Inclosures, Vagueness, and Self-Reference

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Abstract   In this paper, I start by showing that sorites paradoxes are inclosure paradoxes. That is, they fit the Inclosure Scheme which characterizes the paradoxes of self-reference. Given that sorites and self-referential paradoxes are of the same kind, they should have the same kind of solution. The rest of the paper investigates what a dialetheic solution to sorites paradoxes is like, connections with a dialetheic solution to the self-referential paradoxes, and related issues—especially so called "higher order" vagueness.

1   Introduction: Vagueness and Self-Reference

Sorites paradoxes and the paradoxes of self-reference are quite different kinds of creature. The first are generated by the fact that some predicates have a certain kind of tolerance to small changes in their range of application. The second are generated by the fact that some things can refer, directly or indirectly, to themselves. True, in the last twenty years some logicians have tried to develop a unified approach to the two kinds of paradox.[1] The motivation is essentially that some kind of indeterminacy is involved in both phenomena (borderline cases, ungroundedness). This is hardly sufficient for putting the two kinds of paradox in the same class, though. Aristotle argued, famously (*De Int.*, Chapter 9), that certain paradoxes of fatalism are best avoided by a kind of indeterminacy concerning future statements. But, even if he is right, few would think that this puts these paradoxes in the same class as sorites or self-referential paradoxes. The absence of future truth-makers seems to be a phenomenon sui generis. So it is with the paradoxes of sorites and self-reference themselves. The mechanisms involved in the generation of contradiction, tolerance and self-reference, would seem to make them quite different kinds of paradox.

Or so it seemed to me until recently. I am now inclined to think differently. The paradoxes of self-reference can naturally be seen as having a form given by the

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Inclosure Schema. In the Schema, a construction is applied to collections of a certain
kind to produce a different object of the same kind. Contradiction arises at the limit
of all things of that kind. Sorites paradoxes can be seen as having exactly the same
form. In this paper, I will start by explaining how. Given that paradoxes of sorites
and self-reference are of the same kind, they should have the same kind of solution.
I hold that a dialetheic solution is the correct one for paradoxes of self-reference. It
follows that a dialetheic solution is therefore appropriate for sorites paradoxes. The
rest of the paper investigates what such a solution is like, and explores related issues,
especially so-called higher order vagueness.

2 The Inclosure Schema

Let us start with the Inclosure Schema and its application to the paradoxes of self-
reference.[2] An inclosure paradox arises when for some monadic predicates φ and θ,
and a one place function δ, there are principles which appear to be true, or a priori
true,[3] and which entail the following conditions. (It is not required, note, that the
arguments entailing the conditions be sound, though dialetheism prominently allows
for this possibility.)

1. There is a set Ω such that $\Omega = \{x : \varphi(x)\}$, and $\theta(\Omega)$ (*Existence*)
2. If $X \subseteq \Omega$ and $\theta(X)$,
    (a) $\delta(X) \notin X$ (*Transcendence*)
    (b) $\delta(X) \in \Omega$ (*Closure*)

(A special case of an inclosure is when $\theta(X)$ is the vacuous condition, $X = X$, and
so mention of it may be dropped.) Given these conditions, a contradiction occurs at
the limit when $X = \Omega$. For then we have $\delta(\Omega) \notin \Omega \land \delta(\Omega) \in \Omega$.

   To illustrate: In the Burali-Forti paradox, $\varphi(x)$ is 'x is an ordinal', so that Ω is the
set of all ordinals, *On*—defined, let us assume, as von Neumann ordinals. $\theta(X)$ is the
vacuous condition, and $\delta(X)$ is the least ordinal greater than every member of X. By
definition, $\delta(X)$ satisfies Transcendence and Closure. The brunt of the Burali-Forti
paradox is exactly in showing that $\delta(X)$ is well defined, even when $X = \Omega$. The
reasoning shows that *On* is itself an ordinal—an ordinal greater than all ordinals.

   In the liar paradox, $\varphi(x)$ is the predicate Tx, 'x is true', so that Ω is the set of true
sentences. $\theta(X)$ is the predicate 'X is definable', that is, is a set that is referred to by
some name. If X is definable, let N be an appropriate name; then $\delta(X)$ is a sentence,
σ, constructed by an appropriate self-referential construction of the form $\langle \sigma \notin N \rangle$.
(I use angle brackets as a name-forming device.) Liar-type reasoning establishes
Transcendence and Closure. The liar paradox arises in the limit. $\Omega = \{x : Tx\}$, and
$\delta(\Omega)$ is a sentence σ of the form $\langle \sigma \notin \{x : Tx\} \rangle$; that is, '$\sigma$ is not true'.

3 Sorites and Inclosures

Let us now see how sorites paradoxes fit the Schema. In a sorites paradox there is a
sequence of objects, $a_0, \ldots, a_n$, and a vague predicate P such that Pa_0 and $\neg Pa_n$;
but for successive members of the sequence there is very little difference between
them with respect to their P-ness, so that if one satisfies P, so does the other—the
principle of tolerance.

   For the Inclosure Schema, let $\varphi(x)$ be Px, so $\Omega = \{x : Px\}$; $\theta(X)$ is the vacuous
condition. Ω is a subset of $A = \{a_0, \ldots, a_n\}$—indeed, a proper subset, since a_n
is not in it—and so we have Existence. If $X \subseteq \Omega$ then, since X is a proper subset

of A, there must be a first member of A not in it. Let this be $\delta(X)$. By definition, $\delta(X) \notin X$. So we have Transcendence. Now, either $\delta(X) = a_0$ (if $X = \phi$), and so $P\delta(X)$, or (if $X \neq \phi$) $\delta(X)$ comes immediately after something in $X \subseteq \Omega$, so $P\delta(X)$, by tolerance. In either case, $\delta(X) \in \Omega$, so we have Closure.

