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$\in_I$: An Intuitionistic Logic without Fregean Axiom and with Predicates for Truth and Falsity

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Abstract We present $\in_I$-Logic (Epsilon-I-Logic), a non-Fregean intuitionistic logic with a truth predicate and a falsity predicate as intuitionistic negation. $\in_I$ is an extension and intuitionistic generalization of the classical logic $\in_T$ (without quantifiers) designed by Sträter as a theory of truth with propositional self-reference. The intensional semantics of $\in_T$ offers a new solution to semantic paradoxes. In the present paper we introduce an *intuitionistic* semantics and study some semantic notions in this broader context. Also we enrich the quantifier-free language by the new connective $<$ that expresses reference between statements and yields a finer characterization of intensional models. Our results in the intuitionistic setting lead to a clear distinction between the notion of denotation of a sentence and the here-proposed notion of *extension of a sentence* (both concepts are equivalent in the classical context). We generalize the *Fregean Axiom* to an intuitionistic version not valid in $\in_I$. A main result of the paper is the development of several model constructions. We construct intensional models and present a method for the construction of standard models which contain specific (self-)referential propositions.

1 Introduction

There are two dogmas in formal logic which are widely accepted:

(i) The "Bedeutung" (the denotation) of a sentence is a truth value.
(ii) A (sufficiently rich) language cannot contain a total truth predicate without producing semantic paradoxes that imply the inconsistency of the underlying logical system.

Dogma (i) is as old as mathematical logic itself and goes back to the groundbreaking work of Frege [8]. The acceptance of dogma (ii) is supported by the works on

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semantic truth theory started by Tarski ([23], [24]). Moreover, it seems that (ii) relies on (i) and on the problematic assumption that sentences (and not what sentences really denote, namely, propositions, statements, situations, or similar entities instead of truth values) are the bearers of truth values. In fact, if we weaken dogma (i) and interpret sentences not only as truth values but as propositions of a given model-theoretic propositional universe, then it makes sense to define reflexive languages[1] where sentences refer directly to their (proper) subsentences. Self-reference then can be managed by introducing an identity connective into the language. If the language contains a predicate that applies to a sentence if and only if the sentence denotes a true proposition of the universe, then this predicate is an adequate truth predicate: the Tarski-biconditionals follow; the truth predicates of object language and meta-language coincide. The truth predicate is total, and self-referential statements can be asserted without restrictions (the language is semantically closed in the sense of Tarski). However, the model-theoretic truth conditions ensure that paradoxical assertions are not satisfied—such assertions are contradictory formulas. The liar, for instance, can be asserted, but no model contains the liar proposition. Under these assumptions Sträter [18] defined classical $\in_T$-Logic (which we will shortly discuss on pages 278 and 279) as a theory of truth and propositions.

It seems that the first logician who proposed the abolition of dogma (i) was Suszko ([19], [20]), inspired by the ontology of Wittgenstein's *Tractatus* (see, for instance, [21], [22], [6], [16], [15] for further information). Frege's principle that *all sentences of the same truth value have the same "Bedeutung"* is called by Suszko the *Fregean Axiom*. Bedeutung in the usual, classical case can be translated as "denotation." As our technical results on $\in_I$-Logic suggest this is no longer tenable in the general intuitionistic case where "denotation" must be replaced by a more general semantic notion. We will tackle this problem by introducing here the more general notion of "extension." We then translate Bedeutung in the intuitionistic context as *extension*.

Suszko introduces the concept of *non-Fregean Logic* which stands for a continuation of Frege's program [8] without the Fregean Axiom. The rejection of the Fregean Axiom implies that there may exist more than two denotations (*verum, falsum*)—the denotation of a sentence is, in general, more than a truth value. Such denotations are usually called *situations, states of affair*, or *facts*. In the present paper we shall use the concepts *statement* and *proposition*, where propositions are those statements which have a truth value, either *true* or *false*. The sentential calculus based on the principles of non-Fregean Logic is the *Sentential Calculus with Identity* (SCI) (see, for instance, Bloom and Suszko [6]). "Identity" refers to the identity connective $\equiv$. The intended meaning of $\equiv$ is the following:

$\varphi \equiv \psi$ is true if and only if the formulas φ and ψ have the same Bedeutung.

So in the *classical case* we read this as "$\varphi \equiv \psi$ is true if and only if the formulas φ and ψ have the same denotation," whereas in the general *intuitionistic case* we propose here the following reading: "$\varphi \equiv \psi$ is true if and only if the formulas φ and ψ have the same extension," assuming our definition of *extension* given below. The Fregean Axiom in the classical case claims that the identity connective coincides with the relation of logical equivalence. In general, the Fregean Axiom holds if and only if all models are *extensional* (see Definition 3.10 below). In particular, universes with only two statements *verum, falsum* are extensional. However, there are nonextensional models; thus, the Fregean Axiom does not hold. The construction of such universes (with nice properties) is in general complex and can be seen as a central challenge in the research on $\in_T$ ([18], [25]) and on $\in_I$. In the present paper, we develop some new constructions of such models—in the classical and in the intuitionistic context. Our models have the additional nice property that they have no "nonstandard elements"; they are standard models.

Now arises the question for an appropriate generalization of the Fregean Axiom to the intuitionistic case. For this we define the notion of *extension* in the following way.

Definition 1.1 The extension of a formula φ (relative to a given world) is the set of formulas ψ such that in every accessible world the formulas φ and ψ denote the same statement.

It follows that in the *classical case* the extension of a formula reduces in some sense to its denotation (there is a one-to-one correspondence; two formulas have the same denotation if and only if they have the same extension), whereas in the general *intuitionistic case* the extension of a formula φ is the set $\{\psi \mid \varphi \equiv \psi$ is true in the given world$\}$, with the intuitionistic reading of $\equiv$. Now we can formalize an *intuitionistic version* of the Fregean Axiom:

$\varphi \equiv \psi$ is true whenever $(\varphi : \text{false} \wedge \psi : \text{false}) \vee (\varphi : \text{true} \wedge \psi : \text{true})$ is true.[2]

That is, formulas of the same truth value have the same extension. This axiom does not hold in $\in_I$. In the classical setting this axiom specializes to the usual form of Fregean Axiom: $\varphi \equiv \psi$ is true whenever $\varphi \leftrightarrow \psi$ is true.

Epsilon Logics (we mean all logics which are extensions or generalizations of original $\in_T$-Logic developed by Sträter [18] and Zeitz [25]) with a truth predicate are related to the semantic truth theory of Tarski ([23], [24]). One reason for the enormous influence and wide acceptance of Tarski's semantic truth theory is probably the fact that it provides rather clear criteria for a satisfactory definition of truth in formal languages. These criteria are "formal correctness" and "material adequacy." Whereas the former concerns certain formal rules that the definition must satisfy, the latter ensures that the definition captures our intuitive notion of the term "true sentence." The definition is material adequate if it implies all equivalences of the form

$$\text{X is true if and only if p.} \tag{1}$$

We get an instance of such an equivalence if we replace "p" by any sentence of the language to which the word "true" refers and replace "X" by the name of this sentence (Tarski requires that the language has names for its sentences in order to establish references.) An example used by Tarski is

"Snow is white" is true if and only if snow is white.

We call the scheme 1 of material adequacy the Tarski-Biconditionals (TB). The truth predicate in the TB applies to sentences which, in Tarski's truth theory, are the bearers of truth values. Indeed, Tarski rejects any other semantic or psychological entity such as proposition or judgment. In [24] he writes

> "...as regards the term "proposition," its meaning is notoriously a subject of lengthy disputations by various philosophers and logicians, and it seems never to have been made quite clear and unambiguous. For several reasons it appears most convenient to apply the term "true" to sentences, and we shall follow this course." (Tarski [24])

If the language is *semantically closed*, then paradoxical sentences, such as the liar sentence, can be formulated. Such paradoxes imply the inconsistency of the underlying logical system. Roughly speaking, a language is *semantically closed* if the following holds:

> *The language contains technical means that enable sentences to refer to sentences. Furthermore, the language contains semantic predicates such as "true" and "not true" (or "false").*

Consequently, Tarski rejects semantically closed languages for an adequate treatment of the liar paradox. His solution consists in a strict distinction between object language and metalanguage. The truth predicate belongs to the metalanguage and applies exclusively to (the names of) sentences of the object language. Thus, the object language is contained in the metalanguage (or can be translated into a sublanguage of the metalanguage). The definition of truth and the Tarski-Biconditionals implied by this definition are expressed in the metalanguage. In this way, one may construct a hierarchy of interpreted languages; each level contains a truth predicate that applies only to sentences of lower levels.

Tarski's conclusions harmonize with his famous result which implies that in the first-order language of arithmetic there is no truth predicate relative to the complete theory of arithmetic T; that is, arithmetic truth is not representable in T. The assumption of such a truth predicate in the object language would lead to the construction of a sentence which is equivalent modulo T to the assertion of its own falsehood, that is, the liar sentence.

Probably the most influential truth theory after Tarski is developed by Kripke [9]. A mathematical treatment of this approach is given by Fitting [7]. Kripke, as well as Tarski, considers sentences as the bearers of truth values. In order to avoid semantic paradoxes he develops a theory of partial valuations of sentences. The set of true sentences (and the set of false sentences) then is a fixed point of a monotonic process of evaluation. Semantic paradoxes, such as the liar sentence, do not appear in such fixed points and are therefore truth value gaps of the partial truth predicate represented by the fixed point.

An approach to a formalized semantic truth theory that relies on essentially different assumptions is developed by Sträter in his doctoral thesis [18]. This work was motivated by attempts to reconstruct natural language semantics via self-referential structures ([13], [12]). More information about the historical background of $\in_T$-Logic can be found in [3]. The language of Sträter's $\in_T$-Logic has variables and constant symbols for propositions, classical connectives, classical first-order quantification over propositions, the identity connective $\equiv$, and symbols $:\mathrm{true}$ and $:\mathrm{false}$ for a truth and a falsity predicate, respectively. The crucial feature of the logic is its model-theoretic semantics which is intensional: expressions are interpreted as propositions in a propositional universe. Because of the reflexivity of the language, expressions may refer directly to their proper subexpressions (there are, however, some restrictions with respect to quantification). Self-reference now can be achieved by the identity connective $\equiv$. For instance, $c : \mathrm{true}$ refers to the subexpression c (saying that c is true). If $c \equiv (c : \mathrm{true})$ is satisfied in a model, then the sentences c and $c : \mathrm{true}$ denote the same statement p (fact, proposition) in this model, which must be a truth-teller. The formula $c \equiv (c : \mathrm{false})$ asserts that c denotes the liar. The truth conditions of the model-theoretic semantics ensure that such a paradoxical equation is never satisfied. The liar can be asserted by the above equation; however, this assertion is always false. No model contains the liar proposition. A proposition is the bearer of a truth value and represents a semantic content which is expressed by the expressions that denote it. For instance, if φ : false denotes proposition p, then p stands for "φ is false". There may exist further expressions that denote p, say $\neg\varphi$. Then p also stands for "not φ," and so forth. Although φ : false and $\neg\varphi$ are logically equivalent in $\in_T$, they have different intensions (expressed by the syntactical form) and therefore denote, in general, different propositions. The same holds for $\varphi \vee \psi$ and $\psi \vee \varphi$, and so on. Sträter constructs extensional and intensional models. The existence of intensional models shows in particular that logically equivalent sentences can be interpreted by different propositions. $\equiv$ cannot coincide with logical equivalence. $\in_T$-Logic is an example of a logic that violates dogma (i); it is a non-Fregean Logic (this fact, however, is not explicitly mentioned in [18]). $\in_T$-Logic can be viewed as semantically closed in Tarski's sense discussed above. The truth predicate of the object language coincides with the truth predicate of the metalanguage—the Tarski Biconditionals hold and can be formulated in the object language. [18] also discusses some specific self-referential statements that can be asserted in $\in_T$ such as the liar by $c \equiv (c$: false), several liar cycles, the truth-teller by $c \equiv (c$: true), Löb's paradox by $c \equiv (c$: true $\rightarrow d)$, and so on. A discussion on such self-references and semantic paradoxes can also be found in Barwise and Etchemendy [4].

