Nagoya Math. J. **201** (2011), 117–131
DOI 10.1215/00277630-2010-018

COHEN-MACAULAY EDGE IDEAL WHOSE HEIGHT IS HALF OF THE NUMBER OF VERTICES

MARILENA CRUPI, GIANCARLO RINALDO, AND NAOKI TERAI

Abstract. We consider a class of graphs G such that the height of the edge ideal $I(G)$ is half of the number $\sharp V(G)$ of the vertices. We give Cohen-Macaulay criteria for such graphs.

§0. Introduction

In this article, a *graph* means a simple graph without loops and multiple edges. Let G be a graph with the vertex set $V(G) = \{x_1, \ldots, x_n\}$ and with the edge set $E(G)$. Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field K. The *edge ideal* $I(G)$, associated to G, is the ideal of S generated by the set of all square-free monomials $x_i x_j$ so that x_i is adjacent to x_j. For this ideal, the following theorem is known.

THEOREM 0.1 (see [5]). *Suppose that G is an unmixed graph without isolated vertices. Then we have* $2\,\mathrm{height}\, I(G) \geq \sharp V(G)$.

In this article, we treat the class of graphs for which the above equality holds; that is, we consider an unmixed graph without isolated vertex with $2\,\mathrm{height}\, I(G) = \sharp V(G)$. Such a class of graphs is rich, because it includes all the unmixed bipartite graphs and all the grafted graphs. Herzog and Hibi [8] gave beautiful theorems on Cohen-Macaulay edge ideals of bipartite graphs. Our purpose in this article is to generalize their results for our class of graphs.

Received May 19, 2009. Revised March 8, 2010. Accepted May 9, 2010.
2000 Mathematics Subject Classification. Primary 05C75; Secondary 05C90, 13H10, 55U10.
Terai's work partially supported by Gruppo Nazionale per le Strutture Algebriche e Geometriche e loro Applicazioni - Instituto Nazionale di Alta Matematica and by KAKENHI-18540041 and KAKENHI-20540047.

It is known that a graph G in our class has a perfect matching (see [6, Remark 2.2]). We may assume that

$$V(G) = X \cup Y, \qquad X \cap Y = \emptyset, \tag{*}$$

where $X = \{x_1, \ldots, x_n\}$ is a minimal vertex cover of G and where $Y = \{y_1, \ldots, y_n\}$ is a maximal independent set of G such that $\{x_1y_1, \ldots, x_ny_n\} \subset E(G)$.

Hence, $\{x_1 - y_1, \ldots, x_n - y_n\}$ is a system of parameters of $S/I(G)$. In Sections 3 and 4, using assumption (*), we give the following characterization of Cohen-Macaulayness, which is similar to the case of bipartite graphs (see [8, Corollary 3.5]).

THEOREM 0.2. *Let G be an unmixed graph with $2n$ vertices, which are not isolated, and with* height $I(G) = n$. *Then the following conditions are equivalent.*

(1) *G is Cohen-Macaulay.*
(2) *$\Delta(G)$ is strongly connected.*
(3) *There is a unique perfect matching in G.*
(4) *G is shellable.*

Note that it includes equivalence between Cohen-Macaulayness and shellability as in the bipartite graphs (see [3]).

We also have a Cohen-Macaulay criterion which is similar to that of Herzog and Hibi [8, Theorem 3.4].

THEOREM 0.3. *Let G be a graph with $2n$ vertices, which are not isolated, and with* height $I(G) = n$. *We assume conditions (*) and*

$$x_iy_j \in E(G) \text{ implies } i \leq j. \tag{**}$$

Then the following conditions are equivalent.

(1) *G is Cohen-Macaulay.*
(2) *G is unmixed.*
(3) *The following conditions hold:*
    (i) *if $z_ix_j, y_jx_k \in E(G)$, then $z_ix_k \in E(G)$ for distinct i, j, k and for $z_i \in \{x_i, y_i\}$;*
    (ii) *if $x_iy_j \in E(G)$, then $x_ix_j \notin E(G)$.*

Although in Herzog and Hibi [8] Alexander duality plays an important role in their proof, we give a direct and elementary proof without it. The

Herzog-Hibi criterion for bipartite graphs is discussed by other authors in the literature that give alternative proofs for it (see [7], [12]).

In Section 5, we introduce a new class of graphs which we call *B-grafted graphs*. They are a generalization of grafted graphs introduced by Faridi [4]. If G is an unmixed B-grafted graph, then we have $2\,\mathrm{height}\, I(G) = \sharp V(G)$. Hence, applying our main result, we show the following.

THEOREM 0.4. *The B-grafted graph $G(H_0; B_1, \ldots, B_p)$ is Cohen-Macaulay (resp., unmixed) if and only if every bipartite graph B_i is Cohen-Macaulay (resp., unmixed) for $i = 1, \ldots, p$.*

See Sections 1 and 5 for undefined concepts and notation.

