what the argument provides, and also that the embedding j has the Hauser property.

Just as unfoldability and strong unfoldability are absolute to L, we now show this for the higher degree analogues.

Theorem 6.6 *If κ is strongly unfoldable of degree α, then it is strongly unfoldable of degree α in L.*

Proof The argument proceeds by induction on α, simultaneously in all transitive models of set theory containing κ and α. (That is, we prove that for any transitive model $W \models \mathrm{ZFC}$, if κ is θ-strongly unfoldable of degree α in W, then it is θ-strongly unfoldable of degree α in L^W.) Suppose that κ is θ-strongly unfoldable of degree α in V and that $A \subseteq \kappa$ is in L. Since κ is weakly compact in L, we may find as in the proof of Lemma 6.5 in L a κ-model M such that $A \in M$ and κ is Σ_2-reflecting in M. Since M has size κ in L, we may find an ordinal $\alpha < \kappa^+$ such that L_α sees that M has size κ. By Lemma 6.5, there is a κ-model $\overline{M}$ with $L_\alpha \in \overline{M}$, a transitive set $\overline{N}$, and an embedding $j : \overline{M} \to \overline{N}$ having critical point κ, with $\max\{\alpha, \theta\} < j(\kappa)$ and $V_\theta \subseteq \overline{N}$, such that κ is strongly unfoldable in $\overline{N}$ of every degree $\beta < \alpha$. It follows that $M \in \overline{M}$, and, moreover, that $M \in L^{\overline{M}}$ and of size κ there. Consider the map $j \restriction M : M \to j(M)$. As $M^{<\kappa} \cap L \subseteq M$, it follows by elementarity that $j(M)$ is highly closed under sequences existing in $L^{\overline{N}}$ and, therefore, that $(V_\theta)^L = (V_\theta)^{L^{\overline{N}}} \subseteq j(M)$. By elementarity, $j(M) \in L^{\overline{N}}$ and hence is in L. Since $\overline{M}$ knows that M has size κ in L, there is a relation $E \subseteq \kappa \times \kappa$ with $E \in L^{\overline{M}}$ such that $\langle \kappa, E\rangle \cong \langle M, \in\rangle$. It follows that if $x \in M$ is coded by ζ with respect to E, then $j(x)$ is coded by $j(\zeta) = \zeta$ with respect to $j(E)$. Thus, $j \restriction M$ is constructible from E and $j(E)$, which are both in L, and consequently $j \restriction M$ is in L. By induction, since κ is strongly unfoldable in $\overline{N}$ of every degree $\beta < \alpha$, it follows that κ is strongly unfoldable in $L^{\overline{N}}$ of every degree $\beta < \alpha$. In particular, κ is ${<}j(\kappa)$-strongly unfoldable in $L^{\overline{N}}$ of every degree $\beta < \alpha$. Since $L^{\overline{N}}$ agrees with $j(M)$ up to $j(\kappa)$, we conclude that κ is ${<}j(\kappa)$-strongly unfoldable of every degree $\beta < \alpha$ in $j(M)$. Since $j(\kappa)$ is Σ_2-reflecting in $j(M)$, it follows by Lemma 6.4 that κ is fully strongly unfoldable in $j(M)$ of every degree $\beta < \alpha$. In conclusion, $j \restriction M : M \to j(M)$ witnesses the desired property in L. $\square$

The argument establishes that if κ is θ-strongly unfoldable of degree α, then κ is θ-strongly unfoldable of degree α in L. If $\alpha = 0$, this simply asserts that θ-strong unfoldability is downward absolute to L. We now provide a natural upper bound for the consistency strength of being strongly unfoldable of every ordinal degree.