It is essentially the intensional semantics, the absence of the Fregean Axiom, and the reflexive syntax which enable $\in_T$, $\in_I$ to deal adequately with truth and self-reference. The absence of Fregean Axiom is also crucial for the expressive power of the language. For instance, under the assumption of Fregean Axiom every equation of the form $\varphi \equiv (\varphi$: true) would be valid since φ and φ : true are logically equivalent expressions. In this case, the truth-teller turns out to be meaningless since every expression denotes a "truth-teller," and there are exactly two, namely, *verum* and *falsum*. A well-known argument against non-Fregean logics is the so-called slingshot argument, developed by Church, Gödel, Quine, and Davidson, which is often considered a formal proof of the Fregean Axiom. In a recent paper of Shramko and Wansing [17], a new version of the slingshot argument is presented and it is shown that under some minimal assumptions the Fregean Axiom can be derived inside some non-Fregean logics. Non-Fregean Logic is understood in [17] as a useful tool for representing the slingshot argument. We argue that non-Fregean logics such as $\in_T$, $\in_I$ may be useful tools not only for dealing adequately with semantic truth theory but also for clarifying semantic concepts in general. In this sense, the significance of non-Fregean logics should be reconsidered. Surprisingly, the obvious relationship between $\in_T$-Logic and Suszko's non-Fregean Logic seems to be unnoticed in all previous works on Epsilon Logic ([18], [25], [14], [1], [2], [10]) and is therefore not yet explicitly explored.

Zeitz [25] further develops the original $\in_T$-Logic under the aspect of a parametrized logic. He studies $\in_T$-Logic as a semantic framework that extends a given abstract logic. As a main result he shows that under certain conditions there is a sound and complete Hilbert-style calculus for the $\in_T$-extension whenever there is such a calculus for the underlying abstract logic. Zeitz is also able to simplify and to improve some aspects of the technical apparatus of Sträter's original $\in_T$-Logic.

The parametrized version of Zeitz $\in_T$-Logic is studied by Mahr and Bab [14] as a formalism for the integration of several object logics. This integration formalism is mainly discussed from the point of view of *software specification* but is motivated by a general scenario of integration where different views on a complex object A given by models of different logics are integrated in a single model of an appropriate integration logic. The paper also proposes an interpretation of $\in_T$ as a logic of judgments whose propositions are expressed in the object logics. Bab ([1], [2]) discusses a scenario of a possible world semantics which is given *inside a single* propositional universe—the possible worlds are represented as subsets of the set of true propositions. Under these assumptions Bab generalizes $\in_T$ and defines $\in_\mu$ [2], a logic which is able to integrate modalities coming from an underlying classical modal object logic. The definition of a possible world semantics differs essentially from our (intuitionistic) approach where each possible world is given by its own propositional universe. Moreover, $\in_\mu$-Logic is classical; intuitionistic logics cannot be handled in $\in_\mu$.

One purpose of the present paper is to introduce intuitionism into Epsilon-Logic (by means of semantics). Our task is to find adequate intuitionistic interpretations for the identity connective, the reference connective, and the predicates for truth and falsity. We define a possible world semantics where each possible world is given by a single propositional universe (essentially an $\in_T$-model with some additional structure). A first new phenomenon that appears in the intuitionistic setting is that there may exist elements of the universe which have no truth value. Since an essential property of a proposition is that it bears a truth value, we call the elements of a universe "statements" and consider propositions as those statements that have a truth value: either true or false. Nevertheless, we still call these universes "propositional universes." We also introduce a new connective, namely, the reference connective $<$. In a classical setting, the intended meaning of the reference connective would be the following:

> $\varphi < \psi$ is true iff the proposition denoted by ψ says something (contains some information) about the proposition denoted by φ.

In the broader intuitionistic setting we propose the following reading:

> $\varphi < \psi$ is true iff in every accessible world the statement denoted by ψ says something (contains some information) about the statement denoted by φ.

Thus, $<$ expresses reference between statements. In particular, self-reference can be expressed. For instance, if $c \equiv (c : \text{true})$ is true in a model (c denotes a truth-teller which is a self-referential proposition), then the model-theoretic semantics ensures that also $c < c$ is true.

For the interpretations of the truth predicate, the connectives of conjunction, disjunction, and implication, we follow the usual conditions of intuitionistic possible world semantics. The falsity predicate assumes the role of intuitionistic negation—interpreted as a set, the set of false propositions. Although we consider propositions as the bearers of truth values we may define "true expression" and "false expression"; that is, we may define the truth predicate and falsity predicate of the *metalanguage*:

(i) we say that φ is true in a given world if φ denotes a true proposition;
(ii) we say that φ is false in a given world if in all accessible worlds φ is not true.

Notice that in an intuitionistic setting "false" is stronger than "not true." Unfortunately, "not true" cannot be expressed in the language of intuitionistic logic: negation is interpreted as "not true in all accessible worlds," that is, as "false." As a consequence, we cannot assert the strengthened liar "This sentence is not true" in $\in_I$-Logic. The liar can be asserted only in the form "This sentence is false."

Now the model-theoretic truth conditions for the truth predicate, : true, and the falsity predicate, : false, of the object language are defined in such a way that they coincide exactly with the respective predicates of the metalanguage. The semantics of the identity and the reference connective is respectively defined in accordance with the above discussions:

(i) we say that $\varphi \equiv \psi$ is true in a given world if in every accessible world φ and ψ denote the same statement;
(ii) we say that $\varphi < \psi$ is true in a given world if in every accessible world $\mathcal{M}$ the following holds: φ and ψ denote statements p, q, respectively, and q is related to p in $\mathcal{M}$ (i.e., $p <^{\mathcal{M}} q$).

The question for the existence of models (with specific properties) is not trivial. The simplest model is the classical extensional one which contains exactly two propositions: the true and the false proposition. The construction of such a model is not difficult and works in all Epsilon Logics in a similar way. As we have seen, if there would exist only such extensional models, then the identity connective $\equiv$ would collapse with logical equivalence and the Fregean Axiom would follow. But there exist more models. Of particular importance are intensional models, that is, models which satisfy the equation $\varphi \equiv \psi$ only if φ, ψ express the same intension (the same sense). The existence of intensional models was first proved by Sträter [18]. His construction is highly complex and extensive. This is mainly due to the impredicativity of the quantifiers and the related difficulty to assign truth values to the expressions. An improvement and considerable simplification of this construction is given by the author in [10]. Zeitz [25] presents a completely different construction method which is shorter and simpler; however, his intensional model has some unintuitive and undesirable properties. The construction of intensional models (with nice properties) is a challenge in research on Epsilon Logics. The method developed in the present paper (in the setting of $\in_I$-Logic) yields intensional models with no "nonstandard elements." Such *standard* models have the best properties. Our intensional models must satisfy a further condition which is related to the here-introduced reference connective: $\varphi < \psi$ is true if and only if φ is a proper subformula of ψ. Thus, in an intensional model, the meaning of $\varphi < \psi$ is that the expression ψ expresses something (depending on the syntactical form of ψ) about the proper subexpression φ.

Finally, we use our intensional classical standard model in order to construct models that contain specific self-referential propositions such as the truth-teller. This is managed by introducing an equivalence relation on the universe that relates those propositions which we wish to identify. The new universe then consists of the equivalence classes of this relation together with a new division into the subsets of true and false propositions. These model constructions are among the main results of the present work. A systematic study of such constructions may help to get an overview of all possible ($\in_T$- and $\in_I$-)models. This is a promising task for future works.

2 Syntax

The language consists of the following symbols:

(i) A set C of constant symbols denoted by $c, d, e, \ldots, c_0, c_1, \ldots$. We assume that the set C contains at least two special constant symbols $\bot, \top \in C$.

(ii) An infinite set $V = \{x_0, x_1, \ldots\}$ of variables for statements. We denote the elements of V by $x, y, z, u, v, x_0, \ldots$.
(iii) Symbols for the logical connectives disjunction, conjunction, and implication: $\vee$, $\wedge$, $\rightarrow$, respectively.
(iv) The symbols : true, : false (in postfix notation) representing the truth- and the falsity-predicate, respectively. : false also stands for the connective of intuitionistic negation.
(v) The connective $<$ for reference between statements, and the connective $\equiv$ for identity between statements.
(ii) Auxiliary symbols:), (, .

Definition 2.1 Let C be any set of constant symbols. The set Expr(C) of expressions (or formulas) over C is the smallest set that contains C and V and is closed under the following condition. If φ and ψ are expressions, then $(\varphi : \text{true})$, $(\varphi : \text{false})$, $(\varphi \vee \psi)$, $(\varphi \wedge \psi)$, $(\varphi \rightarrow \psi)$, $(\varphi \equiv \psi)$, and $(\varphi < \psi)$ are expressions. The set of sentences, denoted by Sent(C), is the set of those expressions in which no variables occur.

Usually we omit outermost parentheses. Sometimes we omit parentheses respecting the following descending priority of symbols: : true, : false, $\vee$, $\wedge$, $\rightarrow$, $\equiv$, $<$. For instance, $x \equiv y \vee c : \text{true}$ is the expression $x \equiv (y \vee (c : \text{true}))$. We use $\varphi \leftrightarrow \psi$ as an abbreviation for $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

Constant symbols may play a special role in $\in_I$. The idea is that these symbols can be viewed as names of sentences of any given (natural or formal) language $\mathcal{L}$. The logic $\in_I$ then turns out to be a metalanguage for the object language $\mathcal{L}$. In this case the truth predicate (falsity predicate) of $\in_I$ is in particular a truth predicate (falsity predicate) for the sentences of $\mathcal{L}$. $\in_I$ may also serve as a metalogic for a given intuitionistic abstract logic such as defined in [11]. Such metalogic aspects (of classical $\in_T$-Logic) were first studied in [25] and [10]. This is, however, not the subject of the present paper. Constant symbols can also be used to denote special propositions such as the truth-teller.

The notion of subexpression is recursively defined as follows.

Definition 2.2 Let φ be an expression.

(i) φ is a subexpression of φ.
(ii) φ is a subexpression of the expressions $\chi : \text{true}$, $\chi : \text{false}$ if φ is a subexpression of χ.
(iii) φ is a subexpression of the expressions $\chi \vee \psi$, $\chi \wedge \psi$, $\chi \rightarrow \psi$, $\chi \equiv \psi$, $\chi < \psi$ if φ is a subexpression of χ or φ is a subexpression of ψ.

The set of subexpressions of φ is denoted by sub(φ). We say that ψ is a proper subexpression of φ and write $\psi \prec \varphi$ if $\psi \in \text{sub}(\varphi) \smallsetminus \{\varphi\}$.

It is clear that $\prec$ is a partial order on the set Expr(C).

We define $\text{var}(\varphi) = \text{sub}(\varphi) \cap V$, the set of all variables that occur in the expression φ. The elements of C and V are also called atomic expressions. If φ is an expression then we define $\text{at}(\varphi) = \text{sub}(\varphi) \cap (C \cup V)$, the set of all atomic expressions occurring in φ. Analogously, we define the set of all constant symbols occurring in φ and denote this set by con(φ).

Substitutions are defined as follows.