The inclosure contradiction is of the form $\delta(\Omega) \notin \Omega \land \delta(\Omega) \in \Omega$. In the case of the sorites paradox, the contradiction is that the first thing in the sequence that is not P is P. Diagonalization takes us out of X, and tolerance keeps us within Ω. We see why a contradiction occurs at the limit of P-things.

4   Indefinite Extensibility

In connection with the paradoxes of self-reference, and following Dummett, a number of philosophers have endorsed the claim that certain set-theoretic concepts are indefinitely extensible.[4] Any determination of a totality of things satisfying such a concept triggers an extension. The ordinals are a paradigm of such a concept. Given any totality of ordinals, we can always extend this by the least ordinal greater than all of them. That some concepts are indefinitely extensible in this sense is correct. To be indefinitely extensible is just to be the φ of some inclosure structure.[5] If $X \subseteq \Omega$, the addition of $\delta(X)$ extends it to a new subset of Ω.[6] What is not correct is the conclusion that most philosophers have drawn concerning such concepts, namely, that there is no determinate totality corresponding to such a notion—no totality of all ordinals, for example. This does not follow. What follows, as the Inclosure Schema makes plain—at least for a dialetheist—is that such a totality has contradictory properties.

In [4], Dummett suggests that indefinite extensibility is a species of vagueness:[7]

> The reason why the concept of 'natural number' is inherently vague is that a central feature of it, which would be involved in any characterization of the concept, is the validity of induction with respect to any well-defined property; and the concept of a well-defined property in turn exhibits a particular variety of inherent vagueness, namely indefinite extensibility. A concept is indefinitely extensible if, for any definite characterization of it, there is a natural extension of this characterization, which yields a more inclusive concept; this extension will be made according to some general principle for generating such extensions, and, typically, the extended characterization will be formulated by reference to the previous, unextended characterization.... An example is the concept of 'ordinal number'. Given any precise specification of a totality of ordinal numbers, we can always form a conception of an ordinal number which is the upper bound of that totality, and hence of a more extensive totality.

He later became more circumspect about the claim that indefinitely extensible concepts are vague:[8]

> Better than describing the intuitive concept of ordinal number as having a hazy extension is to describe it as having an increasing sequence of extensions: what is hazy is the length of the sequence, which vanishes into the indiscernible distance.

And apparently for good reasons. As Shapiro and Wright note,[9] to be an ordinal is not like being, say, red. As one progresses down the ordinals, numbers don't start off as being ordinals, and get less and less so until they are not ordinals, in the way that, in a sorites progression from red to blue, things start out as being red, gradually becoming less and less so, until they are not red at all.[10]

What Section 3 shows, however, is that there is a very intimate connection be-tween indefinite extensibility and vagueness. Vague concepts are one kind of indef-initely extensible concept. The conditions that give rise to an inclosure may be of various kinds: set-theoretic, semantic, and—now we see—tolerant. But always, at the limit, an irresistible force meets an immovable object. This does not show that set-theoretic (or semantic) concepts are vague, any more than it shows that vague concepts are set-theoretic (or semantic). But it does show that we are dealing with essentially the same kind phenomenon in the self-referential and sorites paradoxes.

The Principle of Uniform Solution, 'same kind of paradox, same kind of solution', tells us that in such circumstances, we should expect the same kind of solution to these paradoxes.[11] I take the correct solution to the paradoxes of self-reference to be a dialetheic one.[12] It follows that the solution to the sorites paradoxes should be so too. A simple-minded thought is this: in the case of the paradoxes of self-reference we endorse the soundness of the arguments. These establish certain contradictions, the trivializing consequences of which are avoided by not endorsing Explosion. We should just do the same in sorites paradoxes: endorse the soundness of the arguments. But this cannot be right. Sorites paradoxes are, in their own right, as near triviality-making as makes no difference. One can prove that an old thing is young, that a red thing is blue, and anything else for which one can postulate an appropriate sorites progression. We must be less simple-minded. What follows is, hopefully, so.

5 The Structure of Sorites Transitions

Come back to the sorites progression of Section 3. Pa_0 is true (and true only); Pa_n is false (and false only). If we write the least-number operator as μ then $\delta(\{x : Px\}) = a_i$, where $i = \mu h(a_h \notin \{x : Px\} = \mu h \neg Pa_h$.[13] The Inclosure Schema tells us that $a_i \in \{x : Px\}$ and $a_i \notin \{x : Px\}$, that is, $Pa_i \wedge \neg Pa_i$. So we know that there is at least one h for which Pa_h is both true and false. For all we have seen so far, there may be more than one. If there are, there is no reason, in principle, why these should be consecutive,[14] but the uniform nature of a sorites progression at least suggests this. Assuming it to be so, the structure of a sorites progression will look like this, where a_k is the last thing that is P, and $i \leq k$.[15]

$$
\begin{array}{ccccccc}
a_0 & \ldots & a_i & \ldots & a_k & \ldots & a_n \\
[- & - & P & - & -] & & \\
& & [- & - & \neg P & - & -]
\end{array}
$$