Theorem 6.7 *If $0^\sharp$ exists, then every Silver indiscernible is strongly unfoldable in L of every ordinal degree α, and a limit of such cardinals, and so on.*

Proof Suppose that $0^\sharp$ exists and that κ is a Silver indiscernible of L. We will prove that κ is strongly unfoldable in L of every ordinal degree α. Classical arguments show that κ is Σ_2-reflecting (and much more) in L. Fix α and any ordinal θ. Let $j : L \to L$ be an elementary embedding with critical point κ and $j(\kappa) > \max\{\theta, \alpha\}$. For any $A \subseteq \kappa$ in L, we may find in L a κ-model M_0 with $A \in M_0$ such that κ is Σ_2-reflecting in M_0. Let $j_0 = j \restriction M_0 : M_0 \to j(M_0)$. Since $j(\kappa)$ is inaccessible in L, it follows that $L_{\beth_\theta^L} \subseteq j(M_0)$ and that $(V_\theta)^L \subseteq j(M_0)$. By induction, κ is strongly unfoldable in L of every degree $\beta < \alpha$. Since $j(M_0)$ and L agree up to $j(\kappa)$, it follows that κ is ${<}j(\kappa)$-strongly unfoldable in $j(M_0)$ of every degree $\beta < \alpha$. By elementarity, $j(\kappa)$ is Σ_2-reflecting in $j(M_0)$, and so by Lemma 6.4 it follows that κ is strongly unfoldable in $j(M_0)$ of every degree $\beta < \alpha$. Finally, since M_0 has size κ in L, there is a relation $E \subseteq \kappa \times \kappa$ in L such that $\langle \kappa, E\rangle \cong \langle M_0, \in\rangle$. As in Theorem 6.6, the map $j_0 = j \restriction M_0$ is constructible from E and $j(E)$ and consequently is in L. In conclusion, the map $j_0 : M_0 \to j(M_0)$ witnesses for A that κ is θ-strongly unfoldable in L of degree α, as desired. The usual reflection arguments now show that κ must also be a limit of such cardinals, and a limit of limits of such cardinals, and so on. $\square$

In particular, every measurable cardinal is strongly unfoldable in L of every ordinal degree α. For another upper bound, recall that a cardinal κ is *subtle* if for any closed unbounded set $C \subseteq \kappa$ and any sequence $\langle A_\alpha \mid \alpha \in C\rangle$ with $A_\alpha \subseteq \alpha$, there is a pair of ordinals $\alpha < \beta$ in C with $A_\alpha = A_\beta \cap \alpha$. It is not difficult to see that every subtle cardinal is necessarily an uncountable regular strong limit cardinal, and consequently inaccessible.

Theorem 6.8 *If κ is subtle, then the set of cardinals γ below κ that are strongly unfoldable in V_κ of every ordinal degree below κ is stationary. In particular, V_κ is a model of set theory having a stationary proper class of cardinals that are strongly unfoldable of every ordinal degree.*

Proof Suppose that κ is subtle and the set of cardinals below κ that are strongly unfoldable in V_κ of every ordinal degree below κ is not stationary. Then there is a closed unbounded set $C \subseteq \kappa$ containing no such cardinals. Since κ is inaccessible, we may assume that all elements of C are beth fixed points and that $V_\gamma \prec V_\kappa$ for every $\gamma \in C$. If $\gamma \in C$, then there is a minimal $\beta_\gamma < \kappa$ for which there is a minimal $\theta_\gamma < \kappa$ such that γ is not θ_γ-strongly unfoldable in V_κ of degree β_γ. Since we can equivalently replace inaccessibility by being a beth fixed point in Definition 6.3 (see there and the remarks after it), this means that there exists some $A_\gamma \subseteq \gamma$ having no transitive set $M \models \mathrm{ZFC}^-$ of size γ with $\gamma, A_\gamma \in M$ with a corresponding embedding $j : M \to N$ with $\mathrm{cp}(j) = \gamma$, $V_{\theta_\gamma} \subseteq N$, $j(\gamma) > \theta_\gamma, \beta_\gamma$ and $N \models \gamma$ is strongly unfoldable of every degree below β_γ. By thinning the club C, we may assume that θ_γ and β_γ are both less than the next element of C.

