

Packing Index of Subsets in Polish Groups

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Abstract For a subset A of a Polish group G , we study the (almost) packing index $\text{pack}(A)$ (respectively, $\text{Pack}(A)$) of A , equal to the supremum of cardinalities $|S|$ of subsets $S \subset G$ such that the family of shifts $\{xA\}_{x \in S}$ is (almost) disjoint (in the sense that $|xA \cap yA| < |G|$ for any distinct points $x, y \in S$). Subsets $A \subset G$ with small (almost) packing index are large in a geometric sense. We show that $\text{pack}(A) \in \mathbb{N} \cup \{\aleph_0, \mathfrak{c}\}$ for any σ -compact subset A of a Polish group. In each nondiscrete Polish Abelian group G we construct two closed subsets $A, B \subset G$ with $\text{pack}(A) = \text{pack}(B) = \mathfrak{c}$ and $\text{Pack}(A \cup B) = 1$ and then apply this result to show that G contains a nowhere dense Haar null subset $C \subset G$ with $\text{pack}(C) = \text{Pack}(C) = \kappa$ for any given cardinal number $\kappa \in [4, \mathfrak{c}]$.

1 Introduction

Given a Polish group G and a nonempty subset $A \subset G$ with nice descriptive properties, we study all possible values of the packing index

$$\text{pack}(A) = \sup \{ |S| : S \subset G \text{ and the family } \{xA\}_{x \in S} \text{ is disjoint} \}$$

of A , which indicate the smallness of the subset A in a geometric sense. Answering a problem of Dikranjan and Protasov [4] the first two authors [1] constructed a subset $A \subset \mathbb{Z}$ such that $\text{pack}(A) = \aleph_0$ but for every infinite subset $S \subset \mathbb{Z}$ the family of shifts $\{xA\}_{x \in S}$ is not disjoint. This example shows that the supremum in the definition of the packing index $\text{pack}(A)$ cannot be replaced with maximum. The difference between max and sup is caught by the *sharp packing index*

$$\text{pack}^\sharp(A) = \sup \{ |S|^+ : S \subset G \text{ and the family } \{xA\}_{x \in S} \text{ is disjoint} \},$$

which can be equivalently defined as the smallest cardinal κ such that for every subset $S \subset G$ of cardinality $|S| \geq \kappa$ the family of shifts $\{xA\}_{x \in S}$ is not disjoint. The sharp

Received May 24, 2009; accepted July 16, 2009; printed January 8, 2010

2010 Mathematics Subject Classification: Primary, 03E15, 03E17, 03E35, 03E50, 03E75, 05D99; Secondary, 22A99, 54H05, 54H11

Keywords: Polish group, packing index, Borel set, Haar null set, Martin axiom, continuum hypothesis

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packing index $\text{pack}^\sharp(A)$ determines the value of $\text{pack}(A)$ because

$$\text{pack}(A) = \sup\{\kappa : \kappa < \text{pack}^\sharp(A)\}.$$

The papers [1], [2], and [8] are devoted to constructing subsets with a given (sharp) packing index. In particular, for every nonzero cardinal number $\kappa \leq \mathfrak{c}$ one can easily construct a subset $A \subset \mathbb{R}$ with $\text{pack}(A) = \kappa$. A less trivial task is constructing a subset $A \subset \mathbb{R}$ with a given sharp packing index $\text{pack}^\sharp(A) = \kappa \in [2, \mathfrak{c}^+]$.

After discussing these results on the topological seminar at Wrocław University, the second author was asked by Omiljanowski about possible restrictions on the packing index $\text{pack}(A)$ of subsets $A \subset \mathbb{R}$ having good descriptive properties (like being compact, σ -compact, Borel, measurable, or meager). This question was probably motivated by the well-known fact that the Continuum Hypothesis (although irresolvable in ZFC) has a positive solution in the realm of Borel subsets of the real line: each uncountable Borel subset $A \subset \mathbb{R}$ has cardinality \mathfrak{c} of continuum.

In this paper we shall give several partial answers to Omiljanowski's question. On the one hand, we shall show that σ -compact subsets A in Polish groups cannot have an intermediate packing index $\aleph_0 < \text{pack}(A) < \mathfrak{c}$. For a Borel subset A of a Polish group we get a weaker result: $\text{pack}^\sharp(A)$ cannot take the value on the interval $\mathfrak{s}q(\Pi_1^1) < \text{pack}^\sharp(A) \leq \mathfrak{c}$, where $\mathfrak{s}q(\Pi_1^1)$ stands for the smallest cardinality κ such that each coanalytic subset $X \subset 2^\omega \times 2^\omega$ contains a square $S \times S$ of size $|S \times S| = \kappa$ provided X contains a square of size $\geq \kappa$. The value of the small uncountable cardinal $\mathfrak{s}q(\Pi_1^1)$ is not completely determined by ZFC axioms: both the equality $\mathfrak{s}q(\Pi_1^1) = \mathfrak{c}$ and the strict inequality $\mathfrak{s}q(\Pi_1^1) < \mathfrak{c}$ are consistent with the Martin Axiom (see [10]).

On the other hand, for every infinite cardinal number $\kappa \leq \mathfrak{c}$ in each nondiscrete Polish Abelian group G we shall construct a nowhere dense Haar null subset $A \subset G$ with $\text{pack}(A) = \text{Pack}(A) = \kappa$. Here

$$\text{Pack}(A) = \sup\{|S| : S \subset G \text{ and } \{xA\}_{x \in S} \text{ is almost disjoint}\}$$

is the *almost packing index* of A . In the above definition, a family of shifts $\{xA\}_{x \in S}$ is said to be *almost disjoint* if $|xA \cap yA| < |G|$, for all distinct $x, y \in S$.

To construct the nowhere dense Haar null subset $A \subset G$ with a given (almost) packing index, in each nondiscrete Polish Abelian group G we first construct a closed nowhere dense Haar null subset $C \subset G$ with $\text{Pack}(C) = 1$. The set C , being nowhere dense and Haar null, is small in the sense of category and measure, but it is large in the geometric sense because for any two distinct points $x, y \in G$ the shifts xC and yC have intersection of cardinality $|xC \cap yC| = \mathfrak{c}$.

In particular, $CC^{-1} = G$ and thus C is a closed nowhere dense Haar null subset that algebraically generates the group G . This result can be seen as an extension of a result of Solecki [11] who proved that each nonlocally compact Polish Abelian group G is algebraically generated by a nowhere dense subset.

It also extends some results of [9, §13]. In fact, the closed Haar null subset $C \subset G$ with $\text{Pack}(C) = 1$ is constructed as the union $C = A \cup B$ of two closed subsets $A, B \subset G$ with $\text{pack}(A) = \text{pack}(B) = \mathfrak{c}$. This shows that the packing index is highly nonadditive.

Notation By ω we shall denote the first infinite ordinal; $\mathbb{N} = \omega \setminus \{0\}$ stands for the set of positive integers. Cardinals will be identified with the initial ordinals of given cardinality; \mathfrak{c} stands for the cardinality of continuum. All topological groups G considered in this paper will be supplied with an invariant metric ρ generating the topology of G . By e we shall denote the identity element of G . For a point $x \in G$ and a real number r by $B(x, r) = \{g \in G : \rho(g, x) < r\}$ we shall denote the open r -ball centered at x . Also for $x \in G$ we put $\|x\| = \rho(x, e)$. The invariance of ρ implies $\|x^{-1}\| = \|x\|$ and $\|xy\| \leq \|x\| + \|y\|$, for all $x, y \in G$.

