Why Does Voting Get So Complicated? A Review of Theories for Analyzing Democratic Participation

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Abstract. The purpose of this article is to present a sample from the panoply of formal theories on voting and elections to *Statistical Science* readers who have had limited exposure to such work. These abstract ideas provide a framework for understanding the context of the empirical articles that follow in this volume. The primary focus of this theoretical literature is on the use of mathematical formalism to describe electoral systems and outcomes by modeling both voting rules and human behavior. As with empirical models, these constructs are never perfect descriptors of reality, but instead form the basis for understanding fundamental characteristics of the studied system. Our focus is on providing a general, but not overly simplified, review of these theories with practical examples. We end the article with a thought experiment that applies different vote aggregation schemes to the 2000 presidential election count in Florida, and we find that alternative methods provide different results.

Key words and phrases: Voting rules, elections, participation, rational choice, spatial models, cost–benefit models, Florida 2000 election.

1. VOTE AGGREGATION AND GROUP PREFERENCES

On a very superficial level, voting seems incredibly simple: count the votes and declare the winner. However, we know from the 2000 presidential election in the United States that even the *counting* part is not always so straightforward. Furthermore, there are actually many means by which votes can be organized and counted. Yet from a very early age, citizens of the United States are indoctrinated with the idea that plurality rule, the single person/proposal that receives the most votes wins the election, is the only truly fair and therefore democratic way to organize elections.

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This system is the norm from grade school elections for class president to congressional elections. However, not only is this merely one of many possible "democratic" procedures, it is also not the only system currently used in political life in the United States and around the world.

The founding fathers (James Madison in particular; see *Federalist* 10) worried about the "tyranny of the majority" and the notion of "mob rule," and accordingly wrote several deliberately antimajoritarian schemes into our constitution such as the Senate (two members per state *regardless* of size), the electoral college for presidential elections and specific qualifications for participation that are no longer law. Today, some local municipalities in the United States set up elections in more complicated ways to assure minority participation on school boards and county commissions. In many other countries parliamentary seats are allocated to political parties according to vote totals for that party, regardless of the success of individual candidates. These schemes are indications that there exist other criteria of importance besides straight majoritar-

ian decision, including minority participation in government, political stability and general diversity.

The means by which individual votes are translated into election outcomes is called *preference aggregation* or *vote aggregation*, and importantly the selection of this method can actually alter these outcomes. Does that mean that some of these preference aggregation methods are democratic and others are not? Actually, what it means is that "preference" is really a more complicated notion than it first appears. Voters can be seen to prefer different outcomes when counted differently. Take the recent 2002 French presidential election as an example. In France there is a general election for all presidential candidates and then, unless a candidate receives more than 50% of the votes (rare), the two candidates who receive the most votes face each other in a runoff one week later. In 2002, if there had not been a runoff, then according to every single credible opinion poll Chirac and Jospin would have received the first and second highest vote totals, respectively. However, many voters already anticipating a Chirac–Jospin runoff in the second round declined to vote in the first round and the extreme right-wing Le Pen beat Jospin to face Chirac in the second round where he lost 82% to 18%, which reflected the support measured in opinion polling. What happened was that the 5.5 million votes Le Pen received in the first round were enough to pass Jospin, but the 5.8 million votes he received in the second round were nowhere near enough to challenge Chirac. So voters demonstrated a different "preference" in the second round than if the rules had been otherwise.

What makes vote aggregation methods and election laws important is that they do in fact determine who wins and who loses. Also, the presence of regular elections alone does not determine the existence of democracy. [It turns out that defining a democracy is actually quite a difficult task. Dahl (1971) tied the definition of democracy to the possession of the "voting franchise" by a substantial proportion of citizens, contested elections and civil liberties. Huntington (1991) required that "decision makers are selected through fair, honest, and periodic elections in which candidates freely compete for votes and in which virtually all the adult population is eligible to vote." Unfortunately, it is easy to name countries which meet these criteria and yet still somehow violate our general normative sense of what it means to be democratic. See Zakaria (1997) for an engaging essay on this problem.] The Soviet Union ran regular elections, but only one party was allowed to put candidates on the ballot. The early election history

of the United States was one that excluded a substantial proportion of its citizens. As a number of countries emerge from communist or totalitarian periods, the decisions they make in establishing democratic institutions, including electoral procedures, will fundamentally determine the type of governance and public policy they will have.