A simple Löwenheim-Skolem argument in V_κ thus shows that we may find for each $\gamma \in C$ a transitive set, call it M_γ, such that $M_\gamma \models \mathrm{ZFC}^-$ of size γ with $\gamma, A_\gamma \in M_\gamma$ and $V_\gamma \subseteq M_\gamma$ such that $M_\gamma \prec V_\kappa$. Note that we cannot insist that M_γ is a γ-model, since elements of C need not satisfy $\gamma^{<\gamma} = \gamma$. Since M_γ has size γ, we may code it with a relation E_γ on γ, so that $\langle \gamma, E_\gamma\rangle \cong \langle M_\gamma, \in\rangle$. The isomorphism π_γ witnessing this is exactly the Mostowski collapse of $\langle \gamma, E_\gamma\rangle$. We may assume that $\pi_\gamma(0) = \gamma$. Let D_γ be a subset of γ coding, in some canonical way, the relation E_γ, the elementary diagram of $\langle \gamma, E_\gamma\rangle$ and the map $\pi_\gamma^{-1} \upharpoonright \gamma$, which maps γ into γ.

Since κ is subtle, there must be a pair $\gamma < \delta$ in C with $D_\gamma = D_\delta \cap \gamma$. Define a map $j : M_\gamma \to M_\delta$ by $j = \pi_\delta \circ \pi_\gamma^{-1}$. Observe that $j(\gamma) = \pi_\delta(\pi_\gamma^{-1}(\gamma)) = \pi_\delta(0) = \delta$. Also, if $\alpha < \gamma$, then because D_γ and D_δ agree up to γ, it follows that $\pi_\gamma^{-1} \upharpoonright \gamma = \pi_\delta^{-1} \upharpoonright \gamma$, and so $j(\alpha) = \alpha$. Thus, the critical point of j is γ. The map j is elementary, because if $M_\gamma \models \varphi[x]$ where $x = \pi_\gamma(\alpha)$, then $\varphi(\alpha)$ is in the elementary diagram of $\langle \gamma, E_\gamma\rangle$, and so it is also in the elementary diagram of $\langle \delta, E_\delta\rangle$, which means $M_\delta \models \varphi[j(x)]$. Note that since γ and δ are both in C, we know that δ is larger than θ_γ and β_γ. It follows that $V_{\theta_\gamma} \subseteq V_\delta \subseteq M_\delta$. Since $M_\delta \prec V_\kappa$, it follows that M_δ agrees that β_γ is least such that γ is not strongly unfoldable of degree β_γ. In particular, γ is strongly unfoldable in M_δ for every degree below β_γ. The embedding $j : M_\gamma \to M_\delta$ therefore contradicts our choice of β_γ, θ_γ and A_γ. So the theorem is proved. □

Next, our Main Theorem in this section provides the exact consistency strength of universal indestructibility for strong unfoldability.

Theorem 6.9 *The following theories are equiconsistent over* ZFC*:*

(1) *There is a strongly unfoldable cardinal and universal indestructibility holds for strong unfoldability.*
(2) *There is a cardinal that is strongly unfoldable of every ordinal degree.*

We prove each direction separately.

Lemma 6.10 *Suppose there is universal indestructibility for strong unfoldability. Then every strongly unfoldable cardinal in V is strongly unfoldable in L of every ordinal degree. More specifically, if κ is any $(\kappa+1+\alpha)$-strongly unfoldable cardinal, then κ is strongly unfoldable in L of degree α.*