2 The Packing Indices of σ -Compact Sets in Polish Groups

In this section we shall show that the packing index of a σ -compact subset in a Polish group cannot take an intermediate value between ω and \mathfrak{c} . First we shall prove the following useful result (proved in analogy with Proposition 5 of [3]).

Lemma 2.1 *Let A be a subset of a Polish group G . If $\text{pack}^\sharp(A) \leq \mathfrak{c}$, then the closure of AA^{-1} contains a neighborhood of the neutral element e of G .*

Proof Fix any complete metric ρ generating the topology of the Polish group G . Assuming that $\overline{AA^{-1}}$ is not a neighborhood of e , we shall construct a perfect subset $K \subset G$ such that the indexed family $\{xA\}_{x \in K}$ is disjoint, which will imply that $\text{pack}^\sharp(A) = |K|^+ = \mathfrak{c}^+$.

Taking into account that the closed subset $\overline{AA^{-1}}$ is not a neighborhood of e in G , we can find for any open neighborhood U of e , a point $b \in U \setminus \overline{AA^{-1}}$ and an open neighborhood V of e such that $V^{-1}bV \subset U \setminus \overline{AA^{-1}}$.

Using this fact, construct by induction a sequence $(b_n)_{n \in \omega}$ of points in G and sequences $(U_n)_{n \in \omega}$ and $(V_n)_{n \in \omega}$ of open neighborhoods of e in G such that

1. $b_n \in U_n = U_n^{-1}$;
2. $V_{n+1}^{-1}b_nV_{n+1} \cap AA^{-1} = \emptyset$;
3. $b_n \notin V_{n+1}V_{n+1}^{-1}$;
4. $\text{diam}_\rho(bV_{n+1}) < 2^{-n}$, for any point $b \in B_n = \{b_0^{\varepsilon_0} \cdots b_n^{\varepsilon_n} : \varepsilon_0, \dots, \varepsilon_n \in \{0, 1\}\}$; and
5. $U_{n+1}^3 \subset V_{n+1} \subset U_n$.

Define a map $f : \{0, 1\}^\omega \rightarrow G$ by assigning to each infinite binary sequence $\vec{\varepsilon} = (\varepsilon_i)_{i \in \omega}$ the infinite product

$$f(\vec{\varepsilon}) = \prod_{i=0}^{\infty} b_i^{\varepsilon_i} = \lim_{n \rightarrow \infty} f_n(\vec{\varepsilon})$$

where $f_n(\vec{\varepsilon}) = \prod_{i=0}^n b_i^{\varepsilon_i}$.

Let us show that the latter limit exists. It suffices to check that $(f_n(\vec{\varepsilon}))_{n \in \omega}$ is a Cauchy sequence in (G, ρ) .

The condition (5) implies that $U_{i+1}^2 \subset U_i$ for all i . This can be used as the inductive step in the proof of the inclusion $U_n \cdots U_m \subset U_n^2$ for all $m \geq n$. Then for every $m \geq n$

$$f_m(\vec{\varepsilon}) \in f_n(\vec{\varepsilon}) \cdot U_{n+1} \cdots U_m \subset f_n(\vec{\varepsilon}) \cdot U_{n+1}^2 \subset f_n(\vec{\varepsilon})V_{n+1}$$

and thus

$$\rho(f_m(\vec{\varepsilon}), f_n(\vec{\varepsilon})) \leq \text{diam}_\rho(f_n(\vec{\varepsilon}) \cdot V_{n+1}) < 2^{-n}$$

by the condition (4). Therefore, the sequence $(f_n(\vec{\varepsilon}))_{n \in \omega}$ is Cauchy and the limit $f(\vec{\varepsilon}) = \lim_{n \rightarrow \infty} f_n(\vec{\varepsilon})$ exists.

Moreover, the upper bound $\rho(f_m(\vec{\varepsilon}), f_n(\vec{\varepsilon})) \leq 2^{-n}$ implies that the map $f : \{0, 1\}^\omega \rightarrow G$ is continuous. On the other hand, the inclusions $f_m(\vec{\varepsilon}) \in f_n(\vec{\varepsilon}) \cdot U_n^2$, $m \geq n$, imply that

$$f(\vec{\varepsilon}) \in f_n(\vec{\varepsilon}) \cdot \overline{U_n^2} \subset f_n(\vec{\varepsilon}) \cdot U_n^3 \subset f_n(\vec{\varepsilon}) \cdot V_{n+1}.$$

This inclusion will be used for the proof of the injectivity of f . We shall prove a little bit more: for any distinct vectors $\vec{\varepsilon}$ and $\vec{\delta}$ in $\{0, 1\}^\omega$, we get $f(\vec{\varepsilon})A \cap f(\vec{\delta})A = \emptyset$.

Find the smallest number $n \in \omega$ such that $\varepsilon_n \neq \delta_n$. We lose no generality by assuming that $\delta_n = 0$ and $\varepsilon_n = 1$. It follows that $f(\vec{\varepsilon}) \in f_n(\vec{\varepsilon})U_{n+1}^3 \subset f_{n-1}(\vec{\varepsilon})b_n V_{n+1}$ while $f(\vec{\delta}) = f_n(\vec{\delta})V_{n+1} = f_{n-1}(\vec{\delta}) \cdot e \cdot V_{n+1} = f_{n-1}(\vec{\delta})V_{n+1}$. Then

$$(f(\vec{\delta}))^{-1} f(\vec{\varepsilon}) \in V_{n+1}^{-1} b_n V_{n+1} \subset G \setminus AA^{-1}$$

by the condition (2) and hence $f(\vec{\varepsilon})A \cap f(\vec{\delta})A = \emptyset$.

Thus for the set $K = f(\{0, 1\}^\omega)$ the indexed family $\{xA\}_{x \in K}$ is disjoint. The injectivity of f implies that $\text{pack}^\sharp(A) \geq |K|^+ = c^+$. □

Now we can prove the main result of this section.

Theorem 2.2 *If A is a σ -compact subset of a Polish group G , then $\text{pack}^\sharp(A) \in \mathbb{N} \cup \{\aleph_0, \aleph_1, c^+\}$. Moreover, if the set A is compact, then*

1. $\text{pack}^\sharp(A) = c^+$ provided G is not locally compact;
2. $\text{pack}^\sharp(A) \in \{\aleph_1, c^+\}$ provided G is locally compact but not compact;
3. $\text{pack}^\sharp(A) \in \mathbb{N} \cup \{c^+\}$ provided G is compact.

Proof If A is σ -compact, then so is the set $AA^{-1} = \{xy^{-1} : x, y \in A\}$ and then the set $(G \setminus AA^{-1}) \cup \{e\}$ is a G_δ -set in G . In its turn, the subset

$$X = \{(x, y) \in G \times G : y^{-1}x \in (G \setminus AA^{-1}) \cup \{e\}\}$$

is of type G_δ in $G \times G$, being the preimage of the G_δ -subset $(G \setminus AA^{-1}) \cup \{e\}$ under the continuous map $g : G \times G \rightarrow G$, $g : (x, y) \mapsto y^{-1}x$.

Assuming that $\text{pack}^\sharp(A) > \aleph_1$, we can find an uncountable subset $S \subset G$ with disjoint family $\{xA\}_{x \in S}$, which implies that $S \times S \subset X$. Since the Polish space X contains the uncountable square $S \times S$, we can apply Theorem 2.2 of [7] to conclude that X contains the square $P \times P$ of a perfect subset $P \subset G$ (the latter means that P is closed in G and has no isolated point). It follows from $P \times P \subset X$ that the family $\{xA\}_{x \in P}$ is disjoint and thus $c^+ = |P|^+ \leq \text{pack}^\sharp(A) \leq |G|^+ = c^+$.