Having just established the importance of studying systems of voting and elections, we have to note that there is a major disconnect in related topical knowledge. Many people who are deeply interested in election procedures around the world do not understand the underlying mathematical principles that govern vote aggregation (this is particularly noticeable with journalists). Conversely, many people who have the prerequisite technical background have not had an appropriate reference that directly explains the theoretical principles. We address this second deficiency here by reviewing the fundamental mathematical and logical precepts of voting and elections theory, without omitting important methodological details.

2. UNDERLYING THEORETICAL AND PRACTICAL PRINCIPLES

There is a core set of assumptions about individual behavior that is necessary to describe systems of aggregate voting in symbolic terms. These fall under the general rubric of *rational choice theory* (also called *public choice theory*), where individuals are assumed to make orderly choices that reflect their personal preferences and desires (philosophically summarized in Buchanan, 1983). This overtly mechanical perspective evokes strong emotions among advocates and critics, but usually the more vitriolic debates stem from an overly canonical interpretation of the paradigm on both sides. Not even the strongest supporters really believe that people are 100% rational, doing detailed research and analysis on every material decision before making a perfectly informed and deliberate decision. In truth, the factors that detractors of rational choice highlight, such as sources from psychological, instinctual, whimsical and unaccounted for sociological characteristics, are certainly a component of human decision making in the political context.

Underlying the study of mass voting is the philosophy of *methodological individualism*, which states that collective social decisions can be modeled by aggregated individual observations (somewhat like the iid assumption in a statistical model). Furthermore, it is recognized that in this process of aggregating individual observations, any discoverable systematic effect

will also be accompanied by a stochastic term because there is an inherent random element in human behavior. The payoff for this perspective is that even if some individual actions are not themselves apparently rational, the full system behaves as a collectively rational entity with an accepted error component derived by deviations from expectations.

2.1 Rationality

Thus far the term *rationality* has been left a little bit vague. Specifically, we mean the following commonly required assumptions, which are standard in the literature:

Utility. Each individual has a relative sense of benefit: some outcomes that might occur have more value than other events that might occur (preference ordering). Utility is typically measured as money in economic models, but more generally it could be termed *satisfaction*. Utility is personal, private, noncomparable across individuals and can be measured only indirectly by observable indicators which partially reveal preferences.

Purposefulness. An individual's actions (choice) are purposeful: directed toward obtaining some increase in utility. Refusing to make a choice (such as not turning out to vote) is also considered a choice.

Certainty. Individuals prefer choice sets where the results of their selection (through the aggregate outcome) are known with greater certainty rather than less. Substantial uncertainty of outcomes conditional on the individual's actions is undesirable: usually termed "decision-making under risk." [We need to add the caveat though that there *are* circumstances where a perfectly rational individual prefers uncertainty over certainty. Consider, e.g., a lottery where the expected value of participating is favorable. The individual may prefer to participate (i.e., purchase a ticket) in this uncertain game rather than accept the absolutely certain result of no gain by abstention, but this implies more individual flexibility than the situation imposed by a public election.]

Sincerity. Voters are said to vote either sincerely or insincerely (strategically). Sincere voters (the default in most models) vote/choose according to their true utility-maximizing preferences in the current election. The opposite of this sincere voting assumption is strategic voting (Farquharson, 1969), where individuals select short-term non-utility-maximizing alternatives to maximize some greater long-term utility (often seen in games with repeated trials).

Comparability. Alternatives are comparable by voters in the sense that for any two choices, *a* and *b*, the voter either prefers *a* over *b*, *b* over *a* or is indifferent between *a* and *b*. Furthermore, this preference is *transitive*: if the voter prefers *a* over *b* and *b* over another alternative *c*, then the voter prefers *a* over *c*, or if the voter is indifferent between *a* and *b* and between *b* and *c*, then the voter is indifferent between *a* and *c*.

Buchanan (1983) and others summarize these assumptions in the form of an assumed rational, self-motivated individual voluntarily entering into economic exchanges (also defined to include political scenarios) and seeking through these exchanges to increase his or her individual utility: so-called species *homo economicus*. We generally rely on these assumptions in the descriptions of voting systems and voting behavior that follow, but deliberately avoid the controversy about broadly assigning universal individual rationality. As with any theory, social science or otherwise, the true test of these theories lies in their empirical verifiability ("instrumentalism"). This debate about the rationality of self-interest in political behavior can be traced as far back as works by Machiavelli, Hobbes and Hume, but for current discussions, see Fiorina (1995), Green and Shapiro (1994) and the essays in Friedman (1996).