Proof We prove the lemma by induction on α, simultaneously in all transitive ZFC$^-$ models containing κ and α. (That is, we prove that in any transitive $W \models$ ZFC$^-$ with universal indestructibility, if κ is $(\kappa + 1 + \alpha)$-strongly unfoldable in W, then it is strongly unfoldable in L^W of degree α.) Suppose that there is universal indestructibility for strong unfoldability in V and that κ is $(\kappa + 1 + \alpha)$-strongly unfoldable for some ordinal α. Fix any $A \subseteq \kappa$ in L and any ordinal θ above κ and α. Since κ is weakly compact in L, we may find as in the proof of Lemma 6.5 in L a κ-model M such that $A \in M$ and κ is Σ_2-reflecting in M. Let $\theta^* = \beth_\theta^L$ and suppose that $G \subseteq \mathrm{Add}(\kappa, \theta^*)$ is V-generic for the forcing to add θ^* many subsets to κ. By universal indestructibility, κ remains $(\kappa+1+\alpha)$-strongly unfoldable in $V[G]$. In $V[G]$, find a κ-model $\overline{M}$ with $M \in L^{\overline{M}}$ of size κ there and an embedding $j : \overline{M} \to \overline{N}$ with $V[G]_{\kappa+1+\alpha} \subseteq \overline{N}$. Since $P(\kappa)^{V[G]}$ has size at least the size of the ordinal θ^*, it follows that $j(\kappa) > \theta^*$ and consequently $(V_\theta)^L \subseteq \overline{N}$. The restricted embedding $j \restriction M : M \to j(M)$ satisfies $(V_\theta)^L \subseteq j(M)$ and exists in L, just as in Theorem 6.6. Since the set A and the ordinal θ were chosen arbitrarily, this proves the lemma for the case $\alpha = 0$. (Alternatively, we could have used Theorem 6.6 directly to see that the lemma holds for the case $\alpha = 0$.) For the general case, note that κ is $<(\kappa+1+\alpha)$-strongly unfoldable in $\overline{N}$, since each instance is witnessed by embeddings coded in $V_{\kappa+1+\alpha}$. Since universal indestructibility holds in V, it also holds in $V_\kappa = \overline{M}_\kappa$, and consequently in $j(\overline{M}_\kappa) = \overline{N}_{j(\kappa)}$. By induction, therefore, κ is strongly unfoldable in $L^{\overline{N}_{j(\kappa)}}$ of every degree $\beta < \alpha$. Since $j(M)$ agrees with $L^{\overline{N}}$ up to $j(\kappa)$, it follows that κ is $<j(\kappa)$-strongly unfoldable in $j(M)$ of every degree $\beta < \alpha$. Since $j(\kappa)$ is Σ_2-reflecting in $j(M)$, this implies by Lemma 6.4 that κ is strongly unfoldable in $j(M)$ of every degree $\beta < \alpha$. The restricted embedding $j \restriction M$ exists in L and witnesses there for the set A that κ is θ-strongly unfoldable of degree α. Since A and θ were chosen arbitrarily, this shows that κ is strongly unfoldable of degree α in L, as desired. □

This establishes the forward direction of Theorem 6.9. We turn now to the converse direction.

Lemma 6.11 *If κ is strongly unfoldable of every ordinal degree, then there is a (possibly proper class) model of* ZFC *having a strongly unfoldable cardinal and universal indestructibility for strong unfoldability.*

Proof Suppose κ is strongly unfoldable of every ordinal degree. We will perform the trial-by-fire forcing for strong unfoldability, adapting [1]. This is the Easton support iteration $\mathbb{P}_\kappa$ of length at most κ, which at each stage γ attempts to destroy as much of the strong unfoldability of γ as is possible. We will have nontrivial forcing at stage γ only when γ is at least weakly compact in $V[G_\gamma]$. If it happens at stage