Assuming that $A \subset G$ is compact, we shall now prove the assertions (1)–(3) of the theorem. The compactness of A implies the compactness of AA^{-1} . If AA^{-1} is not a neighborhood of e , then we can apply Lemma 2.1 to conclude that $\text{pack}^\sharp(A) = c^+$. This holds if the group G is not locally compact. So, we assume that AA^{-1} is a neighborhood of e . In this case the group is locally compact and we can take a neighborhood $U \subset G$ of e with $UU^{-1} \subset AA^{-1}$.

Therefore, for every $B \subset G$ with $B^{-1}B \cap AA^{-1} = \{e\}$ we get $B^{-1}B \cap UU^{-1} = \{e\}$, which implies that the indexed family $\{xU\}_{x \in B}$ is disjoint and the set B is at

most countable, being discrete in the Polish space G . This gives the upper bound $\text{pack}^\sharp(A) \leq \aleph_1$.

If the group G is not compact, then using the Zorn Lemma, we can find a maximal set $B \subset G$ with $B^{-1}B \cap AA^{-1} = \{e\}$. We claim that $BAA^{-1} = G$.

Assuming to the contrary, we can find a point $b \in G \setminus BAA^{-1}$. Then the set bA is disjoint from the set BA and hence we can enlarge the set B to the set $\tilde{B} = B \cup \{b\}$ so that $(xA)_{x \in \tilde{B}}$ is disjoint.

The latter is equivalent to $\tilde{B}^{-1}\tilde{B} \cap AA^{-1} = \{e\}$ and this contradicts the maximality of B . The compactness of AA^{-1} and noncompactness of $G = BAA^{-1}$ implies that B is infinite and thus $\text{pack}^\sharp(A) \geq |B|^+ \geq \aleph_1$. This completes the proof of the second assertion of the theorem.

To prove the third assertion, assume that G is compact. In this case G carries a Haar measure μ which is a unique probability invariant σ -additive Borel measure on G . If AA^{-1} is not a neighborhood of e , then $\text{pack}^\sharp(A) = c^+$ by a preceding case. So we assume that AA^{-1} is a neighborhood of e and take another neighborhood U of e with $UU^{-1} \subset AA^{-1}$.

Since finitely many shifts of U cover the group G , we get $\mu(U) > 0$. Now given any subset $B \subset G$ with $B^{-1}B \cap AA^{-1} = \{e\}$, we get $B^{-1}B \cap UU^{-1} = \{e\}$. The latter equality implies that the family $\{xU\}_{x \in B}$ is disjoint and then $1 = \mu(G) \geq \mu(BU) = |B|\mu(U)$ implies that $|B| \leq 1/\mu(U)$. Consequently, the packing index $\text{pack}(A) \leq 1/\mu(U)$ is finite and so is its sharp version $\text{pack}^\sharp(A)$. \square

The equality $\text{pack}(A) = \sup\{\kappa : \kappa < \text{pack}^\sharp(A)\}$ and Theorem 2.2 imply the following.

Corollary 2.3 *If A is a σ -compact subset A of a Polish group G , then $\text{pack}(A) \in \mathbb{N} \cup \{\aleph_0, c\}$. Moreover, if the set A is compact, then*

1. $\text{pack}(A) = c$ provided G is not locally compact;
2. $\text{pack}(A) \in \{\aleph_0, c\}$ provided G is locally compact but not compact;
3. $\text{pack}(A) \in \mathbb{N} \cup \{c\}$ provided G is compact.

In light of Corollary 2.3, the following two open questions arise naturally.

Question 2.4 *Are there a compact group G and a σ -compact subset $A \subset G$ with $\text{pack}(A) = \aleph_0$?*

Question 2.5 *Are there a Polish group G and a Borel subset $A \subset G$ with $\aleph_0 < \text{pack}(A) < c$?*

The latter question does not reduce to the σ -compact case because of the following example (in which $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ stands for the circle).

Proposition 2.6 *The countable product $G = \mathbb{T}^\omega$ contains a G_δ -subset $A \subset G$ such that $\text{pack}(A) = c$; however, each σ -compact subset $B \subset \mathbb{T}^\omega$ containing A has $\text{pack}(B) < \aleph_0$.*

Proof Let $q : \mathbb{R} \rightarrow \mathbb{T} = \mathbb{R}/\mathbb{Z}$ denote the quotient map, $J = q([0, \frac{1}{2}))$ be the half-circle, $I = \bar{J} = q([0, \frac{1}{2}])$ be its closure, and $D = q(\{0, \frac{1}{2}\})$ be two opposite points on \mathbb{T} . It is clear that $D^{-1}D \cap JJ^{-1} = \{e\}$ while $I \cdot I^{-1} = \mathbb{T}$.

It follows that $A = J^\omega$ is a G_δ -subset of \mathbb{T}^ω with $\text{pack}(A) = |D^\omega| = c$ because $(D^\omega)^{-1}D^\omega \cap AA^{-1} = \{e\}$.

Now given any σ -compact subset $B \supset A$ in \mathbb{T}^ω , we must check that $\text{pack}(B) < \aleph_0$. Replacing B by $B \cap I^\omega$, if necessary, we can assume that $B \subset I^\omega$. Since $B \subset I^\omega$ contains the dense G_δ -subset J^ω of I^ω , the standard application of the Baire Theorem yields a nonempty open subset $U \subset I^\omega$ with $U \subset B$. Without loss of generality we can assume that U is of the basic form $U = V \times I^{\omega \setminus n}$ for some $n \in \omega$ and some open set $V \subset I^n$. Observe that

$$UU^{-1} = VV^{-1} \times I^{\omega \setminus n}(I^{\omega \setminus n})^{-1} = VV^{-1} \times \mathbb{T}^{\omega \setminus n}$$

is an open neighborhood of e in \mathbb{T}^ω . Consequently, $BB^{-1} \supset UU^{-1}$ is also a neighborhood of e in \mathbb{T}^ω . Proceeding as in the proof of the last assertion of Theorem 2.2, we can see that

$$\text{pack}(B) \leq 1/\mu(V \times \mathbb{T}^{\omega \setminus n}) < \aleph_0. \quad \square$$

3 The Packing Indices of Analytic Sets in Polish Groups

In this section we shall give a partial answer to Question 2.5 related to the packing indices of Borel subsets in Polish groups. It is well known that each Borel subset of a Polish space is analytic. We recall that a metrizable space X is *analytic* if X is a continuous image of a Polish space. A space X is *coanalytic* if for some Polish space Y containing X the complement $Y \setminus X$ is analytic. The classes of analytic and coanalytic spaces are denoted by Σ_1^1 and Π_1^1 , respectively. It is well known that the intersection $\Delta_1^1 = \Sigma_1^1 \cap \Pi_1^1$ coincides with the class of all absolute Borel (metrizable separable) spaces. By \mathcal{K}_σ and \mathcal{G}_δ we denote the classes of σ -compact and Polish spaces, respectively.

We shall say that a subset $X \subset 2^\omega \times 2^\omega$ contains a square of size κ if there is a subset $A \subset 2^\omega$ with $A \times A \subset X$ and $|A \times A| = \kappa$.

Given a class \mathcal{C} of spaces denote by $\mathfrak{s}q(\mathcal{C})$ the smallest cardinal κ such that each subspace $X \in \mathcal{C}$ of $2^\omega \times 2^\omega$ that contains a square of size κ contains a square of size \mathfrak{c} . Theorem 2.2 of [7] (applied in the proof of Theorem 2.2) guarantees that $\mathfrak{s}q(\mathcal{G}_\delta) = \aleph_1$. On the other hand, $\mathfrak{s}q(\mathcal{K}_\sigma) \geq \min\{\aleph_2, \mathfrak{c}\}$; see [6]. It is clear that $\mathfrak{s}q(\mathcal{K}_\sigma) = \mathfrak{s}q(\Sigma_1^1) = \mathfrak{s}q(\Pi_1^1) = \mathfrak{c}$ under the Continuum Hypothesis. Yet, the strict inequality $\mathfrak{s}q(\Pi_1^1) < \mathfrak{c}$ is consistent with ZFC+MA (see [10, 1.9, 1.10]).