2.2 Districting Systems

The results of voting systems are contingent on the nature of the underlying districting system. A districting system maps electoral regions to legislative representation: how such districts are represented in government. In elections for members of legislatures, the outcome can be contingent on whether there are single-member voting districts (one representative only per district), multimember districts (more than one representative per district) or proportional representation (general party representation).

In the United States, congressional representation is a mixture: House of Representatives districts are served by one member from each district, but each Senate district is served by two senators (a variation on multimember districts because the two Senate seats in each state are not simultaneously contested). Actually the United States has institutionalized malapportionment since the "one-person, one-vote" norm is not held through the Senate, and noninstitutionalized malapportionment since it is impossible to configure exactly equal House districts across states. Every 10 years a

complicated and high-stakes game occurs when House districts are redrawn by the states to reflect changes from the census (when this redistricting process is done to advantage a particular racial group or political party, it is referred to as *gerrymandering*). Conversely, in proportional representation systems, the focus is on the parties with the often intended effect of ensuring broad representation across widely disparate groups in society. Here candidates are simply advocates of their party, and the legislature is divided roughly in proportion to the total electoral fortune of the parties (typically with a minimum threshold).

Duverger (1963) contended that single-member district systems favor a two-party system; in other words, the likely outcome of these type systems is a legislative body that is dominated by two narrowly separated parties on policy issues. Riker (1986) qualitatively tested the obvious counterpart to this theory, that proportional representation systems encourage multiparty systems. His findings indicated mixed support for this proposition: single-member voting districts *tend* to conform to Duverger's principle, but not uniformly.

On the other hand, a consistent criticism of proportional representation (PR) is that it fosters fragmented legislative politics since voters may find satisfaction in minority representation by a political party defined along narrow policy interests. The aggregation of these narrow interests may result in legislatures split among many groups with little or no incentive to cooperate and form coalitions. This is a problem because in PR systems a majority vote in the legislature is required to "form a government," meaning fill the executive branch positions (ministers). Italy is often held up as a classic example since PR has contributed there to a change of government roughly every year since World War II. There are many variations of proportional representation (see Farrell, 2001, for a list), but the general effect is to empower smaller, more particularistic groups at the expense of larger, more broadly-based parties.

3. INSTITUTIONAL VOTING SYSTEMS

As the previously discussed French case illustrates, the structure of the electoral system plays a significant role in the election outcome. Early work (Rae, 1971; Fishburn, 1971; Straffin, 1980; Fishburn and Brams, 1981; Riker, 1982; Nurmi, 1983) noticed that multicandidate elections (a term indicating more than two candidates) were particularly affected by the form of electoral rules. In this section we describe the primary forms of electoral systems in use around the world and characterize how they affect outcomes. These systems differ in some important ways: some allow voters to reveal the intensity of their preferences, some are designed to elect multiple candidates and one can even fail to produce a winner under certain circumstances.

3.1 Unanimity Rule

Unanimity as an advocated procedure is generally attributed to Wicksell (1896), who saw the combination of unanimous consent combined with line-item taxation as the best way to mandate public policy expenditures in English society. Consider first an n-person population where each individual has an identified income, I_i, which is fully spent each year on either private goods, Pr, or public goods, Pu. Naturally then each person has a utility function that defines his or her two-dimensional spending preferences, $U_i(Pr_i, Pu_i)$, depending on the utility received in the two dimensions. While choice of spending on private goods can be determined independently for each individual, the provision of public goods is through a regularly paid tax requiring agreement among citizens. Thus each voter will have his or her own individually preferred budget line which is a function of his or her utility for public goods and his or her income, as well as his or her tolerance for taxes as a way to pay for the public goods.

The real problem with the unanimity rule is now quite obvious: you have to get everyone to agree, and each voter will have a different preference structure. Suppose that the mandated level of public spending Pu induces different utilities across the n individuals. That is, one person almost certainly receives higher utility at this public spending and associated taxation level than the others. Such disagreement then needs to be worked out politically such that the combined utility (public and private) is sufficient for all voters to approve. This may not be difficult with a small number of voters, but the possibility of this cooperative outcome drops sharply as the number of voters increases, all but ensuring no unanimous agreement for a reasonably sized electorate.

3.2 Majorities and Pluralities

The *simple majority rule* system requires little description here because its definition is essentially contained within its name: the candidate, with votes x_j, who carries at least one more vote than 50% of the electorate wins the election. The simple majority rule can present problems in multicandidate elections, however,

because it is possible that no candidate exceeds the required threshold ($x_j > 50\%$). In these cases, some systems are set up to require a runoff: a new election between the top two vote-getting candidates, the *simple majority rule with runoff* system. It is then inevitable in the second stage that one of the candidates will exceed the 50% threshold ($x_1 > 50\%$ or $x_2 > 50\%$) because of the restriction to two participating candidates (except of course for small elections with an even number of voters).