Definition 2.3 A substitution σ is a function $\sigma : C \cup V \to \mathrm{Expr}(C)$. If $A \subseteq C \cup V$ and $\sigma(u) = u$ for all $u \in (C \cup V) \setminus A$, then we write $\sigma : A \to \mathrm{Expr}(C)$. If $A = V$, then σ is called a variable substitution (or a substitution of variables).

If σ is a substitution, $u_0, \ldots, u_n \in C \cup V$ and $\varphi_0, \ldots, \varphi_n \in \mathrm{Expr}(C)$, then the substitution $\sigma[u_0 := \varphi_0, \ldots, u_n := \varphi_n]$ is defined by the following equation:

$$\sigma[u_0 := \varphi_0, \ldots, u_n := \varphi_n](u) = \begin{cases} \varphi_i & \text{if } u = u_i, \text{ for some } i \leq n \\ \sigma(u) & \text{else.} \end{cases}$$

The substitution given by the identity $u \mapsto u$, $u \in C \cup V$, is denoted by ε. Instead of $\varepsilon[u_0 := \varphi_0, \ldots, u_n := \varphi_n]$ we also write $[u_0 := \varphi_0, \ldots, u_n := \varphi_n]$.

A substitution σ extends in a canonical way to a function $[\sigma] : \mathrm{Expr}(C) \to \mathrm{Expr}(C)$ (we use postfix notation for $[\sigma]$):

$$\begin{aligned} u[\sigma] &:= \sigma(u) \qquad\qquad \text{for all } u \in C \cup V \\ (\varphi : \text{true})[\sigma] &:= \varphi[\sigma] : \text{true} \\ (\varphi : \text{false})[\sigma] &:= \varphi[\sigma] : \text{false} \\ (\varphi \vee \psi)[\sigma] &:= \varphi[\sigma] \vee \psi[\sigma] \\ (\varphi \wedge \psi)[\sigma] &:= \varphi[\sigma] \wedge \psi[\sigma] \\ (\varphi \to \psi)[\sigma] &:= \varphi[\sigma] \to \psi[\sigma] \\ (\varphi \equiv \psi)[\sigma] &:= \varphi[\sigma] \equiv \psi[\sigma] \\ (\varphi < \psi)[\sigma] &:= \varphi[\sigma] < \psi[\sigma]. \end{aligned}$$

The composition of two substitutions σ and τ is the substitution $\sigma \circ \tau$ defined by

$$(\sigma \circ \tau)(u) = \sigma(u)[\tau],$$

for $u \in C \cup V$. If $\delta_0, \ldots, \delta_n$ are substitutions and φ is an expression, then we write $\varphi[\delta_0][\delta_1] \ldots [\delta_n]$ for the substitution $(((\varphi[\delta_0])[\delta_1]) \ldots)[\delta_n]$.

In the following we collect some useful properties of substitutions that are easy to prove (usually by induction on the expressions).

Lemma 2.4 *Let φ be an expression and let σ, τ be substitutions. If $\sigma(u) = \tau(u)$ for all $u \in \mathrm{at}(\varphi)$, then $\varphi[\sigma] = \varphi[\tau]$.*

Lemma 2.5 *Let φ be an expression and let σ, τ be substitutions. Then*

$$\varphi[\sigma \circ \tau] = \varphi[\sigma][\tau].$$

Corollary 2.6 *For all substitutions σ, τ, δ,*

$$\sigma \circ (\tau \circ \delta) = (\sigma \circ \tau) \circ \delta.$$

Proof Suppose that $u \in C \cup V$. Then we get

$$\begin{aligned} (\sigma \circ (\tau \circ \delta))(u) = \sigma(u)[\tau \circ \delta] = \sigma(u)[\tau][\delta] = \\ ((\sigma \circ \tau)(u))[\delta] = ((\sigma \circ \tau) \circ \delta)(u). \end{aligned}$$

□

Corollary 2.7 *Let φ be an expression. Suppose that $\sigma_1, \ldots, \sigma_n$ and $\tau_1, \ldots, \tau_m$ are substitutions such that $(u)[\sigma_1] \ldots [\sigma_n] = (u)[\tau_1] \ldots [\tau_m]$ for all $u \in \mathrm{at}(\varphi)$. Then $\varphi[\sigma_1] \ldots [\sigma_n] = \varphi[\tau_1] \ldots [\tau_m]$.*

Lemma 2.8 *If $\varphi \prec \psi$ and σ is a substitution, then $\varphi[\sigma] \prec \psi[\sigma]$.*

3 Semantics

We wish to define a possible world semantics over $\in_T$-models. So the following definition of a world is similar to the definition of an $\in_T$-model given in [18] and [25]. We add here the semantic reference relation $<^{\mathcal{M}}$, introduced in this paper, and the resulting reference property. Furthermore, we do not require that the union of the sets of true and false propositions constitutes the whole universe.

Definition 3.1 A world is a structure $\mathcal{M} = (M, \text{TRUE}, \text{FALSE}, <^{\mathcal{M}}, \Gamma)$ such that the following hold:

(i) M is a nonempty set of statements. $\text{TRUE}, \text{FALSE} \subseteq M$ are the sets of those statements which have a truth value, "true," "false," respectively. Statements with a truth value are called propositions. We require $\text{TRUE} \cap \text{FALSE} = \varnothing$. Even if there may exist statements with no truth value (in the case $M \smallsetminus (\text{TRUE} \cup \text{FALSE}) \neq \varnothing$), we call M the propositional universe of $\mathcal{M}$.

(ii) The binary relation $<^{\mathcal{M}} \subseteq M \times M$ is called the reference relation.

(iii) The semantic function $\Gamma : \text{Expr}(C) \times M^V \to M$ maps each expression φ to its denotation: a statement $\Gamma(\varphi, \gamma) \in M$. Γ depends on assignments $\gamma : V \to M$ of statements to variables. If $\gamma \in M^V$ and δ is a substitution, then $\gamma\delta \in M^V$ denotes the assignment defined by $\gamma\delta(x) = \Gamma(\delta(x), \gamma)$. If $x \in V, m \in M$, then γ_x^m is the assignment defined by

$$\gamma_x^m(y) := \begin{cases} m & \text{if } x = y \\ \gamma(y) & \text{else.} \end{cases}$$

The semantic function Γ satisfies the following structure properties:

(EP) for all $x \in V$ and all assignments $\gamma \in M^V$, $\Gamma(x, \gamma) = \gamma(x)$;

(CP) if φ is an expression and $\gamma, \gamma' \in M^V$ are assignments with $\gamma(x) = \gamma'(x)$ for all $x \in \text{var}(\varphi)$, then $\Gamma(\varphi, \gamma) = \Gamma(\varphi, \gamma')$;

(SP) if φ is an expression, $\gamma \in M^V$ an assignment and $\sigma : V \to \text{Expr}(C)$ is a substitution of variables, then $\Gamma(\varphi[\sigma], \gamma) = \Gamma(\varphi, \gamma\sigma)$;

(RP) if $\varphi \prec \psi$, then $\Gamma(\varphi, \gamma) <^{\mathcal{M}} \Gamma(\psi, \gamma)$, for all expressions φ, ψ and all assignments γ.

(EP) is the extension property. The coincidence property (CP) ensures that the semantics of an expression depends only on the interpretation of those variables that occur in φ. Note that this justifies writing $\Gamma(\varphi)$ instead of $\Gamma(\varphi, \gamma)$, if φ is a sentence. The substitution property (SP) guarantees (see the following Substitution Lemma) that the denotation of an expression is invariant under the substitution of subexpressions by expressions of the same semantics. We require here the substitution property only for substitutions of variables. However, we will see (Substitution Lemma) that this condition is sufficient. Finally, the reference property (RP) ensures that the relation $\prec$ is mirrored semantically as reference between respective statements. If ψ says something about φ, that is, φ is a proper subexpression of ψ, then the statement denoted by ψ refers to the statement denoted by φ.

Definition 3.2 Let I be a nonempty set and let $R \subseteq I \times I$ be a reflexive, transitive, and antisymmetric relation (I is partially ordered by R). For each $i \in I$ let

$\mathcal{M}_i = (M_i, \text{TRUE}_i, \text{FALSE}_i, <_i, \Gamma_i)$ be a world and $\gamma_i : V \to M_i$ a respective assignment. The elements of I are called nodes. We also assume here that there is a bottom node; that is, there is some $0 \in I$ such that $0Ri$ for all $i \in I$. The structure $\mathcal{F} = ((\mathcal{M}_i, \gamma_i)_{i \in I}, R)$ is called a frame over (I, R) if for all $i \in I$ the following truth conditions are satisfied. For all expressions $\varphi, \psi \in \text{Expr}(C)$ and for all $x \in V$ and for all $c \in C$,

(i) $\gamma_i(x) \in \text{TRUE}_i \Rightarrow$ for all $j \in I$ with iRj: $\gamma_j(x) \in \text{TRUE}_j$;

(ii) $\Gamma_i(c) \in \text{TRUE}_i \Rightarrow$ for all $j \in I$ with iRj, $\Gamma_j(c) \in \text{TRUE}_j$;

(iii) $\Gamma_i(\varphi : \text{true}, \gamma_i) \in \text{TRUE}_i \Leftrightarrow \Gamma_i(\varphi, \gamma_i) \in \text{TRUE}_i$;

(iv) $\Gamma_i(\varphi : \text{false}, \gamma_i) \in \text{TRUE}_i \Leftrightarrow \Gamma_i(\varphi, \gamma_i) \in \text{FALSE}_i$;

(v) $\Gamma_i(\varphi \vee \psi, \gamma_i) \in \text{TRUE}_i \Leftrightarrow \Gamma_i(\varphi, \gamma_i) \in \text{TRUE}_i$ or $\Gamma_i(\psi, \gamma_i) \in \text{TRUE}_i$;

(vi) $\Gamma_i(\varphi \wedge \psi, \gamma_i) \in \text{TRUE}_i \Leftrightarrow \Gamma_i(\varphi, \gamma_i) \in \text{TRUE}_i$ and $\Gamma_i(\psi, \gamma_i) \in \text{TRUE}_i$;

(vii) $\Gamma_i(\varphi \to \psi, \gamma_i) \in \text{TRUE}_i \Leftrightarrow$ for all $j \in I$ with iRj: $\Gamma_j(\varphi, \gamma_j) \notin \text{TRUE}_j$ or $\Gamma_j(\psi, \gamma_j) \in \text{TRUE}_j$;

(viii) $\Gamma_i(\varphi \equiv \psi, \gamma_i) \in \text{TRUE}_i \Leftrightarrow$ for all $j \in I$ with iRj: $\Gamma_j(\varphi, \gamma_j) = \Gamma_j(\psi, \gamma_j)$;

(ix) $\Gamma_i(\varphi < \psi, \gamma_i) \in \text{TRUE}_i \Leftrightarrow$ for all $j \in I$ with iRj: $\Gamma_j(\varphi, \gamma_j) <_j \Gamma_j(\psi, \gamma_j)$;

(x) $\Gamma_i(\varphi, \gamma_i) \in \text{FALSE}_i \Leftrightarrow$ for all $j \in I$ with iRj: $\Gamma_j(\varphi, \gamma_j) \notin \text{TRUE}_j$;

(xi) $\Gamma_i(\top) \in \text{TRUE}_i$ and $\Gamma_i(\bot) \in \text{FALSE}_i$.

These conditions are called the truth conditions of a frame.

If $\mathcal{F}$ as above is a frame, then we call the $\mathcal{M}_i$ worlds of the frame, and R is called the accessibility relation. The tuples $\mathcal{I}_i = (\mathcal{M}_i, \gamma_i)$ are called interpretations (of the frame $\mathcal{F}$). A frame that contains only one world is called a singleton. We identify a singleton with its unique interpretation $\mathcal{I} = (\mathcal{M}, \gamma)$.