Proposition 3.1 *Let A be an analytic subset of a Polish group G . If $\text{pack}^\sharp(A) > \mathfrak{s}q(\Pi_1^1)$, then $\text{pack}^\sharp(A) = \mathfrak{c}^+$ and $\text{pack}(A) = \mathfrak{c}$.*

Proof Using the fact that each Polish space is a continuous one-to-one image of a zero-dimensional Polish space, we can show that $\mathfrak{s}q(\Pi_1^1)$ coincides with the smallest cardinal κ such that for any Polish space X a coanalytic subset $C \subset X \times X$ contains a square of size \mathfrak{c} provided that C contains a square of size $\geq \kappa$.

Given an analytic subset A of a Polish group G we can see that both sets AA^{-1} and $AA^{-1} \setminus \{e\}$ are analytic and thus the set $C = \{(x, y) \in G \times G : y^{-1}x \notin AA^{-1} \setminus \{e\}\}$ is coanalytic.

Assuming that $\text{pack}^\sharp(A) > \mathfrak{s}q(\Pi_1^1)$, we can find a subset $S \subset G$ of size $|S| \geq \mathfrak{s}q(\Pi_1^1)$ such that the family $\{xA\}_{x \in S}$ is disjoint. The latter is equivalent to $S^{-1} \cdot S \subset G \setminus (AA^{-1} \setminus \{e\})$ and to $S \times S \subset C$. By the equivalent definition of $\mathfrak{s}q(\Pi_1^1)$ (with 2^ω replaced by any Polish space), the coanalytic subset $C \subset G \times G$ contains a square $K \times K$ of size \mathfrak{c} (because it contains the square $S \times S$ of cardinality

$|S \times S| \geq \mathfrak{s}q(\Pi_1^1)$). It follows from $K \times K \subset C$ that the family $\{xA\}_{x \in K}$ is disjoint and thus $\text{pack}^\sharp(A) \geq |K|^+ = c^+$ and $\text{pack}(A) = c$. \square

A similar result holds for the *almost packing index*

$$\text{Pack}(A) = \sup\{|S| : S \subset G \text{ and } \{xA\}_{x \in S} \text{ is almost disjoint}\}$$

and the *sharp almost packing index*

$$\text{Pack}^\sharp(A) = \sup\{|S|^+ : S \subset G \text{ and } \{xA\}_{x \in S} \text{ is almost disjoint}\}$$

of A . We recall that $\{xA\}_{x \in S}$ is *almost disjoint* in G if $|xA \cap yA| < |G|$ for any distinct points $x, y \in S$.

In the proof of the following theorem we shall use a known fact of the Descriptive Set Theory saying that for a Borel subset $A \subset X \times Y$ in the product of two Polish spaces the set $\{y \in Y : |A \cap (X \times \{y\})| \leq \aleph_0\}$ is coanalytic in Y (see [5, 18.9]).

Proposition 3.2 *Let A be a Borel subset of a Polish group G . If $\text{Pack}^\sharp(A) > \mathfrak{s}q(\Pi_1^1)$, then $\text{Pack}^\sharp(A) = c^+$ and $\text{Pack}(A) = c$.*

Proof It follows from $\text{Pack}^\sharp(A) > \mathfrak{s}q(\Pi_1^1) > \aleph_0$ that the Polish space G is uncountable and hence $|G| = c$.

Let us show that the subset $C = \{x \in G : |A \cap xA| \leq \aleph_0\}$ is coanalytic. Consider the homeomorphism $h : G \times G \rightarrow G \times G, h : (x, y) \mapsto (x, y^{-1}x)$, and the Borel subset $B = h(A \times A) \subset G \times G$. Since $C = \{z \in G : |B \cap (G \times \{z\})| \leq \aleph_0\}$, we can apply the mentioned result [5, 18.9] to conclude that the set C is coanalytic. Then the set $D = \{(x, y) \in G \times G : y^{-1}x \in C \cup \{e\}\}$ is coanalytic being the preimage of the coanalytic subset under a continuous map between Polish spaces.

Assuming that $\text{Pack}^\sharp(A) > \mathfrak{s}q(\Pi_1^1)$, we can find a subset $S \subset X$ such that $|S| \geq \mathfrak{s}q(\Pi_1^1)$ and the family $\{xA\}_{x \in S}$ is almost disjoint. Then for any distinct $x, y \in S$ the intersection $xA \cap yA$, being a Borel subset of cardinality $|xA \cap yA| < |G| = c$, is at most countable. Consequently, $y^{-1}x \in C$ and thus $S \times S \subset D$. By the equivalent definition of $\mathfrak{s}q(\Pi_1^1)$, the coanalytic set D contains a square $K \times K$ of size c . It follows from $K^{-1}K \subset C$ that the family $\{xA\}_{x \in K}$ is almost disjoint. Consequently, $c^+ = |K|^+ \leq \text{Pack}^\sharp(A) \leq |G|^+ = c^+$ and $\text{Pack}(A) = c$. \square

It would be interesting to study the cardinals $\mathfrak{s}q(\mathcal{C})$ for various descriptive classes \mathcal{C} . If such a class \mathcal{C} contains the square of a countable metrizable space, then $\aleph_1 \leq \mathfrak{s}q(\mathcal{C}) \leq c$ and thus $\mathfrak{s}q(\mathcal{C})$ falls into the category of the so-called small uncountable cardinals; see [14]. However, unlike other typical small uncountable cardinals, $\mathfrak{s}q(\mathcal{C})$ does not collapse to c under the Martin Axiom (see [10]).

Problem 3.3 Explore possible values and inequalities between classical small uncountable cardinals and the cardinals $\mathfrak{s}q(\mathcal{C})$ for various descriptive classes \mathcal{C} .

4 The Packing Index and Unions

Let us note that the union of two subsets $A, B \subset G$ with large packing index can have the smallest possible packing index. A suitable example is given by the sets $A = \mathbb{R} \times \{0\}$ and $B = \{0\} \times \mathbb{R}$ on the plane \mathbb{R}^2 . They have packing indices $\text{pack}(A) = \text{pack}(B) = c$ but $\text{pack}(A \cup B) = 1$. The same is true for each nondiscrete Polish Abelian group.

Theorem 4.1 *Each nondiscrete Polish Abelian group G contains two closed subsets $A, B \subset G$ such that $\text{pack}^\sharp(A) = \text{pack}^\sharp(B) = c^+$ but $\text{pack}(A \cup B) = \text{Pack}(A \cup B) = 1$.*

Proof Fix an invariant complete metric ρ generating the topology of the Polish Abelian group G ; see [5, §9.A]. Since G is Abelian, we use the additive notation for the group operation on G . The neutral element of G will be denoted by 0.

We define a subset D of G to be ε -separated if $\rho(x, y) \geq \varepsilon$ for any distinct points $x, y \in D$. By the Zorn lemma, each ε -separated subset can be enlarged to a maximal ε -separated subset of G .