Returning to the French presidential election of 2002, consider a voter who strongly supports Chirac but may vote for Le Pen in the first round. Why would this person be inclined to vote for a candidate so far from his or her ideal candidate's policy position? Consider that Jospin was expected to challenge Chirac to a much greater extent in the runoff than Le Pen and that as incumbent president with a well-established base of support, Chirac was virtually assured of making it to the second round. Then this hypothetical *strategic* voter may vote for Le Pen in the first round in the hopes of seeing the weaker candidate face Chirac in the second: classic strategic voting. This example highlights a constant danger with such strategic voting. Suppose Le Pen had somehow beaten Chirac in the second round. Then the hypothetical strategic voter would have to accept a president far less palatable than Jospin.

The *plurality system* is identical to simple majority rule except that the winner is merely selected by attaining the most votes without the necessity of passing the 50% threshold and, therefore, there is no need for a runoff (majority voting is a special case of plurality voting). This is often referred to as *first past the post* since in the consecutive counting of votes, as soon as one of the candidates meets this criterion then all of the subsequent votes are immaterial to the electoral decision. While this system is much easier to implement because there is no longer the necessity to resolve nonmajority outcomes, it is often considered bad for the resulting government because the winner can take office having only minority support. Also, plurality voting does not take into account the *intensity* of people's feelings about candidates. For instance, Jesse "The Body" Ventura won the governorship of Minnesota as a third party candidate in 1998 with only 37% of the vote. It appears from academic and journalistic accounts that the 63% who did not vote for Ventura had more intense negative feelings than the intensity of the positive feelings of the 37% who did vote for him. However, the 63% group had their votes

split across the Democratic and Republican candidates in such a way that neither party candidate surpassed 37%. Interestingly, it is probable that Ventura would have lost in a hypothetical runoff vote against the second highest vote-getter from the first round.

A related vote counting methodology is the *antiplurality* (blackball) system. In this procedure, voters are asked to vote against one candidate on the ballot, and the candidate with the fewest (now necessarily negatively interpreted) votes wins the election (see Saari, 2001, for nuances). This method produces the "least objectionable" candidate across the voting population. There are also some slight variations on this scheme, including multiple rounds of voting, but the general result is typically to promote candidates who are not necessarily the best for office or the most dynamic, but instead those who offend the fewest voters.

The plurality system has some interesting and unexpected consequences. Cox (1997) developed a formal model that demonstrates for single-member voting districts under a plurality system there will be only two sustained, enduring parties. Assume that voters have *defined preferences* (the election outcome affects their utility), *incomplete information* about the preferences of other voters, *expectations* about the viability of candidates (partly a function of media coverage, which introduces, not necessarily accurate, aggregate information) and rational voting intentions. The result of these precepts is that internal conflicts between issue/policy preferences and viability are most often resolved in favor of viability: more voters derive greater utility by voting for the imperfect but acceptable mainstream candidate over the ideal but long-shot candidate. As a consequence, nations such as the United States end up with only two enduring political parties, whose policy positions are well known and very close to each other in issue space, since politicians and political parties recognize voter acknowledgment of viability.

3.2.1 *Plurality voting and efficiency.* Economists have noted that varieties of plurality voting can actually be *inefficient*. The efficiency standard is measured in total benefits to the society at large, a deliberately vague definition but one that can be substituted using total money or total utility. The inefficiency occurs because the complete distribution of benefits is unlikely to match up with the way votes occur. For instance, consider the following five-voter, two-alternative election to determine some public policy decision, with stipulated benefits shown in Table 1.

It is apparent from the way that the utilities are set up that policy B provides greater total benefit to

Voter	Benefits from policy A	Benefits from policy B
1	3	0
2	3	0
3	3	2
4	0	5
5	0	5
Total	9	12

society, but policy A will win because voter 3 slightly prefers this outcome. Since the less efficient alternative to society as a whole wins, this is considered an economically inefficient outcome. Suppose we altered the vote criteria slightly to include a measurement of personal utility loss. In each case add up the negative utility differential from each individual outcome. Thus policy A provides $0 + 0 + 0 - 5 - 5 = -10$ and policy B provides $-3 - 3 - 1 + 0 + 0 = -7$. So if some sense of relative utility were included in the voting, then outcomes can differ. These sorts of approaches underlie many of the different voting schemes that we now discuss.