Definition 3.3 Let $\mathcal{F}$ be a frame over (I, R). The satisfaction relation between interpretations and expressions is defined as follows. For $i \in I$,

$$(\mathcal{M}_i, \gamma_i) \vDash \varphi :\Longleftrightarrow \Gamma_i(\varphi, \gamma_i) \in \text{TRUE}_i.$$

The interpretation $\mathcal{I}_i = (\mathcal{M}_i, \gamma_i)$ is a model of the expression φ if $\mathcal{I}_i \vDash \varphi$. For a set Φ of expressions we define $\mathcal{I}_i \vDash \Phi :\Longleftrightarrow \mathcal{I}_i \vDash \varphi$ for all $\varphi \in \Phi$. $\mathcal{I}_i$ is called a model of the set Φ if $\mathcal{I}_i \vDash \Phi$.

Furthermore, we say that a frame $\mathcal{F}$ is a model of a set of formulas Φ if every interpretation in $\mathcal{F}$ is a model of Φ. This is the same as to say that the interpretation at the bottom node is a model of Φ.

Lemma 3.4 *Let $\mathcal{F} = ((\mathcal{M}_i, \gamma_i)_{i \in I}, R)$ be a frame. Then for all nodes $i \in I$ and for all expressions φ,*

(i) $(\mathcal{M}_i, \gamma_i) \vDash \varphi \Longleftrightarrow \Gamma_i(\varphi, \gamma_i) \in \text{TRUE}_i$,

(ii) $(\mathcal{M}_i, \gamma_i) \vDash \varphi : false \Longleftrightarrow \Gamma_i(\varphi, \gamma_i) \in \text{FALSE}_i$,

(iii) $\Gamma_i(\varphi, \gamma_i) \in \text{TRUE}_i \Longleftrightarrow$ *for all* $j \in I$ *with* iRj, $\Gamma_j(\varphi, \gamma_j) \in \text{TRUE}_j$,

(iv) $\Gamma_i(\varphi, \gamma_i) \in \text{FALSE}_i \Longleftrightarrow$ *for all* $j \in I$ *with* iRj, $\Gamma_j(\varphi, \gamma_j) \in \text{FALSE}_j$.

Proof The last assertion follows from truth condition (x) and transitivity of R. Now the third assertion follows by induction on φ. The first assertion is simply the definition of the satisfaction relation. Finally, the second assertion follows from the definition of the satisfaction relation and truth condition (iv). $\square$

Definition 3.5 Let $\mathcal{F} = ((\mathcal{M}_i, \gamma_i)_{i \in I}, R)$ be a frame. Then we call the four biconditionals of Lemma 3.4 the *Axioms of Adequacy*.

The first axiom is the definition of the satisfaction relation; it implies in particular the following: $(\mathcal{M}_i, \gamma_i) \vDash \varphi : \text{true} \iff \Gamma_i(\varphi, \gamma_i) \in \text{TRUE}_i$. So the first two Axioms of Adequacy ensure that the truth predicate and the falsity predicate satisfy our intuition: The truth predicate applies to an expression if and only if the expression denotes a true proposition; the falsity applies to an expression if and only if this expression denotes a false proposition. Indeed, the first axiom implies the Tarski-Biconditionals (see below). Moreover, the Axioms of Adequacy express that a true (a false) expression remains true (false) in all accessible worlds, respectively. This is in accordance with the intuition behind intuitionistic semantics: our "knowledge" is increasing at each successor node of a frame.

Remark 3.6 Let $\mathcal{F} = ((\mathcal{M}_i, \gamma_i)_{i \in I}, R)$ be a frame. Then the truth conditions together with the definition of the satisfaction relation imply the following equivalences, for every node $i \in I$.

$$(\mathcal{M}_i, \gamma_i) \vDash x \iff \gamma_i(x) \in \text{TRUE}_i$$
$$(\mathcal{M}_i, \gamma_i) \vDash c \iff \Gamma_i(c) \in \text{TRUE}_i$$
$$(\mathcal{M}_i, \gamma_i) \vDash \varphi : \text{true} \iff (\mathcal{M}_i, \gamma_i) \vDash \varphi$$
$$(\mathcal{M}_i, \gamma_i) \vDash \varphi : \text{false} \iff \text{ for all } j \text{ with } iRj : (\mathcal{M}_j, \gamma_j) \nvDash \varphi$$
$$(\mathcal{M}_i, \gamma_i) \vDash \varphi \vee \psi \iff (\mathcal{M}_i, \gamma_i) \vDash \varphi \text{ or } (\mathcal{M}_i, \gamma_i) \vDash \psi$$
$$(\mathcal{M}_i, \gamma_i) \vDash \varphi \wedge \psi \iff (\mathcal{M}_i, \gamma_i) \vDash \varphi \text{ and } (\mathcal{M}_i, \gamma_i) \vDash \psi$$
$$(\mathcal{M}_i, \gamma_i) \vDash \varphi \rightarrow \psi \iff \text{ for all } j \text{ with } iRj : (\mathcal{M}_j, \gamma_j) \nvDash \varphi \text{ or } (\mathcal{M}_j, \gamma_j) \vDash \psi$$
$$(\mathcal{M}_i, \gamma_i) \vDash \varphi \equiv \psi \iff \text{ for all } j \in I \text{ with } iRj : \Gamma_j(\varphi, \gamma_j) = \Gamma_j(\psi, \gamma_j)$$
$$(\mathcal{M}_i, \gamma_i) \vDash \varphi < \psi \iff \text{ for all } j \in I \text{ with } iRj : \Gamma_j(\varphi, \gamma_j) <^{\mathcal{M}_j} \Gamma_j(\psi, \gamma_j)$$
$$(\mathcal{M}_i, \gamma_i) \vDash \top : \text{true}$$
$$(\mathcal{M}_i, \gamma_i) \vDash \bot : \text{false}\,.$$

On the other hand, if we define the satisfaction relation inductively in this way, then the truth conditions of a frame follow from this alternative definition and the Axioms of Adequacy.

Hence, the connectives have, in effect, the expected intuitionistic behavior.

Definition 3.7 Let $\Phi \cup \{\varphi\}$ be a set of expressions. We say that Φ entails φ (or φ is a consequence of Φ) and write $\Phi \Vdash \varphi$ if every model of Φ is a model of φ. That is, $\Phi \Vdash \varphi$ if and only if for every frame $\mathcal{F}$ and every interpretation $\mathcal{I}$ in $\mathcal{F}$, $\mathcal{I} \vDash \varphi$ whenever $\mathcal{I} \vDash \Phi$. If the empty set entails φ, then we write $\Vdash \varphi$.

Let $\mathcal{F} = ((\mathcal{M}_i, \gamma_i)_{i \in I}, R)$ be a frame. Recall that we consider propositions as the bearers of truth values. We say that an expression is true (is false) at node i if it denotes a true (a false) proposition in the respective interpretation. This harmonizes with our intuitive notions of truth and falsity which are determined by the notion of model-theoretic satisfaction: φ *is true at node* i if and only if $(\mathcal{M}_i, \gamma_i) \vDash \varphi$, and φ *is false at node* i if and only if φ is not true at j, for all j with iRj. We say that the expression φ has a truth value at node i if φ is true at i or φ is false at i.

The validity of the Tarski-Biconditionals in $\in_T$-Logic was proven in [18]. We show here that they also hold in our intuitionistic setting.

Theorem 3.8 (Tarski-Biconditionals) *The truth predicate and the falsity predicate on the object language coincide with our intuitive notions of truth and falsity, that is, with the truth predicate and the falsity predicate of the metalanguage, respectively. More precisely, if $\mathcal{F}$ is a frame over (I, R), then we have the following for each interpretation $(\mathcal{M}_i, \gamma_i)$ of $\mathcal{F}$.*

$$(\mathcal{M}_i, \gamma_i) \vDash \varphi : \text{true} \Longleftrightarrow \text{“}\varphi \text{ is true at } i\text{,”}$$
$$(\mathcal{M}_i, \gamma_i) \vDash \varphi : \text{false} \Longleftrightarrow \text{“}\varphi \text{ is false at } i\text{,”}$$

for each expression φ. In particular, the logic $\in_I$ satisfies the Tarski-Biconditionals which are expressible on the object level:

$$\Vdash \varphi : \text{true} \leftrightarrow \varphi,$$

for each expression φ.

Proof This follows from Remark 3.6. □

Definition 3.9 A world $\mathcal{M}$ is called classical if the propositional universe M is the disjunct union of the sets TRUE and FALSE. An interpretation $(\mathcal{M}, \gamma)$ of a given frame $\mathcal{F}$ is called classical if the world $\mathcal{M}$ is classical.

In a classical interpretation we have $(\mathcal{M}, \gamma) \vDash \varphi \vee (\varphi : \text{false})$ for all expressions φ. If a classical interpretation $\mathcal{I}$ appears at a node which is not maximal in a given frame, then this implies that all interpretations accessible from $\mathcal{I}$ satisfy exactly the same set of expressions.

If we consider exactly those frames that contain only classical worlds (in particular, all singletons), then we obtain the classical sublogic of $\in_I$. (We understand “sublogic” in a model-theoretic sense. Roughly speaking, $\mathcal{L}'$ is a sublogic of $\mathcal{L}$ if every model of $\mathcal{L}'$ is a model of $\mathcal{L}$.) “φ is false” is the same as “φ is not true” in the classical sublogic. Thus, the falsity predicate is the connective for classical negation and $\varphi \vee (\varphi : \text{false})$ is valid, for any expression φ.

Extensional and intensional $\in_T$-models are defined in [18]. Adapting these notions we use our new reference connective $<$ to establish a refinement of the concept of intensional model.

Definition 3.10 Let $\mathcal{F}$ be a frame over (I, R).

(i) An interpretation $(\mathcal{M}_i, \gamma_i)$ of the frame $\mathcal{F}$ is called extensional if for all expressions φ, ψ the following holds:

$$(\mathcal{M}_i, \gamma_i) \vDash ((\varphi : \text{true}) \rightarrow (\varphi \equiv \top)) \wedge ((\varphi : \text{false}) \rightarrow (\varphi \equiv \bot)).$$

(ii) A world $\mathcal{M}_i$ of the frame $\mathcal{F}$ is called intensional if for any two sentences φ, ψ the following hold:
   (a) if $\mathcal{M}_i \vDash \varphi \equiv \psi$, then $\varphi = \psi$,
   (b) if $\mathcal{M}_i \vDash \varphi < \psi$, then $\varphi \prec \psi$.

The frame $\mathcal{F}$ is said to be extensional (intensional) if all its interpretations (worlds) are extensional (intensional), respectively. A nonstandard element of a world $\mathcal{M}$ is a statement $p \in M$ such that no sentence denotes p. A world with no nonstandard elements is called a standard model.

Lemma 3.11 *Let $\mathcal{F}$ be a frame over (I, R) and let $i \in I$. If $\mathcal{M}_i$ is an intensional world, then for every $j \in I$ with jRi, the world $\mathcal{M}_j$ is intensional. If $(\mathcal{M}_i, \gamma_i)$ is extensional, then for all $j \in I$ such that iRj, the interpretation $(\mathcal{M}_j, \gamma_j)$ is extensional.*

Proof Let $\mathcal{M}_i$ be intensional and jRi. Suppose that $\mathcal{M}_j$ is not intensional. Then we distinguish two possible cases: There are sentences $\varphi_1 \neq \varphi_2$ such that $\mathcal{M}_j \vDash \varphi_1 \equiv \varphi_2$ or there are sentences $\psi_1 \not\prec \psi_2$ such that $\mathcal{M}_j \vDash \psi_1 < \psi_2$. Since the truth of these sentences is preserved in accessible worlds, we get, in the first case, $\mathcal{M}_i \vDash \varphi_1 \equiv \varphi_2$ and, in the second case, $\mathcal{M}_i \vDash \psi_1 < \psi_2$. This contradicts the assumption that $\mathcal{M}_i$ is intensional. The second assertion follows immediately from the definition and from the fact that truth of expressions is preserved in all accessible interpretations. □

We introduce the semantic concept of *extension of an expression* in the following way.