Put $\varepsilon_{-1} = \varepsilon_0 = 1$ and choose a maximal $2\varepsilon_0$ -separated subset $H_0 \subset G$ containing zero. Proceeding by induction we shall define a sequence $(h_n)_{n \in \mathbb{N}} \subset G$ of points, a sequence $(\varepsilon_n)_{n \in \omega}$ of positive real numbers and a sequence $(H_n)_{n \in \omega}$ of subsets of G such that for every $n > 0$,

- (i) $B(0, \varepsilon_{n-1}) \setminus B(0, 33\varepsilon_n)$ is nonempty and $\varepsilon_n < 2^{-6}\varepsilon_{n-1}$;
- (ii) $\|h_n\| = 5\varepsilon_n$; and
- (iii) $H_n \supset \{0, h_n\}$ is a maximal $2\varepsilon_n$ -separated subset in $B(0, 8\varepsilon_{n-1})$.

It follows from (i) that the series $\sum_{n \in \omega} \varepsilon_n$ is convergent and thus for any sequence $x_n \in H_n$, $n \in \omega$, the series $\sum_{n \in \omega} x_n$ is convergent (because $\|x_n\| < 8\varepsilon_{n-1}$ for all $n \in \mathbb{N}$ according to (iii)). Therefore, the following sums are well defined:

$$\Sigma_0 = \left\{ \sum_{n \in \omega} x_{2n} : (x_{2n})_{n \in \omega} \in \prod_{n \in \omega} H_{2n} \right\},$$

$$\Sigma_1 = \left\{ - \sum_{n \in \omega} x_{2n+1} : (x_{2n+1})_{n \in \omega} \in \prod_{n \in \omega} H_{2n+1} \right\}.$$

Let A and B be the closures of the sets Σ_0 and Σ_1 in G . It remains to prove that the sets A, B have the desired properties: $\text{pack}^\sharp(A) = \text{pack}^\sharp(B) = c^+$ and $\text{Pack}(A \cup B) = 1$. This will be done in three steps.

1 First we prove that $\text{pack}^\sharp(A) = c^+$. By Lemma 2.1, this equality will follow as soon as we check that $\overline{A - A}$ is not a neighborhood of the neutral element 0 in G . It suffices to find, for every $k \in \omega$, a point $g \in B(0, \varepsilon_{2k}) \setminus \overline{A - A}$. By condition (i), there is a point $g \in G$ with $33\varepsilon_{2k+1} \leq \|g\| < \varepsilon_{2k}$. We claim that $g \notin \overline{A - A} = \overline{\Sigma_0 - \Sigma_0}$. More precisely,

$$\text{dist}(g, \overline{A - A}) = \text{dist}(g, \Sigma_0 - \Sigma_0) \geq \min\{\varepsilon_{2k+1}, \varepsilon_{2k}/2\}.$$

Take any two distinct points $x, y \in \Sigma_0$ and find infinite sequences $(x_{2n})_{n \in \omega}, (y_{2n})_{n \in \omega} \in \prod_{n \in \omega} H_{2n}$ with $x = \sum_{n \in \omega} x_{2n}$ and $y = \sum_{n \in \omega} y_{2n}$.

Let $m = \min\{n \in \omega : x_{2n} \neq y_{2n}\}$. If $m \geq k + 1$, then

$$\begin{aligned} \|x - y\| &= \left\| \sum_{n \geq m} x_{2n} - y_{2n} \right\| \leq \sum_{n \geq m} \|x_{2n}\| + \|y_{2n}\| \leq \\ &\leq 2 \sum_{n \geq m} 8\varepsilon_{2n-1} \leq 32 \varepsilon_{2m-1} \leq 32 \varepsilon_{2k+1} < \|g\| - \varepsilon_{2k+1} \end{aligned}$$

and hence $\rho(x - y, g) \geq \varepsilon_{2k+1}$.

If $m \leq k$, then

$$\begin{aligned} \|x - y\| &= \|(x_{2m} - y_{2m}) + \sum_{n>m} (x_{2n} - y_{2n})\| \geq \\ &\geq \|x_{2m} - y_{2m}\| - \sum_{n>m} (\|x_{2n}\| + \|y_{2n}\|) \geq \\ &\geq 2\varepsilon_{2m} - 32\varepsilon_{2m+1} \geq \frac{3}{2}\varepsilon_{2m} > \|g\| + \frac{1}{2}\varepsilon_{2m} \end{aligned}$$

and again $\rho(x - y, g) \geq \frac{1}{2}\varepsilon_{2m} \geq \frac{1}{2}\varepsilon_{2k}$.

2 In the same manner we can prove that $\text{pack}^\sharp(B) = c^+$.

3 It remains to prove that $\text{Pack}(A \cup B) = 1$. First we recall some standard notation. Denote by $2^{<\omega} = \bigcup_{n \in \omega} 2^n$ the set of finite binary sequences. For any sequence $s = (s_0, \dots, s_{n-1}) \in 2^{<\omega}$ and $i \in 2 = \{0, 1\}$ by $|s| = n$ we denote the length of s and by $s \hat{\ } i = (s_0, \dots, s_{n-1}, i)$ the concatenation of s and i . For a finite or infinite binary sequence $s = (s_i)_{i < n}$ and $l \leq n$, let $s|l = (s_i)_{i < l}$. The set 2^ω is a tree with respect to the partial order: $s \leq t$ if and only if $s = t|l$ where $l = |s| \leq |t|$.

The equality $\text{pack}(A \cup B) = 1$ will follow as soon as we prove that $|A \cap (g + B)| \geq c$ for all $g \in G$. We shall construct a sequence of points $\{x_s\}_{s \in 2^{<\omega}}$ such that for every sequence $s \in 2^{<\omega}$ the following conditions hold:

1. $x_s \in H_{|s|} \subset B(0, 8\varepsilon_{|s|-1})$;
2. $\|x_{s \hat{\ } 0} - x_{s \hat{\ } 1}\| > \varepsilon_n$; and
3. $\|g - \sum_{t \leq s} x_t\| < 7\varepsilon_{|s|}$.

We start by choosing a point $x_\emptyset \in H_0$ with $\rho(x_\emptyset, g) < 2\varepsilon_{-1} = 2\varepsilon_0$. Such a point x_\emptyset exists because H_0 is a maximal $(0, 2\varepsilon_0)$ -separated set in G . Next, we proceed by induction.

Suppose that for some n the points $x_s, s \in 2^{<n}$, have been constructed. Given a sequence $s \in 2^{n-1}$ we need to define the points $x_{s \hat{\ } 0}$ and $x_{s \hat{\ } 1} \in H_n$. Let $g_s = g - \sum_{t \leq s} x_t$. Since $\|g_s + h_n\| \leq \|g_s\| + \|h_n\| < 7\varepsilon_{n-1} + 5\varepsilon_n < 8\varepsilon_{n-1}$ and H_n is a maximal $2\varepsilon_n$ -separated subset in $B(0, 8\varepsilon_{n-1})$, there are two points $x_{s \hat{\ } 0}, x_{s \hat{\ } 1} \in H_n$ with $\rho(g_s, x_{s \hat{\ } 0}) < 2\varepsilon_n$ and $\rho(g_s + h_n, x_{s \hat{\ } 1}) < 2\varepsilon_n$. The condition (2) follows from

$$\begin{aligned} \|x_{s \hat{\ } 0} - x_{s \hat{\ } 1}\| &\geq \|g_s - (g_s + h_n)\| - \|g_s - x_{s \hat{\ } 0}\| - \|g_s + h_n - x_{s \hat{\ } 1}\| > \\ &> 5\varepsilon_n - 2\varepsilon_n - 2\varepsilon_n = \varepsilon_n. \end{aligned}$$