§1. Preliminaries

In this section, we recall some concepts and a notation on graphs and on simplicial complexes that we use in the article.

Let G be a graph with the vertex set $V(G) = \{x_1, \ldots, x_n\}$ and with the edge set $E(G)$. The *induced subgraph* $G|_W$ by $W \subset V(G)$ is defined by

$$G|_W = \big(W, \{e \in E(G); e \subset W\}\big).$$

For $W \subset V(G)$, we denote $G|_{V(G)\setminus W}$ by $G - W$. For $F \subset E(G)$, we denote $(V(G), E(G) \setminus F)$ by $G - F$. For a family F of two-element subsets of $V(G)$, we denote $(V(G), E(G) \cup F)$ by $G + F$.

A subset $C \subset V(G)$ is a *vertex cover* of G if every edge of G is incident with at least one vertex in C. A vertex cover C of G is called *minimal* if there is no proper subset of C which is a vertex cover of G. A subset A of $V(G)$ is called an *independent set* of G if no two vertices of A are adjacent. An independent set A of G is *maximal* if there exists no independent set which properly includes A. Observe that C is a minimal vertex cover of G if and only if $V(G) \setminus C$ is a maximal independent set of G. And also note that $\mathrm{height}\, I(G)$ is equal to the smallest number $\sharp C$ of vertices among all the minimal vertex covers C of G. A graph G is called *unmixed* if all the minimal vertex covers of G have the same number of elements. A graph G is called *Cohen-Macaulay* if $S/I(G)$ is a Cohen-Macaulay ring, where $S = K[x_1, \ldots, x_n]$ is a polynomial ring for a field K.

Finally, a subgraph H of a graph G with $V(G) = V(H)$ is called a *perfect matching* if every connected component of H is a 2-complete graph.

See [2] and [13] for detailed information on this subject.

Set $V = \{x_1, \ldots, x_n\}$. A *simplicial complex* Δ on the vertex set V is a collection of subsets of V such that (i) $\{x_i\} \in \Delta$ for all $x_i \in V$ and (ii) $F \in \Delta$ and $G \subseteq F$ imply $G \in \Delta$. An element $F \in \Delta$ is called a *face* of Δ. For $F \subset V$, we define the *dimension* of F by $\dim F = \sharp F - 1$, where $\sharp F$ is the cardinality of the set F. A maximal face of Δ with respect to inclusion is called a *facet* of Δ. If all facets of Δ have the same dimension, then Δ is called *pure*.

A pure simplicial complex Δ is called *shellable* if the facets of Δ can be given a linear order $F_1, \ldots, F_m$ such that for all $1 \le j < i \le m$, there exist some $v \in F_i \setminus F_j$ and some $k \in \{1, \ldots, i-1\}$ with $F_i \setminus F_k = \{v\}$.

Moreover, a pure simplicial complex Δ is *strongly connected* if for every two facets F and G of Δ there is a sequence of facets $F = F_0, F_1, \ldots, F_m = G$ such that $\dim(F_i \cap F_{i+1}) = \dim \Delta - 1$ for each $i = 0, \ldots, m-1$.

If G is a graph, we define the *complementary simplicial complex* of G by

$$\Delta(G) = \big\{A \subseteq V(G) : A \text{ is an independent set in } G\big\}.$$

Observe that $\Delta(G)$ is the Stanley-Reisner simplicial complex of $I(G)$.

A graph G is called *shellable* if $\Delta(G)$ is a shellable simplicial complex.

§2. Unmixedness

In this section, we survey unmixed graphs whose edge ideals have the height that is half of the number of vertices.

LEMMA 2.1. *Let G be an unmixed graph with nonisolated $2n$ vertices and with* height $I(G) = n$. *Then G has a perfect matching.*

This fact is written in [6, Remark 2.2]. By the lemma for an unmixed graph G with $2n$ vertices, which are not isolated, and with height $I(G) = n$, we may assume that

$$V(G) = X \cup Y, \qquad X \cap Y = \emptyset, \tag{*}$$

where $X = \{x_1, \ldots, x_n\}$ is a minimal vertex cover of G and where $Y = \{y_1, \ldots, y_n\}$ is a maximal independent set of G such that $\{x_1y_1, \ldots, x_ny_n\} \subset E(G)$.

For the remainder of this article, set $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ for a field K, and $I(G)$ is an ideal of S. By Lemma 2.1, we have the following ring-theoretic properties of $S/I(G)$.