Definition 3.12 Let $\mathcal{F} = ((\mathcal{M}_i, \gamma_i)_{i \in I}, R)$ be a frame and let $\mathcal{I} = (\mathcal{M}_i, \gamma_i)$ be an interpretation. The extension of an expression φ at node i is the set of expressions $\{\psi \mid \mathcal{I} \vDash \varphi \equiv \psi\}$.

Now we have three different notions that we may assign to an expression: every expression has an intension, a denotation, and an extension (in a given interpretation). Recall that the *intension* (or the sense) of an expression is given by its syntactical form; thus it is independent of any ambient model. The *denotation* of an expression φ in a given interpretation is the statement denoted by φ, that is, the image of φ under the Γ-function. Also note that in the classical sublogic the concepts of extension and denotation are equivalent in the following sense: two expressions have the same denotation if and only if they have the same extension. In the general intuitionistic context, however, these notions are not equivalent. Two sentences may denote the same proposition in some world $\mathcal{M}_i$ but may denote distinct propositions in an accessible world $\mathcal{M}_j$. In this case, the sentences have the same denotation in $\mathcal{M}_i$ but not the same extension in $\mathcal{M}_i$.

If we speak about an extensional model, then usually we mean a model with exactly two propositions, *verum*, *falsum* (i.e., the strong form of extensional model). Nevertheless, an extensional model $(\mathcal{M}_i, \gamma_i)$ can contain more than two propositions. This is the case if there is some proposition $p \in M_i$ such that no expression denotes p (we cannot say nothing about p). In all extensional models the extension of $\top$ is the set of all true expressions, and the extension of $\bot$ is the set of all false expressions. On the other hand, in an intensional model, extension of a sentence can be identified with its intension.

If the world $\mathcal{M}$ is classical and intensional, then the following holds for all sentences φ, ψ.

(i) $\Gamma(\varphi) = \Gamma(\psi) \Longrightarrow \varphi = \psi$, and
(ii) $\Gamma(\varphi) <^{\mathcal{M}} \Gamma(\psi) \Longrightarrow \varphi \prec \psi$.

In such a model denotation, extension, and intension of sentences are in one-to-one correspondence and can be considered essentially equivalent notions (i.e., two sentences have the same denotation if and only if they have the same extension if and only if they have the same intension if and only if they are identical). Moreover, if $\mathcal{M}$, in addition, has no nonstandard elements, then the Γ-function can be seen as an order isomorphism from the partial order (Sent(C), $\prec$) onto (M, $<^{\mathcal{M}}$). That is, the relation $\prec$ on the sentences is semantically mirrored by the reference relation on M.

In an extensional interpretation $(\mathcal{M}_i, \gamma_i)$ holds also the following:

$$(\mathcal{M}_i, \gamma_i) \vDash \varphi < \psi,$$

for all expressions φ, ψ which have any (not necessarily the same) truth value. In particular,

$$(\mathcal{M}_i, \gamma_i) \vDash \varphi < \varphi,$$

for all expressions φ that have a truth value at node i. Let us show this. We have $\top \prec \top$: true, $\top \prec \top$: false, $\bot \prec \bot$: true, $\bot \prec \bot$: false. The reference property (RP) of a world forces, respectively, the validity of the following sentences: $\top < (\top$: true), $\top < (\top$: false), $\bot < (\bot$: true), $\bot < (\bot$: false). Observe that $\top$: true and $\bot$: false are valid, whereas $\top$: false and $\bot$: true are contradictory. Thus, in an extensional world we have $\top \equiv (\top$: true), $\top \equiv (\bot$: false), $\bot \equiv (\top$: false), $\bot \equiv (\bot$: true). Hence, $\top < \top$, $\top < \bot$, $\bot < \top$, $\bot < \bot$ are true in an extensional model. Since every expression with a truth value is in the extension of either $\top$ or $\bot$, the assertion follows.

Definition 3.13 Let $\mathcal{F} = ((\mathcal{M}_i, \gamma_i)_{i \in I}, R)$ be a frame and let $(\mathcal{M}_i, \gamma_i)$ be any interpretation. We say that a statement $p \in M_i$ refers to a statement $q \in M_i$ if there are expressions φ, ψ such that φ denotes p and ψ denotes q and $(\mathcal{M}_i, \gamma_i) \vDash \varphi < \psi$. In particular, a statement $p \in M_i$ is self-referential if some φ denotes p and $(\mathcal{M}_i, \gamma_i) \vDash \varphi < \varphi$.

Note that in a nonclassical context $p <^{\mathcal{M}_i} q$ is in general not sufficient for the fact that statement q refers to statement p. If φ, ψ are expressions denoting p, q, respectively, then there may exist an accessible interpretation $(\mathcal{M}_j, \gamma_j)$ where $\Gamma_j(\varphi, \gamma_j) <^{\mathcal{M}_j} \Gamma_j(\psi, \gamma_j)$ does not hold. In this case, $(\mathcal{M}_i, \gamma_i) \nvDash \varphi < \psi$.

The proof of the following Substitution Lemma relies on ideas due to Zeitz who proved a similar Substitution Lemma for $\in_T$-Logic [25]. Also the following theorem is, reduced to the context of $\in_T$, implicitly contained in [25]. Note that—in contrast to the substitution property (SE)—the assertions of the lemma are not restricted to substitutions of variables but work with arbitrary substitutions.

Lemma 3.14 (Substitution Lemma) *Let $\mathcal{M} = (M, \text{TRUE}, \text{FALSE}, <^{\mathcal{M}}, \Gamma)$ be a world and let φ be an expression.*

(i) *Suppose that σ and σ' are substitutions and $\gamma, \gamma' \in M^V$ are assignments such that $\Gamma(\sigma(u), \gamma) = \Gamma(\sigma'(u), \gamma')$, for all $u \in$ at(φ). Then $\Gamma(\varphi[\sigma], \gamma) = \Gamma(\varphi[\sigma'], \gamma')$.*

(ii) *Suppose that $\gamma \in M^V$ is an assignment and σ is a substitution such that $\Gamma(c) = \Gamma(\sigma(c), \gamma)$, for every $c \in$ con(φ). Then $\Gamma(\varphi[\sigma], \gamma) = \Gamma(\varphi, \gamma\sigma)$.*

Proof The idea of the proof is to substitute all constants occurring in φ by variables of the same semantics. Then we may apply the substitution property (SP) from which the assertion will follow.

(i) For every $c \in \text{con}(\varphi)$, let $x_c \in V \smallsetminus (\text{var}(\varphi) \cup \text{var}(\varphi[\sigma]))$ such that $x_c \neq x_d$ for $c \neq d \in \text{con}(\varphi)$. We define three substitutions τ, ϱ, and ϱ' by

$$\tau : \text{con}(\varphi) \to V, c \mapsto x_c;$$

$$\varrho : V \to \text{Expr}(C), x \mapsto \begin{cases} \sigma(c) & \text{if } x = x_c, \text{ for some } c \in \text{con}(\varphi) \\ \sigma(x) & \text{else;} \end{cases}$$

$$\varrho' : V \to \text{Expr}(C), x \mapsto \begin{cases} \sigma'(c) & \text{if } x = x_c, \text{ for some } c \in \text{con}(\varphi) \\ \sigma'(x) & \text{else.} \end{cases}$$

Then by the previous results about properties of substitutions we get

$$\varphi[\tau][\varrho] = \varphi[\tau \circ \varrho] = \varphi[\sigma] \quad \text{and} \quad \varphi[\tau][\varrho'] = \varphi[\tau \circ \varrho'] = \varphi[\sigma'].$$

We can apply (SP) and get

$$\Gamma(\varphi[\sigma], \gamma) = \Gamma(\varphi[\tau][\varrho], \gamma) \overset{\text{(SP)}}{=} \Gamma(\varphi[\tau], \gamma\varrho) \text{ and}$$

$$\Gamma(\varphi[\sigma'], \gamma') = \Gamma(\varphi[\tau][\varrho'], \gamma') \overset{\text{(SP)}}{=} \Gamma(\varphi[\tau], \gamma'\varrho'). \quad (2)$$

By (CP) it is sufficient to show that $\gamma\varrho(x) = \gamma'\varrho'(x)$, for all $x \in \text{var}(\varphi[\tau])$. Let $x \in \text{var}(\varphi[\tau])$. If $x = x_c$ for some $c \in \text{con}(\varphi)$, then

$$(\gamma\varrho)(x) = \Gamma(\varrho(x), \gamma) = \Gamma(\sigma(c), \gamma) = \Gamma(\sigma'(c), \gamma') = \Gamma(\varrho'(x), \gamma') = (\gamma'\varrho')(x).$$

If $x \neq x_c$ for all $c \in \text{con}(\varphi)$, then

$$(\gamma\varrho)(x) = \Gamma(\sigma(x), \gamma) = \Gamma(\sigma'(x), \gamma') = (\gamma'\varrho')(x).$$

Thus, (CP) and (2) yield $\Gamma(\varphi[\sigma], \gamma) = \Gamma(\varphi[\sigma'], \gamma')$.

(ii) Put $\gamma' := \gamma\sigma$ and $\sigma' := \varepsilon$ the identity $u \mapsto u$. Then, on the one hand, we have for all $y \in \text{var}(\varphi)$, $\Gamma(\sigma'(y), \gamma') = \Gamma(y, \gamma\sigma) = \gamma\sigma(y) = \Gamma(\sigma(y), \gamma)$. On the other hand, we have for all $c \in \text{con}(\varphi)$, $\Gamma(\sigma'(c), \gamma') = \Gamma(c) = \Gamma(\sigma(c), \gamma)$. Hence, the assumption of (i) is satisfied. Now follows $\Gamma(\varphi, \gamma\sigma) = \Gamma(\varphi[\sigma'], \gamma') = \Gamma(\varphi[\sigma], \gamma)$. □

The Substitution Lemma implies the following Substitution Principle. The following corollary can be seen as an intensional version of Frege's well-known principle of substitution (or replacement).

Theorem 3.15 (Substitution Principle) *Let $\mathcal{M} = (M, \text{TRUE}, \text{FALSE}, <^{\mathcal{M}}, \Gamma)$ be a world. Let $\varphi_1, \varphi_2, \psi_1, \psi_2$ be expressions and let $\gamma, \gamma' : V \to M$ be assignments. Suppose that $\Gamma(\varphi_1, \gamma) = \Gamma(\psi_1, \gamma')$ and $\Gamma(\varphi_2, \gamma) = \Gamma(\psi_2, \gamma')$. Then the following hold:*

$$\Gamma(\varphi_1 : \text{true}, \gamma) = \Gamma(\psi_1 : \text{true}, \gamma')$$
$$\Gamma(\varphi_1 : \text{false}, \gamma) = \Gamma(\psi_1 : \text{false}, \gamma')$$
$$\Gamma(\varphi_1 \vee \varphi_2, \gamma) = \Gamma(\psi_1 \vee \psi_2, \gamma')$$
$$\Gamma(\varphi_1 \wedge \varphi_2, \gamma) = \Gamma(\psi_1 \wedge \psi_2, \gamma')$$
$$\Gamma(\varphi_1 \to \varphi_2, \gamma) = \Gamma(\psi_1 \to \psi_2, \gamma')$$
$$\Gamma(\varphi_1 \equiv \varphi_2, \gamma) = \Gamma(\psi_1 \equiv \psi_2, \gamma')$$
$$\Gamma(\varphi_1 < \varphi_2, \gamma) = \Gamma(\psi_1 < \psi_2, \gamma').$$

Proof We only show the third item. The other cases follow similarly. Let σ, σ' be substitutions such that $\sigma(x) = \varphi_1$, $\sigma(y) = \varphi_2$, $\sigma'(x) = \psi_1$, $\sigma'(y) = \psi_2$. Then by the assumptions of the theorem and (i) of the Substitution Lemma,

$$\Gamma(\varphi_1 \vee \varphi_2, \gamma) = \Gamma((x \vee y)[\sigma], \gamma) = \Gamma((x \vee y)[\sigma'], \gamma') = \Gamma(\psi_1 \vee \psi_2, \gamma').$$

□

Corollary 3.16 *Let $(\mathcal{M}, \gamma)$ be a world and suppose that $\varphi, \varphi', \psi, \psi'$ are expressions such that $\psi \prec \varphi$. Furthermore, let χ be any expression and let x be any variable.*

(i) *If ψ and ψ' have the same denotation in $(\mathcal{M}, \gamma)$ and we replace an occurrence of ψ in φ by ψ', then the result φ' has the same denotation as φ.*
(ii) $\Vdash (\psi \equiv \psi') \wedge (\varphi \equiv \chi[x := \psi]) \wedge (\varphi' \equiv \chi[x := \psi']) \rightarrow \varphi \equiv \varphi'$.