The condition (3) follows from the estimates

$$\|g - \sum_{t \leq s \hat{\ } 0} x_t\| = \|g - x_{s \hat{\ } 0} - \sum_{t \leq s} x_t\| = \|g_s - x_{s \hat{\ } 0}\| < 2\varepsilon_n = 2\varepsilon_{|s \hat{\ } 0|}$$

and

$$\begin{aligned} \|g - \sum_{t \leq s \hat{\ } 1} x_t\| &= \|g - x_{s \hat{\ } 1} - \sum_{t \leq s} x_t\| = \|g_s + h_n - x_{s \hat{\ } 1} - h_n\| \leq \\ &\leq \|g_s + h_n - x_{s \hat{\ } 1}\| + \|h_n\| < 2\varepsilon_n + 5\varepsilon_n = 7\varepsilon_{|s \hat{\ } 1|}. \end{aligned}$$

After completing the inductive construction, we can use the condition (3) to conclude that for every infinite binary sequence $s \in 2^\omega$ the following holds:

$$g = \sum_{n \in \omega} x_{s|n} = \sum_{n \in \omega} x_{s|2n} + \sum_{n \in \omega} x_{s|2n+1}.$$

We claim that the set

$$D_0 = \left\{ \sum_{n \in \omega} x_{s|2n} : s \in 2^\omega \right\}$$

lies in the intersection $A \cap (g + B)$. It is clear that $D_0 \subset \Sigma_0 \subset A$. To see that $D_0 \subset g + B$, take any point $x \in D_0$ and find an infinite binary sequence $s \in 2^\omega$ with $x = \sum_{n \in \omega} x_{s|2n}$. Then

$$x = \sum_{n \in \omega} x_{s|2n} + \sum_{n \in \omega} x_{s|2n+1} - \sum_{n \in \omega} x_{s|2n+1} \in g + \Sigma_1 \subset g + B.$$

It remains to prove that $|D_0| \geq c$. Note that the set D_0 , being a continuous image of the Cantor cube 2^ω , is compact. Now the equality $|D_0| = c$ will follow as soon as we check that D_0 has no isolated points. Given any sequence $s \in 2^\omega$ and $\delta > 0$, we must find a sequence $t \in 2^\omega$ such that

$$0 < \left\| \sum_{n \in \omega} x_{s|2n} - \sum_{n \in \omega} x_{t|2n} \right\| < \delta.$$

Find even number $2m \in \omega$ such that $\sum_{n \geq m} 20\varepsilon_{2n-1} < \delta$ and take any sequence $t \in 2^\omega$ such that $t|2m - 1 = s|2m - 1$ but $t|2m \neq s|2m$. Then

$$\begin{aligned} \left\| \sum_{n \in \omega} x_{s|2n} - \sum_{n \in \omega} x_{t|2n} \right\| &= \left\| \sum_{n \geq m} x_{s|2n} - \sum_{n \geq m} x_{t|2m} \right\| \leq \\ &\leq \sum_{n \geq m} \|x_{s|2n}\| + \|x_{t|2m}\| \leq \sum_{n \geq m} 32\varepsilon_{2n-1} < \delta. \end{aligned}$$

On the other hand, the lower bound $\|x_{s|2m} - x_{t|2m}\| > \varepsilon_{2m}$, supplied by (2), implies

$$\begin{aligned} \left\| \sum_{n \in \omega} x_{s|2n} - \sum_{n \in \omega} x_{t|2n} \right\| &= \left\| \sum_{n \geq m} x_{s|2n} - \sum_{n \geq m} x_{t|2n} \right\| \geq \\ &\geq \|x_{s|2m} - x_{t|2m}\| - \left\| \sum_{n > m} (x_{s|2n} - x_{t|2n}) \right\| > \\ &> \varepsilon_{2m} - \sum_{n > m} 16\varepsilon_{2n-1} > \varepsilon_{2m} - 32\varepsilon_{2m+1} > 0 \end{aligned}$$

(the latter two inequalities follow from (i)). Now we see that $|D_0| = c$ and thus $|A \cap (g + B)| \geq |D_0| = c$, which implies that $\text{Pack}(A \cup B) = 1$. □

5 Relation of the Packing Index to Other Notions of Smallness

Taking into account that a subset A with large packing index $\text{pack}(A)$ is geometrically small, it is natural to consider the relation of the packing index to other known concepts of smallness, in particular, the smallness in the sense of category or measure.

We recall that a subset A of a topological space X is said to be *meager* if A can be written as a countable union of nowhere dense subsets. We shall need the following classical fact.

Proposition 5.1 (Banach-Kuratowski-Pettis) *For any analytic nonmeager subset A of a Polish group G , the set AA^{-1} contains a neighborhood of the neutral element of G .*

A similar result holds for analytic subsets that are not Haar null. We recall that a subset A of a topological group G is called *Haar null* if there is a Borel probability measure μ on G such that $\mu(xAy) = 0$ for all $x, y \in G$. This notion was introduced by Christensen [3] and thoroughly studied in [13] and [12]. In particular, a subset A of a locally compact group G is Haar null if and only if A has zero Haar measure. Nevertheless, Haar null sets still exist in nonlocally compact groups (admitting no invariant measure).

Proposition 5.2 (Christensen) *If an analytic subset A of a Polish Abelian group G is not Haar null, then AA^{-1} contains a neighborhood of the neutral element of G .*

We shall use these propositions to prove the following theorem.

Theorem 5.3 *If an analytic subset A of a Polish Abelian group G has packing index $\text{pack}(A) > \aleph_0$, then A is meager and Haar null.*

Proof Suppose not. Then we can apply Propositions 5.1 or 5.2 to conclude that AA^{-1} contains a neighborhood U of the neutral element e of G . Find another neighborhood $V \subset G$ of e with $VV^{-1} \subset U \subset AA^{-1}$.

Since $\text{pack}(A) > \aleph_0$, there is an uncountable subset $S \subset X$ such that the family $\{xA\}_{x \in S}$ is disjoint, which is equivalent to $S^{-1}S \cap AA^{-1} = \{e\}$. It follows by the choice of V that $S^{-1}S \cap VV^{-1} \subset S^{-1}S \cap AA^{-1} = \{e\}$ and thus the family $\{xV\}_{x \in S}$ is disjoint. Since V is an open neighborhood of e , the set S , being discrete in G , is at most countable. This contradiction completes the proof. \square

It should be mentioned that Theorem 5.3 cannot be reversed.

Theorem 5.4 *Each nondiscrete Polish Abelian group G contains a closed nowhere dense Haar null subset C with $\text{Pack}(C) = 1$.*

Proof By Theorem 4.1, the group G contains two closed subsets $A, B \subset G$ with $\text{pack}(A) = \text{pack}(B) = c$ and $\text{Pack}(A \cup B) = 1$. By Theorem 5.3, the sets A, B are Haar null. Then the union $C = A \cup B$ is Haar null, and, being closed in G , is nowhere dense. \square

6 Constructing Small Subsets with a Given (Sharp) Packing Index

In this section we shall develop Theorem 5.3 and shall prove that nondiscrete Polish Abelian groups contain nowhere dense Haar null sets of arbitrary (sharp) packing index.

In the following theorem, we put $[G]_p = \{x \in G : x^p = e\}$ for a group G , where $p \in \mathbb{N}$.

Theorem 6.1 *For a nondiscrete Polish Abelian group G and a cardinal $\kappa \in [2, c^+]$ the following conditions are equivalent:*

- (1) *there is a subset $A \subset G$ with $\text{pack}^\sharp(A) = \kappa$;*
- (2) *there is a nowhere dense Haar null subset $A \subset G$ with $\text{pack}^\sharp(A) = \text{Pack}^\sharp(A) = \kappa$; and*
- (3) *if $|G/[G]_2| \leq 2$, then $\kappa \neq 4$; if $G = [G]_3$, then $\kappa \neq 3$.*

Taking into account that $\text{pack}(A) = \sup\{\kappa : \kappa < \text{pack}^\sharp(A)\}$, we can apply Theorem 6.1 to deduce the following corollary.