COROLLARY 2.2. *Let G be an unmixed graph with $2n$ vertices, which are not isolated, and with* height $I(G) = n$. *We assume condition (*). Then,*

(i) *each minimal prime ideal of* $I(G)$ *is of the form*

$$(x_{i_1}, \ldots, x_{i_k}, y_{i_{k+1}}, \ldots, y_{i_n}),$$

where $\{i_1, \ldots, i_n\} = \{1, \ldots, n\}$*;*

(ii) $\{x_1 - y_1, \ldots, x_n - y_n\}$ *is a system of parameters of* $S/I(G)$*.*

For later use we give a characterization of the unmixedness for our graphs, that is, a more detailed description, but a special case of a more general result (see [10, Theorem 2.9] and see [14, Theorem 1.1] for the bipartite case).

Proposition 2.3. *Let* G *be a graph with* $2n$ *vertices, which are not isolated, and with* $\operatorname{height} I(G) = n$*. We assume condition (*). Then* G *is unmixed if and only if the following conditions hold.*

(i) *If* $z_i x_j$*,* $y_j x_k \in E(G)$*, then* $z_i x_k \in E(G)$ *for distinct* i*,* j*,* k *and for* $z_i \in \{x_i, y_i\}$*.*

(ii) *If* $x_i y_j \in E(G)$*, then* $x_i x_j \notin E(G)$*.*

§3. Cohen-Macaulayness

In this section, we give combinatorial characterizations of Cohen-Macaulay graphs whose edge ideals have the height that is half of the number of vertices.

First, we introduce an operator that allows us to construct a new graph. Let G be a graph with $2n$ vertices, which are not isolated, and with $\operatorname{height} I(G) = n$. We assume condition (*).

For any $i \in [n] := \{1, \ldots, n\}$, set

$$E_i := \big\{k \in [n] : x_k y_i \in E(G)\big\} \setminus \{i\},$$

and define the graph $O_i(G)$ by

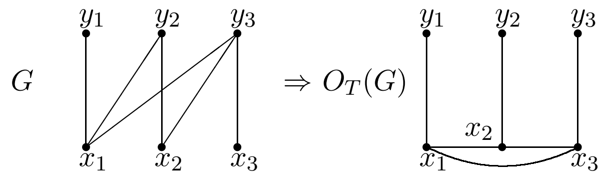
$$O_i(G) := G - \{x_k y_i : k \in E_i\} + \{x_k x_i : k \in E_i\}.$$

Then, for every nonempty subset $T := \{i_1, \ldots, i_\ell\}$ of the set $[n]$, we define

$$O_T(G) = O_{i_1} O_{i_2} \cdots O_{i_\ell}(G).$$

Moreover, if $T = \emptyset$, then we set $O_T(G) = G$. Note that $O_T(G)$ is a graph with $2n$ vertices, which are not isolated, and with $\operatorname{height} I(G) = n$, satisfying condition (*).

Example 3.1. Let $T = \{2,3\}$; then



The next proposition shows that the Cohen-Macaulayness of G can be checked by the unmixedness of all the deformations $O_T(G)$ of G.

Proposition 3.2. *Let G be an unmixed graph with $2n$ vertices, which are not isolated, and with* height $I(G) = n$. *We assume condition (*). Then the following conditions are equivalent.*

(1) *G is Cohen-Macaulay.*
(2) *$O_T(G)$ is Cohen-Macaulay for every subset T of $[n]$.*
(3) *$O_T(G)$ is unmixed for every subset T of $[n]$.*

Proof. Set $S = K[x_1,\dots,x_n,y_1,\dots,y_n]$, set $S_k = K[x_1,\dots,x_n,y_{k+1},\dots,y_n]$, and set $G_k = O_{T_k}(G)|_{X\cup\{y_{k+1},\dots,y_n\}}$.

(1) $\Longrightarrow$ (2). By relabeling, we may assume that $T = [k]$. Let G be a Cohen-Macaulay graph. Then

$$S/\big(I(G) + (x_1 - y_1,\dots,x_k - y_k)\big) \simeq S_k/\big(I(G_k) + (x_1^2,\dots,x_k^2)\big)$$

is Cohen-Macaulay. Since the polarization preserves Cohen-Macaulayness,

$$S/\big(I(G_k) + (x_1^2,\dots,x_k^2)\big)^{\mathrm{pol}} = S/\big(I(G_k) + (x_1y_1,\dots,x_ky_k)\big) = S/I\big(O_T(G)\big)$$

is Cohen-Macaulay, where $(x_1^2,\dots,x_k^2)^{\mathrm{pol}}$ stands for the polarization of $(x_1^2,\dots,x_k^2)$. See [11] for basic properties of polarization.

(2) $\Longrightarrow$ (3). Every Cohen-Macaulay ideal is unmixed (see [1]).