Proof Choose $\gamma = \gamma'$ in Theorem 3.15 and show the assertion inductively on the construction of expressions. □

Roughly speaking, the second item is a generalization of the first item to a broader context substituting "denotation" by "extension." Note that the usual principle of replacement in extensional logics fails: if ψ is a subexpression of φ, and ψ, ψ' are logically equivalent, then the substitution of some occurrence of ψ by ψ' in φ yields a formula φ' which, in general, is not logically equivalent with φ. Consider, for instance, $x \equiv (x : \text{true})$. If we replace $x : \text{true}$ by the logically equivalent expression x, then the result $x \equiv x$ is obviously not equivalent with the original expression which asserts a truth-teller. The failure of the extensional replacement principle reveals the intensional character of the logic. We refer the reader to Béziau ([5], p. 5) where the construction of such an intensional logic is supposed to be an open problem.

4 Some Model Constructions

The existence of models is not obvious; some effort must be spent in order to construct structures that satisfy the properties of a world and of a frame. We will start with the simplest case, the construction of an extensional classical world which guarantees the existence of frames. Constructions of extensional models usually follow the same strategy and can be found in similar forms in [18] and [25]. All other construction methods presented here are new. Of course, the absence of quantifiers simplifies matters (see the discussion on page 281). In particular, we present a construction of intensional (classical) standard worlds. We develop a general construction that builds a new frame from a set of given ones. A curious phenomenon in the intuitionistic case is that an intensional world may contain only three statements (in the classical case intensional models are infinite). Last, we show away that a world can be constructed that satisfies specific given nontrivial equations; that is, we may specify (up to a certain degree) which (self-)referential propositions the world will contain.

4.1 An extensional classical $\in_I$-model Let C be a set of constant symbols (recall that we require $\top, \bot \in C$). We choose a partition $C = C_T \cup C_F$ on C with $\top \in C_T$, $\bot \in C_F$. We define $\text{TRUE} := \{\top\}$, $\text{FALSE} := \{\bot\}$ and $M := \text{TRUE} \cup \text{FALSE}$. Furthermore, we put $<^{\mathcal{M}} := M \times M$. Let $\gamma : V \rightarrow M$ be any assignment. The semantic function is inductively defined as follows.

$$\Gamma(x,\gamma) = \gamma(x), \text{ for } x \in V$$

$$\Gamma(c,\gamma) = \begin{cases} \top & \text{if } c \in C_T \\ \bot & \text{if } c \in C_F \end{cases}$$

$$\Gamma(\varphi : \text{true}, \gamma) = \begin{cases} \top & \text{if } \Gamma(\varphi,\gamma) = \top \\ \bot & \text{else} \end{cases}$$

$$\Gamma(\varphi : \text{false}, \gamma) = \begin{cases} \top & \text{if } \Gamma(\varphi,\gamma) = \bot \\ \bot & \text{else} \end{cases}$$

$$\Gamma(\varphi \vee \psi, \gamma) = \begin{cases} \top & \text{if } \Gamma(\varphi,\gamma) = \top \text{ or } \Gamma(\psi,\gamma) = \top \\ \bot & \text{else} \end{cases}$$

$$\Gamma(\varphi \wedge \psi, \gamma) = \begin{cases} \top & \text{if } \Gamma(\varphi,\gamma) = \top \text{ and } \Gamma(\psi,\gamma) = \top \\ \bot & \text{else} \end{cases}$$

$$\Gamma(\varphi \rightarrow \psi, \gamma) = \begin{cases} \top & \text{if } \Gamma(\varphi,\gamma) = \bot \text{ or } \Gamma(\psi,\gamma) = \top \\ \bot & \text{else} \end{cases}$$

$$\Gamma(\varphi \equiv \psi, \gamma) = \begin{cases} \top & \text{if } \Gamma(\varphi,\gamma) = \Gamma(\psi,\gamma) \\ \bot & \text{else} \end{cases}$$

And finally,

$$\Gamma(\varphi < \psi, \gamma) = \top.$$

Let us show that $\mathcal{M} = (M, \text{TRUE}, \text{FALSE}, <^{\mathcal{M}}, \Gamma)$ is a world in the sense of Definition 3.1. Clearly, (EP) holds. (CP) and (SP) follow easily by induction on the expressions. (RP) is trivially satisfied. By construction, $(\mathcal{M}, \gamma)$ also satisfies the truth conditions of a frame (a singleton). It is easy to see that the world is extensional. We have proved the following.

Theorem 4.1 (Existence of models) *For every set of constant symbols C there exist (extensional classical) models (frames) with respect to the language* $\mathrm{Expr}(C)$.

4.2 Constructing a new frame from a set of given frames The above construction yields only singletons, that is, classical worlds. In the following we show how to construct a new frame from a set of given frames. The idea is to integrate the given frames into a new frame by adding a new world as the bottom world. In the nontrivial case (i.e., if there is at least one expression that distinguishes two of the given frames) the new frame will contain a world (the bottom world) which is not classical.

Definition 4.2 Let $\mathcal{F} = ((\mathcal{M}_i, \gamma_i)_{i \in I}, R)$ be a frame over (I, R) and let $i \in I$. Consider the set $I_i = \{j \in I \mid iRj\}$. It is clear that the structure $\mathcal{F}' = ((\mathcal{M}_j, \gamma_j)_{j \in I_i}, R_i)$ is a frame over (I_i, R_i) (with bottom node i), where R_i is the restriction of the order R onto I_i. We say that $\mathcal{F}'$ is a subframe of $\mathcal{F}$.

Theorem 4.3 *Let C be a set of constant symbols. Let $\beta > 0$ be an ordinal and let $(\mathcal{F}_\alpha \mid \alpha < \beta)$ be a sequence of frames over (I_α, R_α), respectively. We assume that the I_α are pairwise disjunct sets. Let 0_α denote the bottom node of (I_α, R_α). Then* *there exists a frame $\mathcal{F}$ with bottom node $0 \neq 0_\alpha$ ($\alpha < \beta$) such that each $\mathcal{F}_\alpha$ is a subframe of $\mathcal{F}$ and each node 0_α is an immediate successor node of the bottom node* 0.

Proof We construct a new frame $\mathcal{F}$ as follows. Let 0 be an element not contained in any I_i. We put $I = \bigcup\{I_\alpha \mid \alpha < \beta\} \cup \{0\}$. Let R be the transitive closure of $\bigcup\{R_\alpha \mid \alpha < \beta\} \cup \{(0,0), (0, 0_\alpha) \mid \alpha < \beta\}$. Then R is a partial order on I and 0 is the bottom node of I. We define a new world $\mathcal{M}_0 = (M_0, \text{TRUE}_0, \text{FALSE}_0, <^{\mathcal{M}_0}, \Gamma_0)$ by $\text{TRUE}_0 = \{\top\}$, $\text{FALSE}_0 = \{\bot\}$, $M_0 = \text{TRUE}_0 \cup \text{FALSE}_0 \cup \{n\}$, where n is a new symbol. Put $<^{\mathcal{M}_0} = M_0 \times M_0$. Let $\gamma_0 : V \to M_0$ be any assignment, $x \in V, c \in C$. The semantic function of $\mathcal{M}_0$ is defined as follows.

$$\Gamma_0(x, \gamma_0) = \gamma_0(x)$$

$$\Gamma_0(c, \gamma_0) = \begin{cases} \top & \text{if for all } j \in I \setminus \{0\} \text{ with } 0Rj : \Gamma_j(c) \in \text{TRUE}_j \\ \bot & \text{if for all } j \in I \setminus \{0\} \text{ with } 0Rj : \Gamma_j(c) \notin \text{TRUE}_j \\ n & \text{else} \end{cases}$$

$$\Gamma_0(\varphi : \text{true}, \gamma_0) = \Gamma_0(\varphi, \gamma_0)$$

$$\Gamma_0(\varphi : \text{false}, \gamma_0) = \begin{cases} \top & \text{if } \Gamma_0(\varphi, \gamma_0) = \bot \\ \bot & \text{if for all } i \in I \text{ with } 0Ri : \Gamma_i(\varphi, \gamma_i) \notin \text{FALSE}_i \\ n & \text{else} \end{cases}$$

$$\Gamma_0(\varphi \vee \psi, \gamma_0) = \begin{cases} \top & \text{if } \Gamma_0(\varphi, \gamma_0) = \top \text{ or } \Gamma_0(\psi, \gamma_0) = \top \\ \bot & \text{if for all } i \in I \text{ with } 0Ri : \Gamma_i(\varphi, \gamma_i) \notin \text{TRUE}_i \text{ and} \\ & \quad \Gamma_i(\psi, \gamma_i) \notin \text{TRUE}_i \\ n & \text{else} \end{cases}$$

$$\Gamma_0(\varphi \wedge \psi, \gamma_0) = \begin{cases} \top & \text{if } \Gamma_0(\varphi, \gamma_0) = \top \text{ and } \Gamma_0(\psi, \gamma_0) = \top \\ \bot & \text{if for all } i \in I \text{ with } 0Ri : \Gamma_i(\varphi, \gamma_i) \notin \text{TRUE}_i \text{ or} \\ & \quad \Gamma_i(\psi, \gamma_i) \notin \text{TRUE}_i \\ n & \text{else} \end{cases}$$

$$\Gamma_0(\varphi \to \psi, \gamma_0) = \begin{cases} \top & \text{if for all } i \in I \text{ with } 0Ri : \Gamma_i(\varphi, \gamma_i) \notin \text{TRUE}_i \text{ or} \\ & \quad \Gamma_i(\psi, \gamma_i) \in \text{TRUE}_i \\ \bot & \text{if for all } i \in I \text{ with } 0Ri \text{ there is some } j \text{ with } iRj : \\ & \quad \Gamma_j(\varphi, \gamma_j) \in \text{TRUE}_j \text{ and } \Gamma_j(\psi, \gamma_j) \notin \text{TRUE}_j \\ n & \text{else} \end{cases}$$

$$\Gamma_0(\varphi \equiv \psi, \gamma_0) = \begin{cases} \top & \text{if for all } i \in I \text{ with } 0Ri : \Gamma_i(\varphi, \gamma_i) = \Gamma_i(\psi, \gamma_i) \\ \bot & \text{if for all } i \in I \text{ with } 0Ri \text{ there is some } j \text{ with } iRj : \\ & \quad \Gamma_j(\varphi, \gamma_j) \neq \Gamma_j(\psi, \gamma_j) \\ n & \text{else} \end{cases}$$

$$\Gamma_0(\varphi < \psi, \gamma_0) = \begin{cases} \top & \text{if for all } i \in I \text{ with } 0Ri : \Gamma_i(\varphi, \gamma_i) <^{\mathcal{M}_i} \Gamma_i(\psi, \gamma_i) \\ \bot & \text{if for all } i \in I \text{ with } 0Ri \text{ there is some } j \text{ with } iRj : \\ & \quad \text{not } \Gamma_j(\varphi, \gamma_j) <^{\mathcal{M}_j} \Gamma_j(\psi, \gamma_j) \\ n & \text{else.} \end{cases}$$