3.3 Approval Voting

Another process by which elections with more than two candidates can be organized is *approval voting* [generally credited to Robert Weber's Ph.D. thesis in 1971, but see also Weber (1995)], where voters are allowed to vote for (approve of) as many candidates as they want, but cannot cast more than one vote for each candidate. The candidate with the highest total number of approval votes in this system is declared the winner. So voters get K votes to distribute across K candidates with no more than one assigned per candidate. This process has been subsequently described and popularized by Brams (1975) and Brams and Fishburn (1978, 1983).

A particularly attractive feature of such a system is that it provides voters with the maximum number of choices in a single-ballot election. If there are $K \geq 3$ candidates, each voter is essentially casting either an "approve" or a "disapprove" vote for each candidate depending on his or her distribution of votes. This seemingly gives the voter 2^K possible strategies, but because an abstention has the same net effect as voting for every candidate on the ballot, the real number of different choices is $2^K - 1$. Approval voting permits more strategies than simple majority rule and simple

plurality rule due not only to this large number of strategies, but also because these choice-sets can be broken down along divisions such as party affiliation, incumbency status and strategic considerations based on expected outcomes.

Yilmaz (1999) formalized approval voting in the following way. Let a, b, c and d be the individual candidates from which a group of voters can choose and let aPb represent a given voter's strict preference for a over b. A multicandidate (strict) preference order is denoted as $aPbPcPd$. A lack of preference between a and b is indicated by aIb, meaning that the voter is indifferent or ambivalent between the two. Given the assumption that strict preference and indifference have transitive relations, $aPb, bPc \rightarrow aPc$ and $aIb, bIc \rightarrow aIc$, then, every possible ordering can be separated into ℓ nonempty subsets, where $[a_1, a_2, \ldots], [b_1, b_2, \ldots], [c_1, c_2, \ldots], [d_1, d_2, \ldots], \ldots, [\ell_1, \ell_2, \ldots]$ are denoted as $A, B, C, D, \ldots, L$. The voter is indifferent among the candidates within any single subset while still strictly preferring every member of that subset to any of the other candidate subsets lower in the preference ordering.

This setup allows us to characterize voting behavior in the following way: if $\ell = 1$, then the voter is referred to as *unconcerned*; the voter is referred to as *dichotomous* if $\ell = 2$, *trichotomous* if $\ell = 3$ and *multichotomous* if $\ell \geq 4$. Therefore, for a voter who is *unconcerned*, there will be no strict preference between the candidates in A_i.

If all voters have a dichotomous preference, then an approval voting system will always produce the selection that is majority preferred, but when all preferences are not dichotomous the process and results become more complicated. In cases such as these there are multiple *admissible voter strategies*. An admissible voting strategy is simply a strategy that conforms to the available options among k alternatives and is not uniformly dominated (preferred in all aspects by the voter) by another alternative. For instance, the preference order aPb with bPc has two admissible strategies where the voter may have given an approval vote for only the top alternative a or for the two top alternatives a and b. Furthermore, with multiple alternatives it becomes possible for voters to cast insincere strategic votes: they may truly prefer candidate a, but select candidate b because they believe that candidate c is likely to receive more approval votes than candidate b and, therefore, be a greater threat to the preferred candidate.

For any two subsets A and B, define $A \cup B = \{a : a \in A \text{ or } a \in B\}$. The subset that contains only

candidate a is denoted $\{a\}$, the subset that contains only candidate b is denoted $\{b\}$, the subset that contains only candidates a and b is denoted $\{a, b\}$ and so forth. A *strategy* is defined as voting for any subset of candidates approved of or disapproved by the voter and is denoted by S (voting for each candidate in S implies strategy S), and the approval voting system as a whole is denoted by s.

Using the notation and construction of Brams and Fishburn, allow the following assumptions and definition for the subsequent theorem under approval voting:

- P: If aPb, then $\{a\}P\{a, b\}$ and $\{a, b\}P\{b\}$.
- I: If $A \cup B$ and $B \cup C$ are not empty and if aIb, bIc and aIc for all $a \in A$, $b \in B$ and $c \in C$, then $(A \cup B)I(B \cup C)$.
- Define $M(P) = A_1$, the subset of the most-preferred candidates under P and define $L(P) = A_n$, the subset of the least-preferred candidates under P.