(3) $\Longrightarrow$ (1). Suppose that G is not Cohen-Macaulay. We want to prove that there exists a subset $T \subset [n]$ such that $O_T(G)$ is not unmixed. Since G is not Cohen-Macaulay, the sequence $\{x_i - y_i : 1 \le i \le n\}$ is not a regular sequence of $S/I(G)$. Hence, there exists $k \ge 1$ such that $\{x_i - y_i : i \in [k-1]\}$ is a regular sequence of $S/I(G)$ and $x_k - y_k$ is not regular on the ring $R := S_{k-1}/(I(G_{k-1}) + (x_1^2,\dots,x_{k-1}^2)) \simeq S/(I(G) + (x_1 - y_1,\dots,x_{k-1} - y_{k-1}))$. Set $J = I(G_{k-1}) + (x_1^2,\dots,x_{k-1}^2)$. Since $x_k - y_k$ is not regular on R, then

$$x_k - y_k \in \bigcup_{P\in \operatorname{Ass} R} P,$$

and there exists an associated prime ideal $\widetilde{P}$ of J such that $x_k - y_k \in \widetilde{P}$. Since $x_k \in \widetilde{P}$ or $y_k \in \widetilde{P}$, we have $x_k, y_k \in \widetilde{P}$. Hence, height $\widetilde{P} > n$. Hence, R is not unmixed. Therefore, $S/(I(G_{k-1}) + (x_1^2, \ldots, x_{k-1}^2))^{\mathrm{pol}} \simeq S/I(O_{T_{k-1}}(G))$ is not unmixed. ∎

For distinct $i_1, i_2, \ldots, i_r \in [n]$, we denote by $C_{i_1 i_2 \cdots i_r}$ the cycle C with

$$V(C) = \{x_{i_1}, y_{i_1}, x_{i_2}, \ldots, x_{i_r}, y_{i_r}\}$$

and

$$E(C) = \{x_{i_1}y_{i_1}, y_{i_1}x_{i_2}, x_{i_2}y_{i_2}, \ldots, y_{i_r}x_{i_r}, y_{i_r}x_{i_1}\}.$$

Proposition 3.3. *Let G be an unmixed graph with $2n$ vertices, which are not isolated, and with* height $I(G) = n$. *We assume condition (*). Then the following conditions are equivalent.*

(1) *The subset $\{x_1y_1, x_2y_2, \ldots, x_ny_n\}$ of $E(G)$ is a unique perfect matching in G.*
(2) *The cycle C_{ij} is not a subgraph of G for any $i < j$.*
(3) *For any $r \geq 2$, the cycle $C_{i_1 i_2 \cdots i_r}$ is not a subgraph of G for any subset $\{i_1, i_2, \ldots, i_r\} \subset [n]$ of cardinality r.*

Proof. (1) $\Longrightarrow$ (2). Suppose that C_{ij} is a subgraph of G. Then we have two perfect matchings in G:

$$\{x_1y_1, x_2y_2, \ldots, x_ny_n\},$$

$$\{x_1y_1, x_2y_2, \ldots, x_{i-1}y_{i-1}, x_iy_j, x_jy_i, x_{i+1}y_{i+1}, \ldots, x_ny_n\}.$$

(2) $\Longrightarrow$ (3). We proceed by induction on r.

For $r = 2$ there is nothing to prove. Assume that $r > 2$, and suppose that $C_{i_1 i_2 \cdots i_r}$ is a subgraph of G. Since $y_{i_{r-1}}x_{i_r}, y_{i_r}x_{i_1} \in E(G)$, we have $y_{i_{r-1}}x_{i_1} \in E(G)$ by Proposition 2.3. Hence, $C_{i_1 i_2 \cdots i_{r-1}}$ is a subgraph of G, which is a contradiction with the inductive hypothesis.

(3) $\Longrightarrow$ (1). Suppose that there exists another perfect matching:

$$\{x_1y_{i_1}, x_2y_{i_2}, \ldots, x_ny_{i_n}\} \subset E(G).$$

Then we define a permutation σ by

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}.$$

Then σ can be decomposed as $\sigma = \prod \sigma_i$, where each σ_i is a cycle of σ. Since σ is not an identity permutation, for some i the cycle σ_i is of the form $(j_1 j_2 \cdots j_r)$ with $r \geq 2$. Then we have that $C_{j_r j_{r-1} \cdots j_1}$ is a subgraph of G. ∎

Now we give characterizations of Cohen-Macaulayness, which is analogous to the corresponding result for bipartite graphs (see [8, Corollary 3.5]).

THEOREM 3.4. *Let G be an unmixed graph with $2n$ vertices, which are not isolated, and with* height $I(G) = n$ *satisfying condition (*). Then the following conditions are equivalent.*

(1) *G is Cohen-Macaulay.*
(2) *$\Delta(G)$ is strongly connected.*
(3) *The cycle C_{ij} is not a subgraph of G for any $i < j$.*

Proof. (1) $\Longrightarrow$ (2). This is well known.