As in the previous construction, the proof that the semantic function satisfies the structure properties (EP), (CP), (SP), (RP) is straightforward. Hence, $\mathcal{M}_0$ is a world. The assignment γ_0 above is arbitrary. Now we define a specific $\gamma_0 : V \to M_0$ by

$$\gamma_0(x) := \begin{cases} \top & \text{if for all } j \in I \setminus \{0\} \text{ with } 0Rj, \gamma_j(x) \in \text{TRUE}_j \\ \bot & \text{if for all } j \in I \setminus \{0\} \text{ with } 0Rj, \gamma_j(x) \notin \text{TRUE}_j \\ n & \text{else.} \end{cases}$$

It follows readily from the construction that the truth conditions hold at $0 \in I$, that is, for Γ_0 and γ_0. By hypothesis, the truth conditions also hold at all nodes $i \in I \setminus \{0\}$. Hence, $\mathcal{F} = ((\mathcal{M}_i, \gamma_i)_{i \in I}, R)$ is a frame. □

Note that the interpretation $(\mathcal{M}_0, \gamma_0)$ constructed above is, in general, not extensional although it contains exactly one true and exactly one false proposition (all expressions with the same truth value have the same denotation in $\mathcal{M}_0$). If an expression φ is true at 0, then $\Gamma_0(\varphi, \gamma_0) = \Gamma_0(\top, \gamma_0)$. But this equation does not necessarily hold in all accessible worlds (which may contain more than one true/false proposition); that is, the equation $\varphi \equiv \top$ is not necessarily satisfied at node 0. Indeed, if at least one of the accessible worlds is intensional, then also $\mathcal{M}_0$ is intensional, by Lemma 3.11. This is a nice example for the distinction between *denotation* and *extension* of expressions. However, if all frames $\mathcal{F}_\alpha$ $(\alpha < \beta)$ are extensional, then $(\mathcal{M}_0, \gamma_0)$ is extensional, too.

The interpretation $(\mathcal{M}_0, \gamma_0)$ is uniquely determined by the given sequence of frames $(\mathcal{F}_\alpha \mid \alpha < \beta)$ and the above construction. This leads us to the following definition.

Definition 4.4 Let $(\mathcal{F}_\alpha \mid \alpha < \beta)$ be a sequence of frames over (I_α, R_α) such as given in Theorem 4.3. We call the frame $\mathcal{F} = ((\mathcal{M}_i, \gamma_i)_{i \in I}, R)$, constructed in the proof of Theorem 4.3, the integration frame of the sequence of frames $(\mathcal{F}_\alpha \mid \alpha < \beta)$.

4.3 Constructions of intensional $\in_I$-models In the following, we present constructions of intensional models. In a first step we construct a classical standard model. Theorem 4.3 (together with Lemma 3.11) then yields intensional models which are not classical.

4.3.1 The intensional classical standard model Let C be a set of constant symbols. We put $M := \text{Sent}(C)$. Observe that in this case an assignment $\gamma : V \to M$ is also a substitution and $\varphi[\gamma]$ is a sentence, for any expression φ. We define the Gamma function by

$$\Gamma(\varphi, \gamma) = \varphi[\gamma].$$

The reference relation $<^{\mathcal{M}}$ on M is given by $\prec$; that is, for sentences φ, ψ we define

$$\varphi <^{\mathcal{M}} \psi :\Longleftrightarrow \varphi \prec \psi.$$

The subsets TRUE, FALSE $\subseteq M$ will be determined later. At this stage we already are able to show the structure properties. (EP) is clear and (CP) follows from

Lemma 2.4. Now suppose that σ is any substitution of variables and γ is an assignment. Recall that the assignment $\gamma\sigma$ is defined by $\gamma\sigma(x) = \Gamma(\sigma(x), \gamma)$. By the definitions, $\gamma\sigma(x) = \sigma(x)[\gamma] = (\sigma \circ \gamma)(x)$ for all $x \in V$. Thus, $\gamma\sigma$ and $\sigma \circ \gamma$ are the same substitutions. Now we get the following:

$$\Gamma(\varphi[\sigma], \gamma) = \varphi[\sigma][\gamma] \overset{(*)}{=} \varphi[\sigma \circ \gamma] = \varphi[\gamma\sigma] = \Gamma(\varphi, \gamma\sigma),$$

where (*) indicates the application of Lemma 2.5. Hence, (SP) holds. In order to show the reference property (RP) suppose that γ is an assignment and $\varphi \prec \psi$ for expressions φ, ψ. By Lemma 2.8, $\varphi[\gamma] \prec \psi[\gamma]$, and (RP) follows.

It remains to assign truth values to the elements of M; that is, we have to determine the subsets TRUE and FALSE. This is managed inductively in the following way. First, we choose a partition $C = C_T \cup C_F$ that divides the set of constant symbols in two disjoint subsets, the "true" and the "false" constants. As usual we assume that $\bot \in C_F$ and $\top \in C_T$. Now we define inductively

$$\begin{array}{rl}
c \in \mathrm{TRUE} & \text{if } c \in C_T. \\
c \in \mathrm{FALSE} & \text{if } c \in C_F. \\
\varphi : \mathrm{true} \in \mathrm{TRUE} & \text{if } \varphi \in \mathrm{TRUE}. \\
\varphi : \mathrm{true} \in \mathrm{FALSE} & \text{if } \varphi \in \mathrm{FALSE}. \\
\varphi : \mathrm{false} \in \mathrm{TRUE} & \text{if } \varphi \in \mathrm{FALSE}. \\
\varphi : \mathrm{false} \in \mathrm{FALSE} & \text{if } \varphi \in \mathrm{TRUE}. \\
\varphi \vee \psi \in \mathrm{TRUE} & \text{if } \varphi \in \mathrm{TRUE} \text{ or } \psi \in \mathrm{TRUE}. \\
\varphi \vee \psi \in \mathrm{FALSE} & \text{if } \varphi \in \mathrm{FALSE} \text{ and } \psi \in \mathrm{FALSE}. \\
\varphi \wedge \psi \in \mathrm{TRUE} & \text{if } \varphi \in \mathrm{TRUE} \text{ and } \psi \in \mathrm{TRUE}. \\
\varphi \wedge \psi \in \mathrm{FALSE} & \text{if } \varphi \in \mathrm{FALSE} \text{ or } \psi \in \mathrm{FALSE}. \\
\varphi \rightarrow \psi \in \mathrm{TRUE} & \text{if } \varphi \in \mathrm{FALSE} \text{ or } \psi \in \mathrm{TRUE}. \\
\varphi \rightarrow \psi \in \mathrm{FALSE} & \text{if } \varphi \in \mathrm{TRUE} \text{ and } \psi \in \mathrm{FALSE}. \\
\varphi \equiv \psi \in \mathrm{TRUE} & \text{if } \varphi = \psi. \\
\varphi \equiv \psi \in \mathrm{FALSE} & \text{if } \varphi \neq \psi. \\
\varphi < \psi \in \mathrm{TRUE} & \text{if } \varphi \prec \psi. \\
\varphi < \psi \in \mathrm{FALSE} & \text{if } \varphi \not\prec \psi.
\end{array}$$

It is obvious that the world $\mathcal{M} = (M, \mathrm{TRUE}, \mathrm{FALSE}, <^{\mathcal{M}}, \Gamma)$ is intensional. It is also easy to see that M is the disjoint union of TRUE and FALSE. Furthermore, for any assignment $\gamma : V \rightarrow M$ the frame consisting of the singleton $(\mathcal{M}, \gamma)$ satisfies the truth conditions. We have proved the following.

Theorem 4.5 *For every set of constant symbols C there exists an intensional $\in_I$-model. Moreover, we may construct an intensional classical world $\mathcal{M}$ with no non-standard elements.*

It is clear that the above-constructed intensional classical model depends only on the set C and on the partition of C into the sets C_T and C_F. For a given set C and a given partition $C = C_T \cup C_F$ we call this model (world) the *intensional classical* *standard model*. Since it is classical we have

$$\mathcal{M} \vDash \varphi \equiv \psi \Longleftrightarrow \Gamma(\varphi \equiv \psi) \in \text{TRUE} \Longleftrightarrow \Gamma(\varphi) = \Gamma(\psi) \Longleftrightarrow \varphi = \psi$$

$$\mathcal{M} \vDash \varphi < \psi \Longleftrightarrow \Gamma(\varphi < \psi) \in \text{TRUE} \Longleftrightarrow \Gamma(\varphi) <^{\mathcal{M}} \Gamma(\psi) \Longleftrightarrow \varphi \prec \psi,$$

for sentences φ, ψ. It follows that the Gamma function is the identity map from $(\text{Sent}(C), \prec)$ onto $(M, <^{\mathcal{M}})$ and, in particular, an isomorphism between the two partial orders. We may therefore assume that the propositional universe of the intensional classical standard model consists exactly of the set of sentences and is partially ordered by $<^{\mathcal{M}} = \prec$.

4.3.2 Nonclassical intensional standard models We obtain nonclassical intensional worlds by applying the construction method of Theorem 4.3. Since in Section 3 we have constructed intensional models, the following is an immediate consequence of Theorem 4.3 and Lemma 3.11.

Theorem 4.6 *There exist intensional nonclassical worlds.*

4.4 An example: Constructing a classical standard world with exactly two self-referential propositions We have constructed extensional and intensional models, and we have developed a general method to construct a new frame from a set of given frames. In an extensional model all equations between true (between false) sentences are satisfied, respectively. On the other hand, in an intensional model only equations between identical sentences are satisfied. Thus, in an intensional standard model there are no self-referential propositions at all. The question arises which intermediate cases exist between these two extremes. Can we construct (infinite) standard models which contain only a few specific (self-)referential propositions? In this section we give a partial answer. We show in the form of an example how to construct models that satisfy some equations which assert specific self-referential propositions. These equations, however, are in general not independent from each other and the question for a general construction method for this kind of models remains open. It would be nice to get an overview of the hierarchy of models between the intensional classical standard model and the extensional classical model. In the following we construct a classical standard model that contains exactly two self-referential propositions, a true truth-teller and a false truth-teller.

Let $C = \{\bot, \top, c, d\}$ be a set of constant symbols and let $\mathcal{M}$ be the intensional classical standard model constructed above in order to prove Theorem 4.5. We assume that the constant symbol c is interpreted as a true proposition and d is interpreted as a false proposition in $\mathcal{M}$. Recall that the universe of $\mathcal{M}$ is the set of sentences: $M = \text{Sent}(C)$. In order to construct a model which identifies c with c : true and d with d : true we define an appropriate equivalence relation on M and interpret the sentences by their equivalence classes. Consider the set of pairs $E = \{(c, c : \text{true}), (d, d : \text{true})\}$ and let E^* be the smallest equivalence relation on $\text{Sent}(C)$ that contains E and is closed under the following condition, called the *congruence property* of E^*. If $(\varphi, \psi) \in E^*$ and $(\varphi', \psi') \in E^*$, then

$$(\varphi : \text{true}, \psi : \text{true}) \in E^*,$$

$$(\varphi : \text{false}, \psi : \text{false}) \in E^*,$$

$$(\varphi \equiv \varphi', \psi \equiv \psi') \in E^*,$$

$$(\varphi < \varphi', \psi < \psi') \in E^*,$$

$$(\varphi \vee \varphi', \psi \vee \psi') \in E^*,$$
$$(\varphi \wedge \varphi', \psi \wedge \psi') \in E^*,$$
$$(\varphi \to \varphi', \psi \to \psi') \in E^*.$$

We call E^* the congruence generated by E. In the rest of this chapter we will prove the following claim.