We can think of the sequence of dialetheic objects as providing a transition from the things that are definitely P to the things that are definitely not P. Many have argued that in sorites progressions there is a borderline area where the relevant statements have truth value gaps. What intuition actually tells us is that in the middle of the progression, things are symmetric with respect to the ends. The statements about the transition objects should therefore be symmetric with respect to the statements about the ends. And from this point of view, being *both* true and false is as good as being *neither*.[16]

Most importantly, however, note the position of a_i. It might be thought that the first thing that is not P should be a_{k+1}, but it is not. The first thing that is *not P* is actually identical with, or to the left of, the last thing which is P![17] a_{k+1} is not the first thing that is not P, but the first thing of which P is not true. We are in the territory of higher order vagueness here. We will turn to that matter later.[18]

6 Sorites Arguments

What does this tell us about sorites arguments? What tolerance tells us is that for some appropriate biconditional $\Leftrightarrow$, $Pa_h \Leftrightarrow Pa_{h+1}$ (for $0 \leq h < n$). The sorites argument is then of the form,

$$\frac{Pa_0 \quad Pa_0 \Leftrightarrow Pa_1}{\dfrac{Pa_1 \quad Pa_1 \Leftrightarrow Pa_2}{Pa_2}}$$

$$\ddots$$

$$\frac{Pa_{n-1} \quad Pa_{n-1} \Leftrightarrow Pa_n}{Pa_n} \ .$$

The next question is what this biconditional is. The correct understanding is, I take it, that it is a material biconditional, $\equiv$. Consecutive sorites statements have the same truth value. This is what tolerance is all about. Thus, where $\alpha \supset \beta$ is $\neg\alpha \vee \beta$, we have $(Pa_h \supset Pa_{h+1}) \wedge (Pa_h \supset Pa_{h+1})$. This is true if Pa_h and Pa_{h+1} are both true or both false. (If one is true and the other is false, it is false as well.)

   Given this understanding of the conditional, every major premise of the argument is true. For every h, Pa_h and Pa_{h+1} are both true or both false. But assuming an appropriate paraconsistent logic,[19] the disjunctive syllogism (DS)—modus ponens (MP) for the material conditional—is invalid: $\alpha, \alpha \supset \beta \nvdash \beta$; and, of course, exactly the same is true of the material biconditional. This is why the sorites fails. This does not mean that the major premises of the sorites are useless. The DS is not valid, but it is an acceptable default inference.[20] This means that we may use it until we run into contradiction. It is therefore quite legitimate to apply it at the beginning and end of the sequence, but not in the middle.

   It should be noted that, though the sorites argument itself is invalid, the situation is still inconsistent. The sorites is generated by the sentences,

$$Pa_0$$
$$\neg Pa_n$$
$$Pa_h \equiv Pa_{h+1} \ (1 \leq h < n).$$

From these, we cannot prove $Pa_h \wedge \neg Pa_h$, for any particular h, but can prove $\bigvee\limits_{1 \leq h \leq n} Pa_h \wedge \neg Pa_h$. To see this, write α_h for Pa_h. Then α_0 and $\alpha_0 \equiv \alpha_1$ give $(\alpha_0 \wedge \neg\alpha_0) \vee \alpha_1$. This plus $\alpha_1 \equiv \alpha_2$ give $(\alpha_0 \wedge \neg\alpha_0) \vee (\alpha_1 \wedge \neg\alpha_1) \vee \alpha_2$ and so on till we have $(\alpha_0 \wedge \neg\alpha_0) \vee \cdots \vee (\alpha_{n-1} \wedge \neg\alpha_{n-1}) \vee \alpha_n$. Whence $\neg\alpha_n$ gives the result. In other words, this information tells us that the inclosure is located somewhere along the track, but it, itself, does not tell us exactly where.

7 Other Sorites

We can, of course, formulate the sorites using a conditional that does satisfy MP, but the argument then has less plausibility, though the matter is not straightforward, since the question of what such conditionals are like is a vexed one. What we can say definitely is that for any conditional that satisfies MP, at least one of the major premises of the sorites must fail. In particular, for at least one h, the $Pa_h \Rightarrow Pa_{h+1}$ half of the biconditional must fail. The question is why?

   In weak relevant logics, of the kind required for paraconsistent set theory and semantics, if $\alpha \rightarrow \beta$ is true, then α entails β. That is, α, on its own, is a logically

sufficient condition for β.[21] The major premises of sorites arguments have little plausibility if conditionality is construed in this fashion. For no h is Pa_h a *logically* sufficient condition for Pa_{h+1}. Another conditional that is deployed frequently in relevant logic is the enthymematic conditional $\alpha \mapsto \beta$, defined as $(\alpha \wedge t) \rightarrow \beta$, where t is a logical constant.[22] Sometimes t is interpreted as the conjunction of all logical truths. In this case, $Pa_h \mapsto Pa_{h+1}$ is no more plausible than $Pa_h \rightarrow Pa_{h+1}$: even if you add all logical truths to Pa_h, this is still not logically sufficient to get you to Pa_{h+1}. t is sometimes interpreted simply as the conjunction of all truths. In that case, some of the major premises are certainly true. For example, since (we may suppose) Pa_1 is true, $t \rightarrow Pa_1$ holds, and so $Pa_0 \mapsto Pa_1$. But clearly the argument will not work for $Pa_k \mapsto Pa_{k+1}$. Since Pa_k is true, nothing true can be added to it to entail Pa_{k+1}, since this is not true.