Corollary 6.2 *For a nondiscrete Polish Abelian group G and a nonzero cardinal $\kappa \leq \mathfrak{c}$, the following conditions are equivalent:*

- (1) *there is a subset $A \subset G$ with $\text{pack}(A) = \kappa$;*
- (2) *there is a nowhere dense Haar null subset $A \subset G$ with $\text{pack}(A) = \text{Pack}(A) = \kappa$; and*
- (3) *if $|G/[G]_2| \leq 2$, then $\kappa \neq 3$ and if $G = [G]_3$, then $\kappa \neq 2$.*

Theorem 6.1 follows immediately from Theorem 5.4 and the following combinatorial result.

Theorem 6.3 *For an infinite Abelian group G , a cardinal $2 \leq \kappa \leq |G|^+$, and a subset $L \subset G$ with $\text{Pack}(L) = 1$ the following conditions are equivalent:*

- (1) *there is a subset $A \subset G$ with $\text{pack}^\sharp(A) = \kappa$;*
- (2) *there is subset $A \subset L$ with $\text{pack}^\sharp(A) = \text{Pack}^\sharp(A) = \kappa$; and*
- (3) *if $|G/[G]_2| \leq 2$, then $\kappa \neq 4$ and if $G = [G]_3$, then $\kappa \neq 3$.*

Proof The implication (2) \Rightarrow (1) is trivial while (1) \Rightarrow (3) was proved in [8]. To prove the implication (3) \Rightarrow (2), assume that the cardinal κ satisfies the condition (3). If $\kappa = |G|^+$, then for the set A we can take a singleton $A = \{a\}$ with $a \in L$. So, we assume that $\kappa \leq |G|$.

The construction of a set $A \subset G$ with $\text{pack}^\sharp(A) = \text{Pack}^\sharp(A) = \kappa$ is based on the following lemma whose proof will be given after the proof of the theorem.

Lemma 6.4 *The group G contains a subset $\mathbb{B}_\kappa = -\mathbb{B}_\kappa$ of G having the following three properties:*

- (a) *for every cardinal $\alpha < \kappa$, there is a subset $B_\alpha \subset G$ of size $|B_\alpha| = \alpha$ with $B_\alpha - B_\alpha \subset \mathbb{B}_\kappa$;*
- (b) *$B - B \not\subset \mathbb{B}_\kappa$ for every subset $B \subset G$ of size $|B| = \kappa$; and*
- (c) *$L \cap (g+L) \not\subset F + \mathbb{B}_\kappa$ for any $g \in G$ and any subset $F \subset G$ of size $|F| < |G|$.*

Without loss of generality we can assume that $0 \in L$.

Let $\mathbb{B}_\kappa^\circ = \mathbb{B}_\kappa \setminus \{0\}$. We shall construct a subset $A \subset L$ such that $(\mathbb{B}_\kappa^\circ + A) \cap A = \emptyset$. Moreover, the subset A will be constructed so that $G \setminus \mathbb{B}_\kappa^\circ \subset A - A$.

Let $\lambda = |G|$ and $G \setminus \mathbb{B}_\kappa^\circ = \{g_\alpha : \alpha < \lambda\}$ be an enumeration of $G \setminus \mathbb{B}_\kappa^\circ$ such that for every $g \in G \setminus \mathbb{B}_\kappa^\circ$ the set $\{\alpha < \lambda : g_\alpha = g\}$ has cardinality λ .

The set A will be of the form $A = \bigcup_{\alpha < \lambda} \{a_\alpha, g_\alpha + a_\alpha\}$ for a suitable sequence $\{a_\alpha\}_{\alpha < \lambda} \subset G$ such that $(\mathbb{B}_\kappa^\circ + A) \cap A = \emptyset$. We define this sequence by induction.

We start with $a_0 = 0$. Assuming that for some ordinal $\alpha < \lambda$ the points $a_\beta, \beta < \alpha$, have been constructed, put $A_\alpha = \{a_\beta, g_\beta + a_\beta : \beta < \alpha\}$. According to the property (c), we can pick a point $a_\alpha \in L \cap (g_\alpha + L)$ so that

$$a_\alpha \notin (A_\alpha + \mathbb{B}_\kappa) \cup (A_\alpha - g_\alpha + \mathbb{B}_\kappa).$$

This gives $(\mathbb{B}_\kappa^\circ + A) \cap A = \emptyset$.

It remains to show that $\kappa \leq \text{pack}^\sharp(A) \leq \text{Pack}^\sharp(A) \leq \kappa$. The inequality $\text{pack}^\sharp(A) \geq \kappa$ will follow as soon as we check that $\text{pack}^\sharp(A) > \alpha$ for all cardinals $\alpha < \kappa$.

According to the property (a) of the set \mathbb{B}_κ , for each cardinal $\alpha < \kappa$ there is a subset $B_\alpha \subset \mathbb{B}_\kappa$ of size α such that $B_\alpha - B_\alpha \subset \mathbb{B}_\kappa$. From the fact that $(\mathbb{B}_\kappa^\circ + A) \cap A = \emptyset$ we conclude that $(b - b' + A) \cap A = \emptyset$ for all distinct $b, b' \in B_\alpha$. Thus the family $\{b + A : b \in B_\alpha\}$ is disjoint, witnessing that $\text{pack}^\sharp(A) \geq \alpha^+ > \alpha$.

The inequality $\text{Pack}^\#(A) \leq \kappa$ will follow as soon as we check that for every subset $B \subset G$ of size $|B| = \kappa$ the indexed family $\{b + A\}_{b \in B}$ is not almost disjoint. Given any subset $B \subset G$ with $|B| = \kappa$, we can use property (b) to find points $b, b' \in B$ such that $b - b' \notin \mathbb{B}_\kappa$. The choice of the enumeration $\{g_\alpha\}_{\alpha < \lambda}$ guarantees that the set $\Lambda = \{\alpha < \lambda : g_\alpha = b - b'\}$ has cardinality $|\Lambda| = \lambda = |G|$. Observe that for every $\alpha \in \Lambda$ we get $a_\alpha \in A \cap (A - g_\alpha) = A \cap (A + b' - b)$. Consequently,

$$|(b + A) \cap (b' + A)| = |A \cap (A + b' - b)| = |\Lambda| = |G|$$

witnessing that the family $\{b + A\}_{b \in B}$ fails to be almost disjoint. □

Proof of Lemma 6.4 Given a cardinal $\kappa \leq |G|$ and a subset $L \subset G$ with $\text{Pack}(L) = 1$, we need to construct a subset $\mathbb{B}_\kappa \subset G$ possessing the properties (a)–(c) of Lemma 6.4. By [8], the group G contains a subset \mathbb{B}_κ of size $|\mathbb{B}_\kappa| = \kappa$ such that

- (i) for every cardinal $\alpha < \kappa$, there is a subset $B_\alpha \subset G$ of size $|B_\alpha| = \alpha$ with $B_\alpha - B_\alpha \subset \mathbb{B}_\kappa$;
- (ii) $B - B \not\subset \mathbb{B}_\kappa$ for every subset $B \subset G$ of size $|B| = \kappa$.

If $\kappa < |G|$ then, this subset \mathbb{B}_κ satisfies the requirements of the lemma. So it remains to consider the case $\kappa = |G|$.