γ that γ is strongly unfoldable in $V[G_\gamma]$ and indestructible by all $<\gamma$-closed γ^+-preserving forcing, then we declare success and output $V[G_\gamma]$ as our final model. Similarly, if there is some ordinal $\lambda > \gamma$ such that $V_\lambda[G_\gamma] \models$ ZFC and in this model γ is strongly unfoldable and indestructible, then we also declare success and output $V_\lambda[G_\gamma]$ as our final model. If we have not declared success at stage γ, then there is some least $\theta \geq \gamma$, necessarily below the next inaccessible cardinal (and therefore also below κ), for which there is some $<\gamma$-closed γ^+-preserving forcing $\mathbb{Q}$ in $V[G_\gamma]$, such that after forcing with $\mathbb{Q}$ the cardinal γ is not θ-strongly unfoldable. We may furthermore assume, by further collapsing of cardinals to $(\beth_\theta)^+$, that $\mathbb{Q}$ is *tidy* in the sense that after forcing with $\mathbb{Q}$ there are also no inaccessible cardinals in the half-open interval $(\theta, |\mathbb{Q}|^{V[G_\gamma]}]$. (This will ensure that the next stage of nontrivial forcing is beyond $|\mathbb{Q}|^{V[G_\gamma]}$ and that all the relevant cardinals in the final extension are treated as a stage of forcing.) Since κ is Σ_2-reflecting, it follows that if there is any such $\mathbb{Q}$, then there is such a $\mathbb{Q}$ of rank less than κ. The stage γ forcing is the lottery sum of all such $<\gamma$-closed γ^+-preserving tidy posets $\mathbb{Q}$, of minimal rank, that work with this minimal θ. It follows inductively that $\mathbb{P}_\gamma$ for $\gamma < \kappa$ is small relative to κ and therefore preserves the strong unfoldability of κ. The key trial-by-fire observation of [1] is that survivors of the firestorm are certifiably fireproof. That is, because the stage γ forcing destroys as much of the strong unfoldability of γ as possible, any surviving degree of strong unfoldability could only have survived because it was indestructible. More specifically, by the minimality of θ, in the case when $\gamma < \theta$, then we know that γ is $<\theta$-strongly unfoldable in $V[G_\gamma]$, and this degree of strong unfoldability is indestructible over $V[G_\gamma]$ by any further $<\gamma$-closed γ^+-preserving forcing, since otherwise we would have destroyed it at stage γ. Since the rest of the forcing after stage γ is $\leq\gamma$-closed, the $<\theta$-strong unfoldability of γ is thus preserved to the final output model, and it is also indestructible there. Moreover, the next stage of forcing after γ occurs at an inaccessible cardinal above θ, and therefore the rest of the forcing after stage γ is $\leq\beth_\theta$-closed and consequently does not turn on the θ-strong unfoldability of γ. In the case when $\gamma \geq \theta$, the forcing at stage γ destroys the weak compactness of γ, and the rest of the forcing after stage γ is sufficiently closed so that γ remains not weakly compact. It follows that if we ever declare success, then the output model will satisfy universal indestructibility for strong unfoldability. In this way, the trial-by-fire iteration systematically ensures universal indestructibility as it proceeds. What remains, of course, is for us to prove that something does in fact survive the iteration, and that we do eventually declare success.

Claim If κ is strongly unfoldable of degree α, then the trial-by-fire forcing $\mathbb{P}_\kappa$ either declares success by stage κ or forces that κ is $(\kappa+\alpha)$-strongly unfoldable and indestructible by all $<\kappa$-closed, κ^+-preserving forcing.

Proof We prove the claim by induction on α, simultaneously for all models of ZFC$^-$ containing κ and α. Suppose that κ is strongly unfoldable of degree α and that $\mathbb{P}_\kappa$ did not declare success at or before stage κ. It follows by induction that α is less than the next inaccessible cardinal above κ, if any, since otherwise we would have declared success. Suppose $G \subseteq \mathbb{P}_\kappa$ is V-generic for the trial-by-fire iteration up to stage κ. The usual Easton support arguments show that $\mathbb{P}_\kappa$ is κ-c.c. and κ is inaccessible in $V[G]$. In fact, it follows from the induction hypothesis that κ is indestructibly $(\kappa+\beta)$-strongly unfoldable in $V[G]$ for all $\beta < \alpha$. We now show that κ is $(\kappa+\alpha)$-strongly unfoldable and indestructible in $V[G]$. If not, there is a $<\kappa$-closed, κ^+-preserving tidy poset $\mathbb{Q} \in V[G]$, of minimal rank, destroying the $(\kappa+\alpha)$-strong unfoldability of κ. Note that forcing with $\mathbb{Q}$ over $V[G]$ destroys as much of the strong unfoldability of κ as is possible. Let $g \subseteq \mathbb{Q}$ be $V[G]$-generic and $A \subseteq \kappa$ any subset of κ in $V[G][g]$. Fix names $\dot{\mathbb{Q}}$ and $\dot{A}$ in V and choose some large $\lambda > \kappa+\alpha$ and some $\overline{X} \prec V_\lambda[G][g]$ of size κ with $\overline{X}^{<\kappa} \subseteq \overline{X}$ in $V[G][g]$ and $\mathbb{P}_\kappa * \dot{\mathbb{Q}}, G * g, \kappa, \alpha, \dot{A}$ all in $\overline{X}$. Necessarily, $\overline{X} = X[G][g]$, where $X = \overline{X} \cap V$. As in the Main Theorem, we make the key observation by Lemma 2.5 that $X \in V$, that X has size κ in V, and that the filter $G * g$ is X-generic for $\mathbb{P}_\kappa * \dot{\mathbb{Q}}$. Moreover, if $\pi : X[G][g] \to \overline{M}$ is the Mostowski collapse of $\overline{X}$, then π is the identity on $\mathbb{P}_\kappa$ and thus on G. If we let $\mathbb{Q}_0 = \pi(\mathbb{Q})$ and $g_0 = \pi(g) = \pi\,"g$, it follows that $\pi(A) = A$ is an element of $\overline{M} = M[G][g_0]$ where M is the Mostowski collapse of X and $G * g_0$ is M-generic for $\mathbb{P}_\kappa * \mathbb{Q}_0$. Since X has size κ in V and is sufficiently closed, it follows that M is a κ-model in V.