Another sort of relevant conditional is the *ceteris paribus* conditional, $\alpha > \beta$: if α is true then, *ceteris paribus*, so is β.[23] To evaluate a *ceteris paribus* conditional, we look at all the "nearest" worlds where α holds, to see whether β holds. If it does so in all of these, the conditional is true; if not, not. If α is true, then the actual world is itself one of those worlds. This is the case for Pa_k; and since Pa_{k+1} is not true, $Pa_k > Pa_{k+1}$ clearly fails.

Leaving the conditional, it should be noted that there are forms of the sorites where the major premises are not (bi)conditionals but identities.[24] Thus, suppose that $a_0, \ldots, a_n$ are the colors of some consecutive strips such that adjacent ones are indistinguishable. Then the sorites argument has the form,

$$\frac{\dfrac{a_0 = a_1 \quad a_1 = a_2}{a_0 = a_2} \quad a_2 = a_3}{a_0 = a_3}$$

$$\ddots$$

$$\frac{a_0 = a_{n-1} \quad a_{n-1} = a_n}{a_0 = a_n} \ .$$

The inference involved here is not MP but the transitivity of identicals (TI). Exactly what to say about this form of the argument depends on the understanding of identity. But identity is very much like a biconditional; in fact, we can effectively reduce the identity form to the biconditional form if identity is given its Leibnizian definition: $x = y$ iff $\forall Z(Zx \Leftrightarrow Zy)$, where the quantifier is second order. TI is then a quantified version of biconditional transitivity: $\alpha \Leftrightarrow \beta, \beta \Leftrightarrow \gamma \vdash \alpha \Leftrightarrow \gamma$. Given this reduction, what to say about the identity sorites follows from what we have already said about the conditional form.

Arguably, if identity is given its Leibnizian definition, the correct biconditional is the material biconditional. For two objects to be identical, we do not need the statements to the effect that each has a certain property to be mutually *entailing*, for example. All we need is that they have the same truth value. With this understanding of identity, we may take all the premises of the identity sorites argument to be true. The argument is simply invalid. Transitivity fails for the material (conditional and) biconditional: $\alpha \equiv \beta, \beta \equiv \gamma \nvdash \alpha \equiv \gamma$. This is for exactly the same reason that MP fails. (If β is both true and false, then both premises are true, whatever the truth values of α and γ.) Hence TI also fails.[25] However, if $\Leftrightarrow$ is taken to be a detachable biconditional, say the $\leftrightarrow$ of some relevant logic, then the truth of the sorites premises are much less plausible. Details are much the same as in the conditional case.

8 "Extended" Paradoxes of Self-Reference

We now come to the vexed question of so-called higher order vagueness. Let me start, for reasons that will become clear later, by talking about an apparently different issue: "extended paradoxes" in the context of the semantic paradoxes. When people offer solutions to the semantic paradoxes of self-reference, it always seems to turn out that the machinery that they deploy to solve them allows the formulation of paradoxes equally virulent—or maybe better, simply moves the old paradox to a new place. Let me illustrate with respect to the liar and truth value gaps.

The semantic paradoxes deploy the T-schema. If we write T for the truth predicate, and angle brackets for naming, then the T-schema is the principle that

$$T\langle\alpha\rangle \leftrightarrow \alpha$$

for every closed sentence α (where $\leftrightarrow$ is an appropriate, detachable, biconditional[26]). Writing F for the falsity predicate, so that $F\langle\alpha\rangle$ is $T\langle\neg\alpha\rangle$, the simple liar paradox is a sentence λ_0, obtained by some technique of self-reference, of the form $F\langle\lambda_0\rangle$. Substituting in the T-schema, we get

$$T\langle\lambda_0\rangle \leftrightarrow F\langle\lambda_0\rangle.$$

The Principle of Bivalence tells us that, for all α,

$$T\langle\alpha\rangle \vee F\langle\alpha\rangle$$

and applying this to λ_0, we infer $T\langle\lambda_0\rangle \wedge F\langle\lambda_0\rangle$: λ_0 is both true and false.

A standard suggestion to avoid this conclusion is to deny the Principle of Bivalence. Sentences are not necessarily true or false; some are neither (N). So the Principle is replaced by

$$T\langle\alpha\rangle \vee F\langle\alpha\rangle \vee N\langle\alpha\rangle.$$

True, we can no longer infer that λ_0 is both true and false, but now we can construct the "extended liar paradox", a sentence λ_1 of the form $F\langle\lambda_1\rangle \vee N\langle\lambda_1\rangle$. Substituting this in the T-schema, we get

$$T\langle\lambda_1\rangle \leftrightarrow (F\langle\lambda_1\rangle \vee N\langle\lambda_1\rangle).$$

And all three of the possibilities lead to trouble.

Such a conclusion is obviously fatal to gap-theories of this kind. Some have thought that extended paradoxes of the same kind sink dialetheic ("glut") theories. That $T\langle\lambda_0\rangle \wedge F\langle\lambda_0\rangle$ is obviously no problem for a glut theory. The extended liar is now a sentence λ_2 of the form $F\langle\lambda_2\rangle \wedge \neg T\langle\lambda_2\rangle$ or given that there are no gaps, so that anything not true is false, just $\neg T\langle\lambda_2\rangle$. Substituting in the T-schema gives

$$T\langle\lambda_2\rangle \leftrightarrow \neg T\langle\lambda_2\rangle$$

and so, given the Law of Excluded Middle, $T\langle\lambda_2\rangle \wedge \neg T\langle\lambda_2\rangle$. But only a little thought suffices to show that this is no problem for a dialetheist. Dialetheism was never meant to give a consistent solution to the paradoxes. (Even in the case of the simple liar, things are inconsistent, since we have $\lambda_0 \wedge \neg\lambda_0$.) The point was to allow contradictions, but in a controlled way. The "extended" argument does show, however, that the very categories deployed in a dialetheic account of the paradoxes are themselves subject to the very sort of inconsistency they characterize. This is, indeed, to be expected. We may show, moreover, that all the inconsistencies generated are under control, by constructing a single "semantically closed" theory, which is inconsistent, but in which the inconsistencies are quarantined. Specifically, we can take a first-order language with a truth predicate T and some form of naming device $\langle.\rangle$. We

can then formulate a theory in this language, which contains all instances of the T-schema, and an appropriate form of self-reference. The theory can be shown to be inconsistent, but nontrivial.[27]