THEOREM 1. *If the voter has candidate preferences given with P and I, then strategy S is admissible for system s and preference order only when S is feasible for s and either $C1$ or $C2$ holds:*

$C1$. *Every candidate in $M(P)$ is in S and S cannot be divided into two nonempty subsets S_1 and S_2 such that S_1 is feasible for S and S_2 is lower (less preferred) for the operation P.*

$C2$. *S contains no candidate from $L(P)$, there is no nonempty subset A of candidates disjoint from S and A is higher (more preferred) for the operation P.*

Applying Theorem 1 to approval voting gives the following results:

- Strategy S is admissible for approval voting and concerns P only when S contains all candidates in $M(P)$ and none in $L(P)$.
- If a voter has a dichotomous preference order P, then he or she has a unique admissible strategy. This subset of most-preferred candidates is a unique strategy.

Why is this interesting? Consider a voter who has the preference order $aPbPcPd$, while all other voters have dichotomous preferences, some being sequentially indifferent (such as aIb and cId) and some strictly prefer a and b to c and d, while the rest prefer c and d to a and b. Each of the other voters uses his or her unique admissible strategy, so that the aggregated preference for a is equal to that of b, $f(a) = f(b)$ and the aggregated preference for c is equal to that of d,

$f(c) = f(d)$. Now assume that the voter with preference $aPbPcPd$ believes that there is at least a one vote difference between a and c, $f(a) > f(c) + 1$; $\{a, c\}$ will probably be the best strategy for this voter because a vote for a ensures that a will receive at least one more vote than b, and a vote for c ensures that c will receive at least one more vote than d. Therefore, $\{a, c\}$ ensures the election of the $aPbPcPd$ voter's most-preferred candidate when $f(a) > f(c) + 1$ and the defeat of this voter's least-preferred candidate when $f(c) > f(a) + 1$. Had this voter selected $\{a, b\}$ and $f(c) > f(a) + 1$ held, then candidate d, the voter's least-preferred candidate potentially could have won. The point here is that an approval voting process can sometimes provide a strong incentive toward strategic behavior.

3.4 Cumulative Voting

A system that is similar to approval voting is one that gives each voter *multiple* votes to distribute across the candidates, where, unlike approval voting, more than one vote by each voter can be assigned to individual candidates. *Cumulative voting* therefore allows voters to assign different numbers of votes to reflect their relative preferences. For example, suppose a voter had 10 votes to distribute across 3 candidates. The truly committed voter will of course assign all 10 votes to his or her (strongly) preferred candidate, but a less ardent voter with ordered preferences may assign them perhaps as 6/3/1, meaning that the first candidate is preferred twice as much as the second and six times more than the third candidate.

The cumulative voting system has been advocated by Lani Guinier, the Clinton administration nominee for Assistant Attorney General for Civil Rights. Guinier (1994) argued that cumulative voting would promote minority representation better than the current gerrymandering of districts (majority–minority schemes) because minority voters could pool their votes and elect the candidate of their choice to one of the seats.

Needless to say, this setup leads to all kinds of strategic opportunities. For instance, suppose one candidate is particularly distasteful to our hypothetical voter. It may make sense to place all 10 votes on the second choice candidate because that candidate has a greater chance of beating the unacceptable choice. In the 2002 French presidential election, this system probably would have made Le Pen's candidacy less viable, since most French voters apparently ranked him a distant third and cumulative voting would have allowed

others to swamp the strong Le Pen supporters by splitting their votes across Chirac and Jospin.

Of course the level and sophistication of the strategies depend in part on the number of cumulative votes that can be cast by each voter. Suppose, in one extreme, that voters only get two votes to split across multiple candidates. This essentially reduces the possible strategies to three: (1) both votes on the number one preferred candidate, (2) split across the top two candidates as a conservative way to fend off the last choice, and (3) both votes on the second choice because that candidate has the greatest (perceived) potential to beat the last choice, and the most-preferred choice is still likely to win without these two votes.

3.5 Condorcet Voting

One system, originally proposed by Condorcet (1785), seeks to find collective consensus by setting up a series of pairwise contests among candidates and selecting the winner as the one who beats each of the others in this round-robin procedure. More formally, if there are K candidates or proposals, $c_1, c_2, \ldots, c_K$, then the Condorcet winner, c_j, receives the greatest number of wins in each of the $K - 1$ round-robin trials: $c_j > c_i$, $\forall i \neq j$, no matter what the order of the trials. It should be obvious that in the absence of strategic voting, the Condorcet winner would also be the plurality winner, and no forced electoral agenda in the form of ordering the trials would alter the result.