Fix any ordinal $\theta > \beth_{\kappa+\alpha}$ large enough so that $\dot{\mathbb{Q}} \in V_\theta$. Since κ is strongly unfoldable of degree α in V, there is an embedding $j : M \to N$ with critical point κ, having $\theta < j(\kappa)$ and $V_{\theta+1} \subseteq N$ such that κ is strongly unfoldable in N of every degree $\beta < \alpha$. By the proof of Lemma 2.2 modified to our context, we may assume without loss of generality that $N^{\beth_\theta} \subseteq N$. As in the Main Theorem, we shall lift the embedding $j : M \to N$ in two steps, first to the extension $j : M[G] \to N[j(G)]$, and then fully to $j : M[G][g_0] \to N[j(G)][j(g_0)]$. This final embedding, we shall argue, will witness the $(\kappa+\alpha)$-strong unfoldability of κ with respect to A in $V[G][g]$.

Since we have not declared success in V by stage κ, it follows that we do not declare success in N before stage $j(\kappa)$. The key step of this argument is now that because κ is strongly unfoldable in N of every degree $\beta < \alpha$, it follows by induction that κ is indestructibly $(\kappa+\beta)$-strongly unfoldable in $N[G]$ for all $\beta < \alpha$. And furthermore, since $N[G]$ agrees with $V[G]$ beyond θ, it sees that $\mathbb{Q}$ destroys the $(\kappa+\alpha)$-strong unfoldability of κ. Thus, $N[G]$ agrees that $\mathbb{Q}$ destroys as much of the strong unfoldability of κ as is possible. Since $\mathbb{Q}$ also has minimal rank and remains tidy, $<\kappa$-closed and κ^+-preserving in $N[G]$, it follows that $\mathbb{Q}$ appears in the stage κ lottery of the trial-by-fire iteration $j(\mathbb{P}_\kappa)$. Below a condition opting for $\mathbb{Q}$ in this lottery, we may factor $j(\mathbb{P}_\kappa)$ as $\mathbb{P}_\kappa * \mathbb{Q} * \mathbb{P}_{\text{tail}}$, where $\mathbb{P}_{\text{tail}}$ is the forcing beyond stage κ up to $j(\kappa)$. Force to add a $V[G][g]$-generic filter $G_{\text{tail}} \subseteq \mathbb{P}_{\text{tail}}$, and lift the embedding to $j : M[G] \to N[j(G)]$ in $V[G][g][G_{\text{tail}}]$, where $j(G) = G * g * G_{\text{tail}}$. Since $\mathbb{P}_\kappa$ is κ-c.c. and $\mathbb{Q}$ is $<\kappa$-closed, it follows that $M[G][g_0]^{<\kappa} \subseteq M[G][g_0]$ in $V[G][g]$. Consequently, as in the Main Theorem, $V[G][g]$ agrees with $M[G][g_0]$ when it thinks that g_0 is a $<\kappa$-closed subset of $\mathbb{Q}_0$. Since g_0 has size κ in $V[G][g]$ and is directed, there is again in $V[G][g]$ a descending, cofinal κ-sequence of conditions generating g_0. Consequently, since $N[j(G)]$ is closed under κ-sequences in $V[G][g][G_{\text{tail}}]$ and $j(\mathbb{Q}_0)$ is $<j(\kappa)$-closed there, we can find a master condition $p^* \in j(\mathbb{Q}_0)$ which lies below every element of $j\,"\,g_0$. Force to add a $V[G][g][G_{\text{tail}}]$-generic filter $g^* \subseteq j(\mathbb{Q}_0)$ containing the condition p^*, and lift the embedding to $j : M[G][g_0] \to N[j(G)][j(g_0)]$ in $V[G][g][G_{\text{tail}}][g^*]$, where $j(g_0) = g^*$. Note that $V[G][g]_\theta = V_\theta[G][g] \subseteq N[G][g]$. Let $\delta = (\beth_{\kappa+\alpha})^{V[G][g]} = (\beth_{\kappa+\alpha})^{N[G][g]}$ and let $j_0 : M[G][g_0] \to N_0$ be the induced extender embedding in $V[G][g][G_{\text{tail}}][g^*]$ with $V_{\kappa+\alpha}^{V[G][g]} \subseteq N_0$, of hereditary size δ. Since $\delta < \theta$ and there are no inaccessible cardinals in the interval $(\kappa, \kappa+\alpha]$, the next nontrivial stage of forcing in $\mathbb{P}_{\text{tail}}$ lies beyond δ. Consequently, $\mathbb{P}_{\text{tail}} * j(\mathbb{Q}_0)$ is $\leq\delta$-closed in $N[j(G)]$ and hence also in $V[G][g][G_{\text{tail}}]$. Thus, the extra forcing with $\mathbb{P}_{\text{tail}} * j(\mathbb{Q}_0)$ could not have added the extender embedding j_0, which must therefore already exist in $V[G][g]$, where it witnesses the $(\kappa+\alpha)$-strong unfoldability of κ for A. Thus, κ is $(\kappa+\alpha)$-strongly unfoldable in $V[G][g]$, contrary to our assumption, and the claim is proved. $\square$

We now complete the proof of Lemma 6.11, which establishes the reverse direction of Theorem 6.9. If κ is strongly unfoldable of every degree α, then by the claim we have either declared success before stage κ, or else κ becomes indestructibly strongly unfoldable in $V[G_\kappa]$, in which case we declare success at stage κ. In any case, therefore, we have produced the desired model. $\square$

We conclude the paper by remarking that if one wants to obtain universal indestructibility for strong unfoldability without the limitation to γ^+-preserving forcing, then one should carry out the corresponding trial-by-fire iteration, which at each stage γ performs the lottery sum of all $<\gamma$-closed forcing (not necessarily preserving γ^+), of minimal rank, destroying as much as possible of the strong unfoldability of γ. If κ is supercompact of every ordinal degree (using the natural analogue of our notion), then this iteration will declare success at or before stage κ.

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Acknowledgments

The research of the first author has been supported in part by grants from the CUNY Research Foundation, the Netherlands Organization for Scientific Research (NWO), and the National Science Foundation. He is grateful to the Institute of Logic, Language and Computation at Universiteit van Amsterdam for the support of a Visiting Professorship during his 2007 sabbatical there. Final revisions of this article were made during the second author's 2009–10 visit to the Kurt Gödel Research Center at the University of Vienna, supported in part by the Austrian Science Fund through grant P20835-N13, the European Science Foundation through the INFTY Research Networking Programme, and a CUNY Scholar Incentive Award. Parts of this article are adapted from the second chapter of the second author's Ph.D. dissertation [14].

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