I conclude this prolegomenon with a comment on the principle,

$$T\langle\neg\alpha\rangle \leftrightarrow \neg T\langle\alpha\rangle.$$

Not all dialetheists subscribe to this biconditional. For example, I do not.[28] But some do.[29] If one does accept this principle, then there is no difference between $F\langle\alpha\rangle$ and $\neg T\langle\alpha\rangle$. So there is no difference between the liar, λ_0, and the "extended liar", λ_2.

9 Higher Order Vagueness

Let us now return to vagueness. Sorites paradoxes occur because the nature of the transition in a sorites progression is problematic. The straight-forward picture,

$$
\begin{array}{cccccc}
a_0 & \dots & \dots & \dots & \dots & a_n \\
[- & P & -] & [- & \neg P & -]
\end{array}
$$

jars because of the counterintuitive nature of the cutoff point between the true and the false. The solution that we have been looking at removes this cutoff point. But though the machinery does so, it produces, instead, two others—one between the true only and the both true and false, and one between the false only and the both true and false,

$$
\begin{array}{ccccccc}
a_0 & \dots & a_i & \dots & a_k & \dots & a_n \\
[- & - & P & - & -] & & \\
 & [- & - & \neg P & - & -] &
\end{array}
$$

and these would seem to jar just as much. As with the extended liar paradox, the machinery of the proposed solution allows us to produce a phenomenon of the same acuity. What is one to say about this?

The natural thought is that these cutoffs should be handled in exactly the same way. Consider, first, the right-hand boundary. This is located between those a of which P is true and those of which it is not. Let us now use T, not for the truth predicate, but for the binary truth-of (satisfaction) relation. Specifically if α is a formula of one free variable, say y, let the S-schema be

$$T\langle\alpha\rangle x \leftrightarrow \alpha_y(x)$$

where the right-hand side is the result of replacing all free occurrences of y with x (clashes of bound variables being handled by suitable relabeling). In particular, for our vague predicate, P, we have $T\langle Py\rangle x \leftrightarrow Px$. When the variable is clear from the context, I will omit it to keep notation simple. Thus, I will write $T\langle Py\rangle$ simply as $T\langle P\rangle$. Then the S-scheme amounts to this:

(*) $T\langle P\rangle x \leftrightarrow Px.$

Now, the predicate $T\langle P\rangle$ would seem to be just as vague as the predicate P. In particular, it would seem to be just as tolerant to small changes in its argument as the predicate P. Indeed, (*) would seem to tell us that the tolerances of P and $T\langle P\rangle$ march together. It follows that the predicate is just as soritical. And just as the original sorites was generated by a set of sentences,

$$Pa_0, \neg Pa_n$$
$$Pa_i \equiv Pa_{i+1}\ (1 \le i < n),$$

so a sorites is generated by the sentences,

$P\langle T\rangle a_0, \neg T\langle P\rangle a_n$

$T\langle P\rangle a_i \equiv T\langle P\rangle a_{i+1}$ $(1 \leq i < n)$.

The predicate $T\langle P\rangle$ also gives rise to an inclosure. Let $\varphi(x)$ be $T\langle P\rangle x$, so $\Omega = \{x : T\langle P\rangle x\}$; $\theta(X)$ is the vacuous condition. Ω is a subset of $A = \{a_0, \ldots, a_n\}$—indeed, a proper subset, since a_n is not in it. If $X \subseteq \Omega$ then, since X is a proper subset of A, there must be a first member of A not in it. Let this be $\delta(X)$. By definition, $\delta(X) \notin X$, Transcendence. Now, either $\delta(X) = a_0$ (if $X = \phi$), and so $T\langle P\rangle\delta(X)$, or (if $X \neq \phi$) $\delta(X)$ comes immediately after something in $X \subseteq \Omega$, so $T\langle P\rangle\delta(X)$, by tolerance. In either case, $\delta(X) \in \Omega$, Closure. The contradiction is that $T\langle P\rangle \, \delta(\Omega) \wedge \neg T\langle P\rangle \, \delta(\Omega)$.

Similar considerations apply at the left-hand boundary. Let us write $F\langle P\rangle x$, 'P is false of x', for $T\langle \neg P\rangle x$. A predicate is vague (tolerant) if and only if its negation is. In particular, $\neg P$ is just as vague as P. And since, as the S-Schema tells us,

$F\langle P\rangle x \leftrightarrow \neg Px$,

$F\langle P\rangle$ is a vague predicate as then is $\neg F\langle P\rangle$. We therefore have a sorites generated by the sentences,

$\neg F\langle P\rangle a_0, F\langle P\rangle a_n$

$F\langle P\rangle a_i \equiv F\langle P\rangle a_{i+1}$ $(1 \leq i < n)$.

Note that $\alpha \equiv \beta$ is logically equivalent to $\neg \alpha \equiv \neg \beta$.