Let $G = \{g_\alpha : \alpha < \kappa\}$ be an enumeration of the group G such that $g_0 = 0$. For every ordinal $\alpha < \kappa$, put $G_\alpha = \{g_\beta, -g_\beta : \beta < \alpha\}$. We put

$$\mathbb{B}_\kappa = \bigcup_{\alpha < \kappa} B_\alpha - B_\alpha$$

where a set $B_\alpha = \{b_\alpha^\beta : \beta < \alpha\} \subset G$ of size $|\alpha|$ will be chosen later.

To simplify notation we shall write $\mathbb{B}_{<\alpha}$ instead of $\bigcup_{\beta < \alpha} (B_\beta - B_\beta)$ and $\mathbb{B}_{>\alpha}$ instead of $\bigcup_{\alpha \leq \beta < \kappa} (B_\beta - B_\beta)$. By $B_\alpha^{<\beta}$ we shall denote the initial interval $\{b_\alpha^\gamma : \gamma < \beta\}$ of B_α .

Now we are in a position to define a sequence of sets B_α forcing the set \mathbb{B}_κ to satisfy the properties (a) and (c) of Lemma 6.4. To ensure property (c) we will also construct a transfinite sequence of points $(h_\alpha)_{\alpha < \kappa}$ of $L \cap (g_\alpha + L)$ such that $h_\alpha \notin G_\alpha + \mathbb{B}_\kappa$.

We start putting $B_0 = \{0\}$ and taking any nonzero point $h_0 \in L$. Assume that for some ordinal $\alpha < \kappa$ the sets B_β and the points h_β , $\beta < \alpha$, have been constructed. Then pick any point $h_\alpha \in L \cap (g_\alpha + L)$ with

$$h_\alpha \notin G_\alpha + \mathbb{B}_{<\alpha}.$$

Such a point exists because the size of the set $G_\alpha + \mathbb{B}_{<\alpha}$ does not exceed $\aleph_0 \cdot |\alpha| < \kappa = |G|$. Let

$$H_\alpha = \{h_\beta, -h_\beta : \beta \leq \alpha\}.$$

Next we define inductively elements of $B_\alpha = \{b_\alpha^\beta : \beta < \alpha\}$. We pick any b_α^0 with $b_\alpha^0 \in G \setminus \mathbb{B}_{<\alpha}$. Next we choose $b_\alpha^\beta \in G$ so that

- (1) $b_\alpha^\beta \notin B_\alpha^{<\beta} + G_\alpha + \mathbb{B}_{<\alpha}$;
- (2) $b_\alpha^\beta \notin B_\alpha^{<\beta} - B_\alpha^{<\beta} + B_\alpha^{<\beta} + G_\alpha$; and
- (3) $b_\alpha^\beta \notin B_\alpha^{<\beta} + G_\alpha + H_\alpha$.

To ensure properties (1), (2), and (3), we have to avoid the sets of size $< |G|$.

Now let us prove that the constructed set \mathbb{B}_κ satisfies the properties (a)–(c) of Lemma 6.4. In fact, the property (a) is evident while (c) follows immediately from (3). It remains to prove the following claim.

Claim *The set \mathbb{B}_κ has property (b).*

Let B_κ be a subset of G of size $|B_\kappa| = \kappa$. Fix any pairwise distinct points $c_1, c_2, c_3 \in B_\kappa$. If $B_\kappa - B_\kappa \subset \mathbb{B}_\kappa$ then $B_\kappa \subset \bigcap_{i=1}^3 (c_i + \mathbb{B}_\kappa)$ and $\kappa = |B_\kappa| \leq |\bigcap_{i=1}^3 (c_i + \mathbb{B}_\kappa)|$.

So to prove our claim it is enough to show that $|\bigcap_{i=1}^3 (c_i + \mathbb{B}_\kappa)| < \kappa$. Find an ordinal $\alpha < \kappa$ such that $c_p - c_q \in G_\alpha$ for any $1 \leq p, q \leq 3$. Assuming that $|\bigcap_{i=1}^3 (c_i + \mathbb{B}_\kappa)| = \kappa$ we may find a point $b \in \bigcap_{i=1}^3 (c_i + \mathbb{B}_{>\alpha}) \setminus \{c_i\}$. A contradiction will be reached in three steps.

Step 1 *First, show that there is $\beta > \alpha$ with $b \in \bigcap_{i=1}^3 (c_i + B_\beta - B_\beta)$.*

Otherwise, $b - c_p \in B_\gamma - B_\gamma$ and $b - c_q \in B_\beta - B_\beta$ for some $\gamma > \beta > \alpha$ and some $p \neq q$. Find $i, j < \gamma$ with $b - c_p = b_\gamma^i - b_\gamma^j$. The inequality $b \neq c_p$ implies $i \neq j$.

If $i < j$, then $b_\gamma^j = b_\gamma^i - b + c_p = b_\gamma^i - b + c_q - c_q + c_p \subset b_\gamma^i - B_\beta + B_\beta + F_\gamma \subset B_\gamma^{<j} + \mathbb{B}_{<\gamma} + F_\gamma$, which contradicts (1). If $i > j$, then $b_\gamma^i = b_\gamma^j + b - c_p = b_\gamma^j + B_\beta - B_\beta + c_q - c_p \subset B_\gamma^{<i} + \mathbb{B}_{<\gamma} + F_\gamma$, which again contradicts (1).

Step 2 *We claim that if $b - c_p = b_\beta^i - b_\beta^j$ and $b - c_q = b_\beta^s - b_\beta^t$ then $\max\{i, j\} = \max\{s, t\}$.*

It follows from the hypothesis that $c_q - c_p = b_\beta^i - b_\beta^j + b_\beta^t - b_\beta^s$. To obtain a contradiction, assume that $\max\{i, j\} > \max\{s, t\}$. If $j < i$, then $b_\beta^i = c_q - c_p + b_\beta^j - b_\beta^t + b_\beta^s \in F_\beta + B_\beta^{<i} - B_\beta^{<j} + B_\beta^{<s}$, which contradicts (2). If $i < j$, then $b_\beta^j = c_p - c_q + b_\beta^i + b_\beta^t - b_\beta^s \in F_\beta + B_\beta^{<j} + B_\beta^{<t} - B_\beta^{<s}$, again a contradiction with (2).

Step 3 *According to the previous step, there exists $\beta > \alpha$ and l such that*

- (i) $b - c_1 = b_\beta^i - b_\beta^j$ where $\max\{i, j\}$ is equal to l ;
- (ii) $b - c_2 = b_\beta^s - b_\beta^t$ where $\max\{s, t\}$ is equal to l ; and
- (iii) $b - c_3 = b_\beta^q - b_\beta^r$ where $\max\{q, r\}$ is equal to l .

In this case we obtain a dichotomy: either among three numbers i, s, q two are equal to l or among j, t, r two are equal to l . In the first case we lose no generality assuming that $i = s = l$; in the second, that $j = t = l$.

In the first case, we get $G_\alpha \ni c_2 - c_1 = b_\beta^t - b_\beta^j$, which contradicts (1). In the second case, we get $G_\alpha \ni c_2 - c_1 = b_\beta^i - b_\beta^s$, which contradicts (1) again. Therefore, there is no $b \in \bigcap_{i=1}^3 (c_i + \mathbb{B}_{>\alpha}) \setminus \{c_i\}$ and hence $|B_\kappa| \leq |\bigcap_{i=1}^3 (c_i + \mathbb{B}_{>\alpha})| < \kappa$. \square

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Acknowledgments

This research was supported the Slovenian Research Agency grants P1-0292-0101, J1-9643-0101, and BI-UA/07-08-001. The authors express their sincere thanks to Heike Mildenerger, Andrzej Rostanowski, and Lubomyr Zdomsky for their help in understanding the Shelah paper [10], to Wiesław Kubiś who informed the authors about his

preprint [6], and to the referee whose valuable comments resulted in improving the presentation of the paper.

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