(2) $\Longrightarrow$ (3). Assume that C_{ij} is a subgraph of G for some $i < j$. Let F be a facet of $\Delta(G)$ such that $x_i \in F$. Since $x_i y_j \in E(G)$, we have $y_j \notin F$, and by the unmixedness of G it follows that $x_j \in F$. Hence, $\{x_i, x_j\} \subset F$. Let F' be a facet of $\Delta(G)$ such that $\{y_i, y_j\} \subset F'$.

We show that there does not exist a chain of facets of $\Delta(G)$ such that

$$F = F_0, F_1, \ldots, F_m = F', \quad \text{with } \sharp(F_i \cap F_{i+1}) = n - 1 \text{ for } i = 1, \ldots, m-1.$$

Every facet $H \in \Delta(G)$ is one of the following forms:

$$H = \{z_1, \ldots, z_{i-1}, x_i, z_{i+1}, \ldots, z_{j-1}, x_j, z_{j+1}, \ldots, z_n\}$$

or

$$H = \{z_1, \ldots, z_{i-1}, y_i, z_{i+1}, \ldots, z_{j-1}, y_j, z_{j+1}, \ldots, z_n\},$$

where $z_k \in \{x_k, y_k\}$, since $\{x_i y_i, x_j y_j, x_i y_j, x_j y_i\} \subset E(G)$. Hence, it is impossible to find such a chain. Hence, $\Delta(G)$ is not strongly connected.

(3) $\Longrightarrow$ (1). In order to prove the statement by Proposition 3.2, it is sufficient to verify that $O_T(G)$ is unmixed for every subset T of $[n]$. In contrast, suppose that there exists $T \subset [n]$ such that $G' := O_T(G)$ is not unmixed. By Proposition 2.3, one of the following cases occurs:

(i.a) there exist distinct i, j, $k \in [n]$ such that $x_i x_j$, $y_j x_k \in E(G')$ but $x_i x_k \notin E(G')$;
(i.b) there exist distinct i, j, $k \in [n]$ such that $y_i x_j$, $y_j x_k \in E(G')$ but $y_i x_k \notin E(G')$;
(ii) there exist distinct i, $j \in [n]$ such that $x_i y_j$, $x_i x_j \in E(G')$.

In case (i.a), since $j \notin T$, we have $y_j x_k \in E(G)$. Moreover, since $j \notin T$, $x_i x_j \in E(G')$ implies that

(i.aa) $x_i x_j \in E(G)$

or

(i.ab) $y_i x_j \in E(G)$ and $i \in T$.

In subcase (i.aa), we have $x_i x_k \in E(G)$ by Proposition 2.3. Hence, $x_i x_k \in E(G')$. This contradicts $x_i x_k \notin E(G')$.

In subcase (i.ab), we have $y_i x_k \in E(G)$ by Proposition 2.3 with $i \in T$. Hence, $x_i x_k \in E(G')$. This contradicts $x_i x_k \notin E(G')$.

In case (i.b), $y_i x_j, y_j x_k \in E(G')$ implies that $i, j \notin T$. Hence, $y_i x_j, y_j x_k \in E(G)$. Then $y_i x_k \in E(G)$ by Proposition 2.3. Hence, $y_i x_k \in E(G')$. This contradicts $y_i x_k \notin E(G')$.

In case (ii), $x_i y_j \in E(G')$ implies that $j \notin T$. Hence, $x_i y_j \in E(G)$. Moreover, $x_i x_j \in E(G')$ implies that

(ii.a) $y_i x_j \in E(G)$ and $i \in T$

or

(ii.b) $x_i x_j \in E(G)$.

In subcase (ii.a), we have $y_i x_j, y_j x_i \in E(G)$. This contradicts the assumption that C_{ij} is not a subgraph of G.

In subcase (ii.b), we have $x_i x_j, x_i y_j \in E(G)$. Hence, G is not unmixed by Proposition 2.3. This contradicts the assumption that G is unmixed. ∎

The next lemma is crucial for giving another criterion for the Cohen-Macaulayness of our graphs.

LEMMA 3.5. *Let G be an unmixed graph with $2n$ vertices, which are not isolated, and with* $\operatorname{height} I(G) = n$. *We assume condition (*).*

If G is a Cohen-Macaulay graph, then there exists a suitable simultaneous change of labeling on both $\{x_i\}$ and $\{y_i\}$ (i.e., we relabel $(x_{i_1}, \ldots, x_{i_n})$ and $(y_{i_1}, \ldots, y_{i_n})$ as $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ at the same time), such that $x_i y_j \in E(G)$ implies that $i \leq j$.