And as is to be expected, this boundary is another inclosure. Let $\varphi(x)$ be $\neg F\langle P\rangle x$, so $\Omega = \{x : \neg F\langle P\rangle x\}$; $\theta(X)$ is the vacuous condition. Ω is a subset of $A = \{a_0, \ldots, a_n\}$—indeed, a proper subset, since a_n is not in it. If $X \subseteq \Omega$ then, since X is a proper subset of A, there must be a first member of A not in it. Let this be $\delta(X)$. By definition, $\delta(X) \notin X$, Transcendence. Now, either $\delta(X) = a_0$ (if $X = \phi$), and so $\neg F\langle P\rangle\delta(X)$, or (if $X \neq \phi$) $\delta(X)$ comes immediately after something in $X \subseteq \Omega$, so $\neg F\langle P\rangle\delta(X)$, by tolerance. In either case, $\delta(X) \in \Omega$, Closure. The contradiction is that $\neg F\langle P\rangle \, \delta(\Omega) \wedge \neg\neg F\langle P\rangle \, \delta(\Omega)$—or just $\neg F\langle P\rangle \, \delta(\Omega) \wedge F\langle P\rangle \, \delta(\Omega)$.[30]

10 The General Case
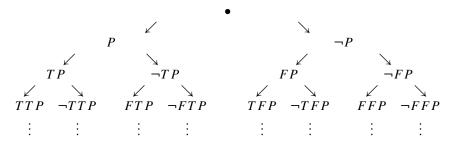
Of course, the situation repeats. We now have some new boundaries. Thus, the next iteration gives us the following.

[−	−	$\neg F\langle P\rangle$		−		−	−	−]				
					[−	−	$F\langle P\rangle$		−	−	−	−	−]
					[−	−	$\neg P$		−	−	−	−	−]
a_0	...		...		...	a_i	...	...	...	a_k	...		...	...	a_n
[−	−		−	P		−	−	−		−	−]
[−	−		−	$T\langle P\rangle$		−	−	−		−	−]
						[−	−	−	$\neg T\langle P\rangle$		−	−]

Look below the as. We have just considered the division between P being true and its not being true. We now have the divisions between $T\langle P\rangle$ being true, and its not being true, and the division between $\neg T\langle P\rangle$ being true and its not being true. The first of these is the same as that between P being true and its not being true, since

P and $T\langle P\rangle$ are coextensional. But the second is now. Above the as we have the symmetrical situation concerning F.

And so it goes on. We need to consider all predicates that can be obtained by iteration. Generally, given the vague predicates Q, $\neg Q$, at the next level we have $T\langle Q\rangle$, $\neg T\langle Q\rangle$, and $T\langle\neg Q\rangle$, $\neg T\langle\neg Q\rangle$ (i.e., $F\langle Q\rangle$, $\neg F\langle Q\rangle$). Thus, the hierarchy of predicates looks as follow. To keep notation simple, I will henceforth omit the angle brackets. (Thus, I will write $F\langle T\langle P\rangle\rangle$ as FTP, and so on.)

$$
\begin{array}{ccccccc}
& & & \bullet & & & \\
& \swarrow & & & & \searrow & \\
& P & & & & \neg P & \\
\swarrow & & \searrow & & \swarrow & & \searrow \\
TP & & \neg TP & & FP & & \neg FP \\
\swarrow\quad\searrow & & \swarrow\quad\searrow & & \swarrow\quad\searrow & & \swarrow\quad\searrow \\
TTP\quad\neg TTP & & FTP\quad\neg FTP & & TFP\quad\neg TFP & & FFP\quad\neg FFP \\
\vdots\qquad\vdots & & \vdots\qquad\vdots & & \vdots\qquad\vdots & & \vdots\qquad\vdots
\end{array}
$$

By exactly analogous consideration, each pair in the family is vague, and each gives rise to an inclosure contradiction.

11   A "Soritically Closed" Language

How do we know that all these contradictions can be accommodated in a uniform way? With the self-referential paradoxes and their extended versions, we know this because we can construct a single semantically closed language, which accommodates all the contradictions in one hit. Exactly the same is true in this case. We can construct a "soritically closed" language. Specifically, we take a language that has the truth-of predicate T and a naming device $\langle.\rangle$. For definiteness, let us suppose that the language contains that for arithmetic and that the naming is obtained by Gödel coding. We suppose, in addition, one vague predicate, P, and a sorites sequence $(a_0, \ldots, a_n)$. Let this be $0, \ldots, n$. Let σ be any string of 'T's and 'F's (including the empty string), and let $\#(\sigma)$ be the number of 'F's in σ. (The parity of this tells us whether we are on the left-hand side of a division, or the right-hand side. Even is left; odd is right.)

Our theory comprises the S-schema, plus the following:

$$\sigma Pa_i \equiv \sigma Pa_{i+1} \qquad (0 \le i < n)$$
$$\sigma Pa_0, \neg\sigma Pa_n \qquad \text{when } \#(\sigma) \text{ is even}$$
$$\neg\sigma Pa_0, \sigma Pa_n \qquad \text{when } \#(\sigma) \text{ is odd.}$$

The theory is inconsistent. For every σ, the theory entails

$$\bigvee_{1 \le i \le n} (\sigma Pa_i \wedge \neg\sigma Pa_{i+1}).$$

(The proof when σ is the empty sequence was already given; in the general case, the argument is exactly the same.)