Condorcet himself felt that enlightened voters will honestly attempt to determine what decision best serves society and that they are more often right than wrong, thus justifying majority rule in principle. He demonstrated his argument using the newly developed calculus of probabilities. While Condorcet concedes that voters will not *always* make the best decision, he argues that because more voters will make the right decision than the wrong decision, the probability of selecting the right candidate is considerably higher than the probability of selecting the wrong candidate.

Condorcet's system is actually built on what he calls opinions, which are an early expression of what we now call utility, except that decisions (and therefore relative opinions) are restricted to pairwise comparisons. Condorcet voting is based on the following internal calculus:

- All possible opinions that do not imply a contradiction reduce to an indication of the order of merit that one judges to exist among the candidates. So for K candidates, an individual faces $K(K-1)/2$ pairwise comparisons (propositions).

- Each voter thus gives his or her opinion by indicating the candidates' order of personal utility. These comparisons can be done individually or by groups. Taking the number of times that each is contained in the opinion of n voters, one will have the number of voices for each proposition.

- One forms an opinion from those $K(K-1)/2$ propositions that agree with the personal utility. If this opinion is among the $K!$ possible opinions, one regards as elected the subject to whom this opinion accords the preference. If this opinion is among the $2^{K(K-1)/2} - K(K-1)\cdots 2$ impossible (contradictory) opinions, then one reverses in that impossible opinion the set of propositions that have the least combined plurality and one adopts the opinion from those that remain (Young, 1988).

One motivation for Condorcet voting is that it reinforces the popularity of the winner, which can assist in governing. However, it is clearly a higher standard than simple aggregation by plurality or majority. In addition, this procedure can lead to a cycling problem called the *Condorcet paradox*, which we describe in Section 4.1.

3.6 Borda Count

Although the Borda procedure is not currently used in any national public elections, it is a well-known ranking procedure (Felsenthal, Maoz and Rapoport, 1993; Cox, 1997). Borda (1781) proposed that voters rank order all the K competing candidates, wherein the candidate ranked first receives from each voter $K - 1$ points and the candidate ranked second gets $K - 2$ points, and so on, until the last candidate receives no points. The points are then summed over all n ballots, and the candidate with the most points wins. There is also the so-called Nanson elimination procedure (Nanson, 1882) which eliminates all candidates who are below some criterion such as the mean Borda score and then does a regular Borda count on the survivors.

It has been argued that the Condorcet and Borda procedures are designed to achieve two different purposes, wherein Condorcet's procedure is intended to provide the greatest overall satisfaction to an absolute majority of the voters and Borda's procedure is designed to provide the greatest overall satisfaction to the entire electorate (Dummett, 1984; Felsenthal, Maoz and Rapoport, 1993). Felsenthal, Maoz and Rapoport demonstrated this point with the following example: suppose there are 100 voters and three candidates a, b

and c, where one winner must be selected, and 66 voters have the ranking $[a, b, c]$ and the remaining 34 voters have the ranking $[b, c, a]$. In this setup a is the Condorcet winner: $a > b$ by 66 to 34, and $a > c$ by 100 to 0. However, b is the Borda count winner since the votes would be assigned as follows: a, $66 \times 2 = 132$; b, $34 \times 2 + 66 \times 1 = 134$; c, $34 \times 1 = 34$.

There are several criticisms of the Borda count. Like the Condorcet procedure, it fails to take into account the intensity of the rankings since ordinal preferences are forced into equal-distance integers. Also, there is great opportunity for strategic voting by individuals who understand the aggregate preferences of the electorate (obviously more difficult in large elections). Returning to the example above, if only 2 of the 66 voters in the first group exchange their second and third place preferences strategically, $[a, b, c]$ to $[a, c, b]$, then a is now the Borda count winner. This is classic strategic voting: why should these voters care what their second and third place preferences are listed as if their preferred candidate wins the election as a result of their insincere switch?

3.7 Hare Procedure

A vote counting method known as the *Hare procedure* was introduced by Hare (1859) and popularized with John Stuart Mill's advocacy. It has become known as the single-transferable vote system (STV; Taylor, 1971; Felsenthal, Maoz and Rapoport, 1993). This process is primarily utilized where it is necessary to elect more than one candidate, but it can also be used to select a single winner. Unlike the similar Borda count, this procedure is employed in several places including Australia, Malta, The Republic of Ireland and Northern Ireland. [One observer has hence referred to STV as the Anglo-Saxon version of proportional representation (Bogdanor, 1984).]