Proof. We can define a partial order $\preceq$ on X by

$$x_i \preceq x_j \quad \text{if and only if } x_i y_j \in E(G).$$

In fact, the reflexivity holds by condition (*), the transitivity holds by unmixedness of G (see Proposition 2.3(i)), and the antisymmetry holds since G contains no cycle C_{ij} for any $i < j$. Take a linear extension of $\preceq$, which we call $\preceq'$. By the linear order $\preceq'$, we have $x_{i_1} \preceq' \cdots \preceq' x_{i_n}$. We relabel them as $x_1 \preceq' \cdots \preceq' x_n$. At the same time, we relabel $y_{i_1}, \ldots, y_{i_n}$ as $y_1, \ldots, y_n$. Then if $x_i y_j \in E(G)$, $x_i \preceq' x_j$. Hence, $i \leq j$. ∎

Hence, for a Cohen-Macaulay graph G with $2n$ vertices, which are not isolated, and with $\operatorname{height} I(G) = n$ satisfying condition (*), we may assume that

$$x_i y_j \in E(G) \text{ implies } i \leq j. \tag{**}$$

Now we state another Cohen-Macaulay criterion on our graphs, which is a generalization of Herzog and Hibi ([8, Theorem 3.4]).

THEOREM 3.6. *Let G be a graph with $2n$ vertices, which are not isolated, and with* $\operatorname{height} I(G) = n$. *We assume conditions (*) and (**). Then the following conditions are equivalent.*

(1) *G is Cohen-Macaulay.*
(2) *G is unmixed.*
(3) *The following conditions hold:*
   (i) *if $z_i x_j, y_j x_k \in E(G)$, then $z_i x_k \in E(G)$ for distinct i, j, k and for $z_i \in \{x_i, y_i\}$;*
   (ii) *if $x_i y_j \in E(G)$, then $x_i x_j \notin E(G)$.*

Proof. (1) $\Longrightarrow$ (2). This is well known.
(2) $\Longrightarrow$ (1). This follows from Theorem 3.4, since we assume condition (**).
(2) $\Longleftrightarrow$ (3). This follows from Proposition 2.3. ∎

We remark that the equivalence between (1) and (2) in Theorem 3.6 is a special case of [9, Theorem 4.3].

As an easy consequence of the previous results, we obtain the upper bound for the minimal number $\mu(I(G))$ of generators of $I(G)$, as follows.

COROLLARY 3.7. *Let G be a graph with $2n$ vertices, which are not isolated, and with* $\operatorname{height} I(G) = n$. *Then we have the following.*

(i) *If G is unmixed, then $\mu(I(G)) \leq n^2$.*
(ii) *If G is Cohen-Macaulay, then $\mu(I(G)) \leq (n(n+1))/2$.*

Proof. The statements are consequences of the criteria for unmixedness and for Cohen-Macaulayness given by Proposition 2.3 and Theorem 3.6. ∎

§4. Shellability and Cohen-Macaulay type

In this section, if G is a graph such that $\sharp V(G) = 2n$ and $\operatorname{height} I(G) = n$, we show the equivalence between the Cohen-Macaulayness and shellability of G. We also express the Cohen-Macaulay type of $S/I(G)$ in a combinatorial way.

THEOREM 4.1. *Let G be an unmixed graph with $2n$ vertices, which are not isolated, and with* height $I(G) = n$. *Then G is Cohen-Macaulay if and only if G is shellable.*

Here we give a proof only of the following lemma. The rest of the proof is almost identical to the proof of [3, Theorem 2.9].

LEMMA 4.2. *Let G be a Cohen-Macaulay graph with $2n$ vertices, which are not isolated, and with* height $I(G) = n$. *Then there exists a vertex $v \in V(G)$ such that* $\deg(v) = 1$.

Proof. Since G is Cohen-Macaulay, it is unmixed. By Lemma 2.1, G has a perfect matching. Then we may assume condition (*). Suppose that each $v \in V(G)$ has at least degree 2. Let $i_1, i_2, \ldots$ be a sequence such that $y_{i_1}x_{i_2}, y_{i_2}x_{i_3}, \ldots \in E(G)$ with $i_j \neq i_{j+1}$. Since the cardinality of Y is finite, there must exist integers $s < t$ such that $i_t = i_s$. We may assume that $i_s, i_{s+1}, \ldots, i_{t-1}$ are distinct. This induces that the cycle $C_{i_s i_{s+1} \cdots i_{t-1}}$ is a subgraph of G. Therefore, G is not Cohen-Macaulay by Proposition 3.3 and Theorem 3.4. $\square$

Now we express the Cohen-Macaulay type of a graph belonging to our class, imitating the bipartite case (see [13, pp. 184–185]).