Moreover, the theory is nontrivial. We can construct an interpretation which shows this, as follows. Start with a language without T. Take an interpretation $\Im$, which is standard with respect to the arithmetic machinery. Let $0 < m < n$. The extension of P is $\{0, \ldots, m\}$, and the anti-extension is $\{m, \ldots, n\}$.[31] So, in the model, $Pm \wedge \neg Pm$ holds, as does every biconditional $Ph \equiv P(h+1)$, for $0 \le h \le n$. We

now construct a model of the S-schema on top of $\Im$ as in Priest [8, 8.2]. (The model constructed there is of the T-schema, but this generalizes to one for the S-schema in an obvious fashion.) In this model, we have not just the S-schema, but its contraposed form. Hence, every $\sigma\alpha$ is logically equivalent to α or $\neg\alpha$, and so all the cases of the axioms where σ is nonempty collapse into the case where it is. Moreover, in the construction of the model, it is only sentences involving 'T' that change their value. So the truth values of all other sentences are as in $\Im$. (In particular, then, any purely arithmetic sentence false in the standard model is not provable in the theory.)

What we see, then, is that from a dialetheic perspective, higher order vagueness is essentially the same as the extended paradoxes of self-reference,[32] can be handled in exactly the same way, and is no more problematic.

One final point on higher order vagueness. As I have already noted, some dialetheists, but not others, endorse the contraposed form of the T-schema. Those who do will also, naturally, endorse the contraposed form of the S-schema. And for those who do, as we have just seen, the higher order sorites will collapse into the standard sorties, just as the extended self-referential paradoxes collapse into the straight liar paradox. There is, then, no such thing as higher order vagueness—as distinct from vagueness.

Note also the following. If α is any formula of one free variable, let us write $|\alpha|^E$ for its extension (the set of things that satisfy it), and $|\alpha|^A$ for its anti-extension (the set of things that anti-satisfy it—make it false). Consider any predicate Q in our hierarchy. Because of the T-schema, $|Q|^E = |TQ|^E$ ($= |TTQ|^E = |TTTQ|^E$, and so on). Without a contraposing S-schema, we will not in general have $|Q|^A = |TQ|^A$. However, if we have half of the contraposed form,[33]

$$\neg TQx \rightarrow T\neg Qx,$$

then, since the S-schema gives us $T\neg Qx \rightarrow \neg Qx$, we have

$$\neg TQx \rightarrow \neg Qx.$$

That is, $|TQ|^A \subseteq |Q|^A$. Similarly, $|TTQ|^A \subseteq |TQ|^A$. So we have the regress: $\cdots \subseteq |TTQ|^A \subseteq |TQ|^A \subseteq |Q|^A$. Since there is only a finite number of objects in a sorites sequence $\{a_0, \dots, a_n\}$, at some stage we must have a predicate Q', the anti-extension of which is minimal. For this Q', applications of T produce a predicate with the same extension and anti-extension. In other words, there is no further higher order vagueness. All higher order vagueness is therefore of finite order.

If we have the other half of the contraposed truth-of schema,

$$T\neg Qx \rightarrow \neg TQx,$$

the set-inclusions are reversed, but the conclusion is the same. If we have neither of these, there is no systematic connection between anti-extensions. Since there is only a finite number of extension/anti-extension pairs to be shared around, some predicates in the sequence, $Q, TQ, TTQ, \dots$ must have the same extension and anti-extension. But the fact that two predicates in the sequence are like this does not imply that their successors are. Anti-extensions can jump all over the place as we run along the sequence.

12 The Soundness of Sorites Arguments

Sorites arguments and self-referential paradoxes are, I have argued, of the same kind: they are all inclosure paradoxes. The Principle of Uniform Solution requires the

same kinds of paradox to have the same kinds of solution. The solutions to the self-referential paradoxes and the sorites paradoxes which I have sketched—are these of the same kind? Yes. True, the paradoxes of self-reference are sound, and sorites arguments are not.[34] But in both cases, as we have seen, the relevant information implies contradictions. In the case of the paradoxes of self-reference, we know exactly which formula is a dialetheia. In the case of the sorites paradox, the information implies only that a contradiction occurs somewhere in the sorites sequence. In both cases, the contradiction occurs at the limit of the inclosure, but in the case of the sorites, the information tells us only that this occurs *somewhere* between the ends.

It remains the case, though, that the sorites arguments themselves are not sound. This raises the question of why it is we are, seemingly ineluctably, drawn to them. Wherein lies their "pull"?—we cannot say that this is in their soundness, as we can with the paradoxes of self-reference.

It is helpful to look at the matter in the following way. The facts of a sorites progression require us to recognize the existence of a cutoff point of some kind. The nature of this cutoff point may be theorized in different ways, but all approaches must postulate a cutoff of some kind. This is the import of the forced-march sorites. I ask you whether Pa_0; you give some answer. I then ask the same question of Pa_1, Pa_2, ... Sooner or later you must give a different answer, or at least an answer not logically equivalent to the first answer. There we have a cutoff. Given that we have no choice but to admit the existence of a cutoff, what is required for a solution to the sorites to do is to theorize its nature and use the theorization to explain why we find its existence so counterintuitive.[35] This latter is exactly the question of explaining the pull of the sorites.

Why, on the present approach, do we find the existence of a cutoff point counterintuitive? It is, I suggest, because, given any statements concerning consecutive members of a sorites sequence, Pa_h and Pa_{h+1} , these have the same truth value: they are both true or both false. Of course, they can have different truth values as well, but it is the identity of the truth values that makes us think that there is no significant change at this point.