Claim 1 There exists a classical standard world $\mathcal{M}^E$ with the property

$$\mathcal{M}^E \vDash \varphi \equiv \psi \iff (\varphi, \psi) \in E^*, \text{ and}$$
$$(\mathcal{M}^E, \gamma) \vDash c : \text{true} \wedge d : \text{false} \wedge ((x < x) \leftrightarrow ((x \equiv c) \vee (x \equiv d))),$$

for all sentences φ, ψ and for all assignments $\gamma : V \to M^E$. In particular, $\mathcal{M}^E$ contains exactly two self-referential propositions: a true truth-teller denoted by c and a false truth-teller denoted by d.

Proof Let φ^E denote the equivalence class of $\varphi \in \text{Sent}(C)$ modulo E^*. We put $M^E := \{\varphi^E \mid \varphi \in \text{Sent}(C)\}$. For an assignment $\gamma : V \to M^E$ let $\delta_\gamma : V \to \text{Sent}(C)$ be a choice function that picks an element $\delta_\gamma(x) \in \gamma(x)$ for each $x \in V$. In particular, δ_γ is a substitution of variables. We define a new Gamma function by

$$\Gamma^E(\varphi, \gamma) := (\varphi[\delta_\gamma])^E,$$

for $\varphi \in \text{Expr}(C)$. That is, each expression φ is mapped to the equivalence class of $\varphi[\delta_\gamma]$ modulo E^*. Γ^E is independent of the particular choice function δ_γ. For let $\delta'_\gamma : V \to \text{Sent}(C)$ be another choice function; that is, $\delta'_\gamma(x) \in \gamma(x)$ for each $x \in V$. Then by induction on φ (using the above congruence property of E^*) one sees that $(\varphi[\delta_\gamma], \varphi[\delta'_\gamma]) \in E^*$, for any expression φ. Thus, $\Gamma^E(\varphi, \gamma)$ does not depend on the choice function δ_γ.

Notice that $\delta_\gamma : V \to M$ is an assignment in the context of the intensional model $\mathcal{M}$. So from the definitions it follows that

$$\Gamma^E(\varphi, \gamma) = \Gamma(\varphi, \delta_\gamma)^E.$$

We define the reference relation $<^E$ on M^E by

$$\varphi^E <^E \psi^E :\iff \text{there are sentences } \varphi' \in \varphi^E \text{ and } \psi' \in \psi^E \text{ such that } \varphi' \prec \psi'.$$

That is, $\varphi^E <^E \psi^E$ if and only if there are sentences $\varphi' \in \varphi^E$ and $\psi' \in \psi^E$ such that $\varphi' <^{\mathcal{M}} \psi'$, where $<^{\mathcal{M}}$ is the reference relation of the model $\mathcal{M}$. It is clear that the relation $\varphi^E <^E \psi^E$ defined in this way is independent of its representatives. The sets TRUE^E, FALSE^E are inductively defined as follows.[3]

$$c^E \in \text{TRUE}^E \text{ and } \top^E \in \text{TRUE}^E.$$
$$d^E \in \text{FALSE}^E \text{ and } \bot^E \in \text{FALSE}^E.$$
$$(\varphi \equiv \psi)^E \in \text{TRUE}^E \qquad \text{if } \varphi^E = \psi^E.$$
$$(\varphi \equiv \psi)^E \in \text{FALSE}^E \qquad \text{if } \varphi^E \neq \psi^E.$$
$$(\varphi < \psi)^E \in \text{TRUE}^E \qquad \text{if } \varphi^E <^E \psi^E.$$
$$(\varphi < \psi)^E \in \text{FALSE}^E \qquad \text{if not } \varphi^E <^E \psi^E.$$

$$
\begin{array}{rl}
(\varphi : true)^E \in \text{TRUE}^E & \text{if } \varphi^E \in \text{TRUE}^E. \\
(\varphi : true)^E \in \text{FALSE}^E & \text{if } \varphi^E \in \text{FALSE}^E. \\
(\varphi : false)^E \in \text{TRUE}^E & \text{if } \varphi^E \in \text{FALSE}^E. \\
(\varphi : false)^E \in \text{FALSE}^E & \text{if } \varphi^E \in \text{TRUE}^E. \\
(\varphi \vee \psi)^E \in \text{TRUE}^E & \text{if } \varphi^E \in \text{TRUE}^E \text{ or } \psi^E \in \text{TRUE}^E. \\
(\varphi \vee \psi)^E \in \text{FALSE}^E & \text{if } \varphi^E \in \text{FALSE}^E \text{ and } \psi^E \in FALSE^E. \\
(\varphi \wedge \psi)^E \in \text{TRUE}^E & \text{if } \varphi^E \in \text{TRUE}^E \text{ and } \psi^E \in TRUE^E. \\
(\varphi \wedge \psi)^E \in \text{FALSE}^E & \text{if } \varphi^E \in \text{FALSE}^E \text{ or } \psi^E \in FALSE^E. \\
(\varphi \rightarrow \psi)^E \in \text{TRUE}^E & \text{if } \varphi^E \in \text{FALSE}^E \text{ or } \psi^E \in TRUE^E. \\
(\varphi \rightarrow \psi)^E \in \text{FALSE}^E & \text{if } \varphi^E \in \text{TRUE}^E \text{ and } \psi^E \in FALSE^E.
\end{array}
$$

By induction on the sentences one easily checks that $\text{TRUE}^E \cap \text{FALSE}^E = \varnothing$ and $\text{TRUE}^E \cup \text{FALSE}^E = M^E$.

In order to see that $\mathcal{M}^E = (M^E, \text{TRUE}^E, \text{FALSE}^E, <^E \Gamma^E)$ is a world we must show that the structure properties and the truth conditions are satisfied. It follows immediately from the definition of the Gamma function and from the inductive definition of the sets TRUE^E, FALSE^E that for every assignment $\gamma : V \rightarrow M^E$ the truth conditions are satisfied. Let us look at the structure properties. Let $x \in V$ and suppose that $\gamma : V \rightarrow M^E$ is an assignment. Then $\Gamma^E(x, \gamma) = (\delta_\gamma(x))^E = \gamma(x)$. Thus (EP) holds. In order to show (CP) let φ be any expression and suppose that γ, γ' are assignments such that $\gamma(x) = \gamma'(x)$ for all $x \in \text{var}(\varphi)$. By induction on φ, considering the above congruence property of E^*, we get $(\varphi[\delta_\gamma], \varphi[\delta_{\gamma'}]) \in E^*$. Hence, $\Gamma^E(\varphi, \gamma) = (\varphi[\delta_\gamma])^E = (\varphi[\delta_{\gamma'}])^E = \Gamma^E(\varphi, \gamma')$.

Now we aim for (SP). Let $\sigma : V \rightarrow \text{Expr}(C)$ be a substitution of variables, φ an expression, and γ an assignment. Recall that by $\gamma\sigma$ we denote the assignment defined by $\gamma\sigma(x) = \Gamma^E(\sigma(x), \gamma)$, for $x \in V$.

Claim 2 $(\varphi[\sigma \circ \delta_\gamma])^E = (\varphi[\delta_{\gamma\sigma}])^E$. In order to prove Claim 2 we use induction on φ. The assertion is clear for $\varphi = e \in C$. Let $\varphi = x \in V$. Then

$$
\begin{aligned}
(x[\sigma \circ \delta_\gamma])^E &= (\sigma(x)[\delta_\gamma])^E \\
&= \Gamma^E(\sigma(x), \gamma) && \text{by definition of the Gamma function} \\
&= \gamma\sigma(x) && \text{by definition of the assignment } \gamma\sigma \\
&= \Gamma^E(x, \gamma\sigma) && \text{by (EP)} \\
&= (x[\delta_{\gamma\sigma}])^E && \text{by definition of the Gamma function.}
\end{aligned}
$$

Now suppose $\varphi = (\psi : \text{true})$. Applying the definition of substitutions, the induction hypothesis and the congruence property of E^*, we get

$$
\begin{aligned}
((\psi : \text{true})[\sigma \circ \delta_\gamma])^E &= (\psi[\sigma \circ \delta_\gamma] : \text{true})^E \\
&= (\psi[\delta_{\gamma\sigma}] : \text{true})^E = ((\psi : \text{true})[\delta_{\gamma\sigma}])^E.
\end{aligned}
$$

The other cases follow in a similar way. We have proved Claim 2.

Now, applying the definition of the Gamma function, Lemma 2.5 and Claim 2, we get

$$\Gamma^E(\varphi[\sigma], \gamma) = (\varphi[\sigma][\delta_\gamma])^E = \varphi[\sigma \circ \delta_\gamma]^E = \varphi[\delta_{\gamma\sigma}]^E = \Gamma^E(\varphi, \gamma\sigma).$$

Thus, (SP) holds.

Let us show that (RP) is satisfied. So let γ be an assignment and suppose $\varphi \prec \psi$ for expressions φ, ψ. By Lemma 2.8 we get $\varphi[\delta_\gamma] \prec \psi[\delta_\gamma]$. The definition of the reference relation $<^E$ yields $\Gamma^E(\varphi, \gamma) = \varphi[\delta_\gamma]^E <^E \psi[\delta_\gamma]^E = \Gamma^E(\psi, \gamma)$. Hence, (RP) holds. □

We may force further identifications. For instance, it might be interesting to get models in which logically equivalent expressions, such as $\varphi \vee \psi$ and $\psi \vee \varphi$, denote the same proposition, and no other expressions are identified. It seems that the construction method of the above example can be generalized up to a certain degree. However, one has to pay attention here. Suppose we wish to construct a world which satisfies the equations $c \equiv (c : \text{true})$ and $d \equiv (c < c)$. Again, we start by the intensional classical standard world $\mathcal{M}$ in which c is interpreted by a true proposition and d is interpreted by a false proposition. At a first glance there seems to be no problem to identify the sentences c, $(c : \text{true})$ and d, $(c < c)$, respectively, since the former have both the truth value "true" and the latter have both the truth value "false" in $\mathcal{M}$. However, if we identify c with $c : \text{true}$, then the equation $c \equiv (c : \text{true})$ holds in the world, and (RP) together with the truth conditions force that $c < c$ is also satisfied in the world. But then d must be true too, since we require that the world satisfies $d \equiv (c < c)$.

A further development of the here-presented constructions, that is, a model theory of $\in_I$-Logic, seems to be an interesting task for future studies. Another aim is the elaboration of a sound and complete calculus. Finally, the extension of this quantifier-free version of $\in_I$ to an intensional intuitionistic logic with first-order quantification over statements is a further interesting challenge.

Notes

1. By *reflexive languages* we mean, roughly speaking, languages where all formulas are terms; that is, there is no distinction between terms and formulas.

2. Notice that we cannot write "$\varphi \equiv \psi$ is true whenever $\varphi \leftrightarrow \psi$ is true." $\varphi \leftrightarrow \psi$ may be true but φ, ψ may have no truth values in the given world.

3. Note that we cannot define $\text{TRUE}^E := \text{TRUE}/E^*$. Consider, for instance, $c \equiv (c : \text{true}) \in \text{FALSE}$. We must have $(c \equiv (c : \text{true}))^E \in \text{TRUE}^E$.

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