LEMMA 4.3. *Let G be a Cohen-Macaulay graph with $2n$ vertices, which are not isolated, and with* height $I(G) = n$. *We assume condition (*). Then*

$$\mathrm{Soc}\big(K[x_1, \ldots, x_n]/(I(O_{[n]}(G)|_X) + (x_1^2, \ldots, x_n^2))\big)$$

is generated by all the monomials $x_{i_1} \cdots x_{i_r}$ such that $\{x_{i_1}, \ldots, x_{i_r}\}$ is a maximal independent set of $O_{[n]}(G)|_X$.

Proof. The ring $A := K[x_1, \ldots, x_n]/(I(O_{[n]}(G)|_X) + (x_1^2, \ldots, x_n^2))$ is spanned as a K-vector space by the image of 1 and the images of the square-free monomials

$$x_{i_1} \cdots x_{i_r}, \quad 1 \le i_1 < i_2 < \cdots < i_r \le n \tag{4.1}$$

such that $x_{i_j}x_{i_k} \notin E(O_{[n]}(G)|_X)$, for $j \neq k$; that is, $\{x_{i_1}, \ldots, x_{i_r}\}$ is an independent set of $O_{[n]}(G)|_X$. Since A is an Artinian positively graded algebra, Soc $A = (0 :_A A_+)$ is generated by the images of the square-free monomials of form (4.1) such that $\{x_{i_1}, \ldots, x_{i_r}\}$ is a maximal independent set of $O_{[n]}(G)|_X$. ∎

Corollary 4.4. *Let G be a Cohen-Macaulay graph with $2n$ vertices, which are not isolated, and with* $\operatorname{height} I(G) = n$. *We assume condition (*). Then we have the following.*

(i) *type $S/I(G) = \sharp\Upsilon(O_{[n]}(G)|_X)$, where $\Upsilon(O_{[n]}(G)|_X)$ is the family of all minimal vertex covers of $O_{[n]}(G)|_X$. In particular, type $S/I(G)$ is independent from the base field K.*
(ii) *G is level if and only if $O_{[n]}(G)|_X$ is unmixed. In particular, the levelness of G is independent from the base field K.*

Proof. Set $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$, and set $S_n = K[x_1, \ldots, x_n]$.

(i) Since G is Cohen-Macaulay and since $\{x_1 - y_1, \ldots, x_n - y_n\}$ is a regular sequence, we have

$$\begin{aligned} \text{type } S/I(G) &= \dim_K \operatorname{Soc} S/\big(I(G) + (x_1 - y_1, \ldots, x_n - y_n)\big) \\ &= \dim_K \operatorname{Soc} S_n/\big(I(O_{[n]}(G)|_X) + (x_1^2, \ldots, x_n^2)\big) \\ &= \sharp\Upsilon\big(O_{[n]}(G)|_X\big) \end{aligned}$$

by Lemma 4.3.

(ii) When G is Cohen-Macaulay, G is level if and only if

$$\operatorname{Soc} S/\big(I(G) + (x_1 - y_1, \ldots, x_n - y_n)\big)$$

is equigenerated. By Lemma 4.3, it is equivalent that $O_{[n]}(G)|_X$ is unmixed. ∎

The next result generalizes [8, Corollary 3.6].

Corollary 4.5. *Let G be a Cohen-Macaulay graph with $2n$ vertices, which are not isolated, and with* $\operatorname{height} I(G) = n$. *We assume condition (*). Then the following conditions are equivalent.*

(1) *G is Gorenstein.*
(2) *$I(G) = (x_1y_1, \ldots, x_ny_n)$.*
(3) *G is a complete intersection.*

Proof. (1) $\Rightarrow$ (2). G is Gorenstein if and only if $S/I(G)$ is Cohen-Macaulay and type $S/I(G) = 1$. Since $1 = \text{type } S/I(G) = \sharp\Upsilon(O_{[n]}(G)|_X)$, it follows that $O_{[n]}(G)|_X$ has a unique minimal vertex cover. Hence, $O_{[n]}(G)|_X$ is isolated n vertices. Hence, $I(G) = (x_1y_1, \ldots, x_ny_n)$.

(2) $\Rightarrow$ (3). This is true from its definition.
(3) $\Rightarrow$ (1). See [1]. ∎

§5. B-grafted graph

In this section, we introduce a new class of graphs G with $\sharp V(G) = 2n$ and with height $I(G) = n$, and we study its Cohen-Macaulayness.

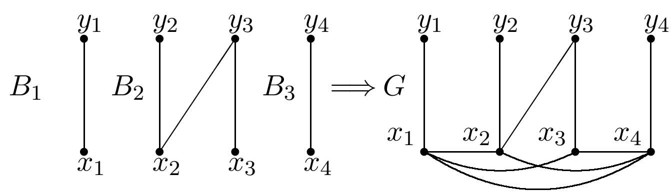
Let H_0 be a graph with the labeled vertices $1, 2, \ldots, p$.