Essentially the same consideration applies in an identity sorites. Given that identity is cashed out in terms of a material biconditional, then for any consecutive members of the sequence, a_h and a_{h+1}, $a_h = a_{h+1}$. Of course, it may be the case that $a_h \neq a_{h+1}$ as well; but it is the identity that makes us think that there is no significant change at this point. In fact, the theory of identity can be applied to give a theory of truth values according to which, in an ordinary sorites, the truth values of Pa_h and Pa_{h+1} are identical. I forego giving details here.[36]

13 Conclusion

Prima facie, sorites arguments and the paradoxes of self-reference are completely distinct. They are certainly distinct. But what I have tried to establish is that, at a fundamental level, they are the same. Both are inclosure paradoxes, where the underlying form is given by the Inclosure Schema. The two kinds of paradox must therefore have the same kind of solution. Given that the correct solution to the paradoxes of self-reference is a dialetheic one, then so must be a solution to the sorites paradoxes. I have discussed such a solution at length, and argued that, despite certain superficial differences, it is also essentially the same.

The fact that one has a single family of paradoxes, and a uniform solution, does not, of course, mean that various subfamilies cannot have their own specificities. Even within the paradoxes of self-reference, the semantic and the set theoretic paradoxes have differences of vocabulary; more importantly, diagonalization may be achieved in various different ways (by employing literal diagonalization, a least number operator, and so on). Thus, it is entirely possible for the sorites paradoxes to have their own specificities, which they do. For example, tolerance plays a distinctive role, and "higher order vagueness" must be accommodated. All this we have seen. But the specificities are superficial, just as the specificities of the set theoretic and semantic paradoxes are superficial when it comes understanding the paradoxes and framing an appropriate solution. Such, at least, has been the import of this paper.

Notes

1. At least the sorites and the semantic paradoxes. See, for example, McGee [7], Field [5].

2. For details of the following, see Priest [12], especially Part 3.

3. Priest [12], p. 277.

4. For example, Shapiro and Wright [14].

5. As Shapiro and Wright [14], Section 6, in effect note.

6. Dummett [3], p. 317, cites Russell as holding a precursor of the view about indefinite extensibility. The reference is to his formulation of "Russell's Schema" for the paradoxes of self-reference, which is certainly a precursor of the Inclosure Schema. See Priest [12], Chapter 9.

7. Dummett [4], pp. 195–6 of reprint.

8. Dummett [3], p. 316f.

9. [14], Section 12.

10. See also Priest [12], 11.3.

11. Priest [12], 11.5, 11.6, 17.6.

12. Priest [11; 12].

13. In naïve set theory, the comprehension schema gives $y \in \{x : Px\} \leftrightarrow Py$, and contraposition gives $y \notin \{x : Px\} \leftrightarrow \neg Py$.

14. The Technical Appendix to Part 3 of Priest [12] constructs models of the Inclosure Schema where some ordinals are consistent and some are not. Section 4 of the Appendix gives a model in which inconsistent ordinals need not be consecutive.

15. It is clear from the diagram that $\{x : Px\} \cap \{x : \neg Px\}$ is not empty. But since this set is $\{x : Px\} \cap \overline{\{x : Px\}}$, it is empty as well. It is difficult to represent this fact in a consistent diagram!

16. See Hyde [6].

17. In this context, note the following. If it were the case that

$$(*) \qquad \{x : Px\} = \{a_0, \dots, a_k\}$$

then it would follow that $\mu h(a_h \notin \{x : Px\}) = \mu h(a_h \notin \{a_0, \dots, a_k\})$, and so that $\delta(\{x : Px\}) = a_{k+1}$. But in a paraconsistent set theory based on a relevant logic (*) need not be true. Sets with the same members are not necessarily identical, since $\alpha, \beta \nvdash \alpha \leftrightarrow \beta$. For example, both of the sets $\{x : x \neq x\}$ and $\{x : \forall y\, x \in y\}$ have no members, but they are not the same set. See Priest [11], 10.1.

18. Note also, for what it is worth, that each a between a_i and a_k (including the endpoints) thinks that it is the first thing satisfying $\neg P$ in the sense that $\neg Pa \wedge \forall x < a\ Px$, where $<$ is the natural ordering on the objects.

19. In what follows, we will take this to be the logic *LP* of Priest [11], Ch. 5, but matters are much the same in virtually every paraconsistent logic.

20. Priest [11], Chapter 16.

21. Priest [13], Chapter 10.

22. Priest [11], 18.3.

23. Priest [13], 10.7.

24. See, for example, Priest [13], Chapter 25.

25. Further on the matter of nontransitive identity and vagueness, see Priest [10].

26. For the sake of definiteness, let this be the conditional of Priest [11], 19.8.

27. Specifically, no inconsistencies involving only the grounded sentences of the language (in the sense of Kripke) are provable. See Priest [8], 8.2.

28. Priest [11], 4.9.

29. See, for example, Beall [1].

30. We can, of course, run the sorites on $F\langle P \rangle$ from the right-hand end. The result is similar.

31. Strictly speaking, $\{x; x \geq m\}$, since every natural number must be in either the extension or the anti-extension of P. But what happens for numbers greater than n is irrelevant for our example.

32. This important observation is due to Colyvan [2].

33. As endorsed in Priest [11], 4.9.

34. Actually, some of the arguments concerning the paradoxes of self-reference may be classically valid but not paraconsistently valid, as well. So exactly the same is true of these. See Priest [12], 9.2.

35. These matters are discussed in much greater detail in Priest [9].

36. Priest [9] shows how to do this in the case of a fuzzy logic. Priest [10] shows how to do this with the material biconditional.

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