For every $i = 1, \ldots, p$, let B_i be a bipartite graph with labeled partition X_i and Y_i such that $\sharp X_i = \sharp Y_i = n_i$. (We do not give a label to each vertex of B_i, but we distinguish the partition X_i and Y_i.) We assume that B_i has no isolated vertex for every $i = 1, \ldots, p$. We define the graph

$$G = G(H_0; B_1, \ldots, B_p)$$

as follows. The vertex set of G is $V(G) := X \cup Y$, where $X = X_1 \cup \cdots \cup X_p$ and $Y = Y_1 \cup \cdots \cup Y_p$. The edge set $E(G)$ of G is defined by $xy \in E(G)$ if and only if either there exist i, j such that $x \in X_i, y \in X_j$, and $ij \in E(H_0)$ or there exists i such that $x \in X_i, y \in Y_i$, and $xy \in E(B_i)$. We call such a graph G the *B-grafted graph*. Note that X is a minimal vertex cover of G and that Y is a maximal independent set of G. Note also that $\sharp V(G) = 2\left(\sum_{i=1}^{p} n_i\right)$.

Example 5.1. Let H_0 be a cycle of length 3. By the following bipartite graphs B_1, B_2, and B_3, we obtain the B-grafted graph G:



Remark 5.2. If B_i is just a complete graph with two vertices, that is, a complete bipartite graph with $\sharp X_i = \sharp Y_i = 1$ for $i = 1, \ldots, p$, then the B-grafted graph G is called a *grafted graph* in [4].

Theorem 5.3. *The B-grafted graph $G(H_0; B_1, \ldots, B_p)$ is Cohen-Macaulay (resp., unmixed) if and only if every bipartite graph B_i is Cohen-Macaulay (resp., unmixed) for $i = 1, \ldots, p$.*

Proof. It is clear from Theorem 3.4 (resp., Proposition 2.3). ∎

Acknowledgment. The third author gratefully acknowledges the hospitality during his stay at the Department of Mathematics of the University of Messina.

References

[1] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Univ. Press, Cambridge, 1997.

[2] R. Diestel, *Graph Theory*, 2nd ed., Grad. Texts in Math. **173**, Springer, Berlin, 2000.

[3] M. Estrada and R. H. Villarreal, *Cohen-Macaulay bipartite graphs*, Arch. Math. (Basel) **68** (1997), 124–128.

[4] S. Faridi, "Cohen-Macaulay properties of square-free monomial ideals" in *Commutative Algebra*, Lect. Notes Pure Appl. Math. **244**, Chapman & Hall, Boca Raton, Fla, 2006, 85–114.

[5] I. Gitler and C. E. Valencia, *Bounds for invariants of edge-rings*, Comm. Algebra **33** (2005), 1603–1616.

[6] I. Gitler and C. E. Valencia, *Bounds for graph invariants*, preprint, arXive:math/0510387v2 [math.CO]

[7] H. Haghighi, S. Yassemi, and R. Zaare-Nahandi, *Bipartite S_2 graphs are Cohen-Macaulay*, to appear in Bull. Math. Soc. Sci. Math. Roumanie (N.S.).

[8] J. Herzog and T. Hibi, *Distributive lattices, bipartite graphs and Alexander duality*, J. Algebraic Combin. **22** (2005), 289–302.

[9] J. Herzog, T. Hibi, and X. Zheng, *The monomial ideal of a finite meet-semilattice*, Trans. Amer. Math. Soc. **358** (2006), 4119–4134.

[10] S. Morey, E. Reyes, and R. H. Villarreal, *Cohen-Macaulay, shellable and unmixed clutters with a perfect matching of König type*, J. Pure Appl. Algebra **212** (2008), 1770–1786.

[11] J. Stückrad and W. Vogel, *Buchsbaum Rings and Applications: An Interaction between Algebra, Geometry and Topology*, Springer, Berlin, 1986.

[12] A. Van Tuyl and R. H. Villarreal, *Shellable graphs and sequentially Cohen-Macaulay bipartite graphs*, J. Combin. Theory Ser. A **115** (2008), 779–814.

[13] R. H. Villarreal, *Monomial Algebras*, Monogr. Textbooks Pure Appl. Math. **238**, Marcel Dekker, New York, 2001.

[14] ———, *Unmixed bipartite graphs*, Rev. Colombiana Mat. **41** (2007), 393–395.

Marilena Crupi
Dipartimento di Matematica
Università di Messina
98166 Messina
Italy
mcrupi@unime.it

Giancarlo Rinaldo
Dipartimento di Matematica
Università di Messina
98166 Messina
Italy
rinaldo@dipmat.unime.it

Naoki Terai
Department of Mathematics
Faculty of Culture and Education
Saga University
Saga 840-8502
Japan
terai@cc.saga-u.ac.jp