The Hare procedure is a form of proportional representation. The first kind of proportional representation involves voters selecting a party and then the seats are apportioned to the parties in accordance with their proportional share of the votes. The executive selected is generally the leader of the party that received the highest proportion of votes, and the legislative seats are often filled according to a rank order list generated by each respective party. The Hare procedure of proportional representation is employed and supported by advocates who are not content with this idea of parties generating rank order lists independently of the voters.

For electing a single candidate, the procedure involves obtaining a complete preference order list or ballot from each voter in the form of an assigned preference from 1 to K for a K-candidate list. If no candidate obtains a majority of first-place votes, the last-place candidate is eliminated and this process is repeated until a candidate has a majority of the first-place votes (Merrill, 1984). This is referred to as a transferable-vote procedure because the votes from the eliminated candidate are essentially transferred to the other candidates when the process is repeated. [Therefore, first past the post with $K > 1$ is just a single-*nontransferable* vote system (SNTV). For example, in a $K = 4$ district, a candidate would need one more vote past 20% to ensure election. SNTV is used in Jordan and Vanuatu, but is best known from its use in Japan from 1948 until 1993.] The same process is employed to select multiple winners, and the iterative process is completed when the target number of candidates has been selected.

Since the Hare procedure is typically applied in multimember constituencies (more than one person per district is elected), some procedure is required such that the correct number of candidates is eventually selected to serve, given the fixed size of the legislative body. A minimum point threshold, called the *droop quota* is calculated to determine a minimum winning number of votes:

$$(3.1) \qquad D_q = \frac{\text{total number of votes}}{\text{total number of seats} + 1} + 1.$$

Suppose that there is a district with 100 voters, all of whom participate, and this district needs to elect 3 representatives. The droop quota is then $100/(3+1)+1 = 26$. That is, every candidate who obtains 26 or more votes is guaranteed a seat in the legislature, and there cannot be more than 3 who meet such a criteria. If less than 3 candidates meet this threshold, then the votes that the winning candidate(s) received are transferred to the other candidates by the order of the voters' stipulated *second most-preferred candidate*. This process then continues until the required allotment of representatives is obtained.

The second type of proportional representation turns the standard logic around and instead of calculating seats based on an apportionment, it determines what each party pays for their seats in stages. The two most common forms are the d'Hondt method and the Sainte-Laguë method. In Cox's (1997) notation, for party i, $a_i(t)$ is ith average seats at period t, $s_i(t)$ is the seats in previous periods and v_i is the total vote. The d'Hondt method allocates seats by a staged process, where the first-stage seats are allocated by

dividing the number of valid votes cast v_i by the total number of seats allocated, $s_i(t) + 1$, where $s_i(1) = 0$ in the first-stage: $a_i(t) = v_i/(s_i(t) + 1)$. The party receiving the highest ratio of votes at time $t = 1$ is allocated one seat and their ratio is now $v_i/2$. In the second stage, the process is repeated with the first-stage losers ratio remaining as $v_i/(s_i(t) + 1)$, and again the party with the highest ratio is allocated one seat, altering their ratio for the next stage. This process is repeated, updating $a_i(t)$ for each party, until all seats are filled. The Sainte-Laguë method is similar but alters the base formula by changing the total number of seats term, resulting in $a_i(t) = v_i/(2s_i(t) + 1)$. These second types of proportional representation systems are known to disadvantage smaller parties relative to the first type because very dominant parties (those with v_i much greater than these small parties) will have slowly eroding $a_i(t)$ values.

3.8 Coombs Procedure

While the Hare method aims at choosing the alternative that is most intensively preferred by the majority of voters, the Coombs procedure (Coombs, 1964) can be interpreted as seeking to select the candidate or set of candidates that is least objectionable by a majority (Nurmi, 1983). This procedure is similar to the Hare procedure in its process, but instead of eliminating the candidate with the least amount of first-rank votes in each iteration, the candidate with the most last-rank votes is eliminated. Table 2, which was designed by Straffin (1980), illustrates the process.

In this example, a would be eliminated because it has the largest number of last-rank votes (8), but using the Condorcet winner selection process a would win. This demonstrates how the choice of a selection process can substantially influence the results. The Coombs procedure can also be repeated in the same fashion as the Hare procedure to reach the targeted number of winners. However, using the Hare procedure, a selected winner in a multimember race may have more last-rank votes than another candidate who would be selected as winner using the Coombs procedure.

4. IRONIES AND COMPLEXITIES OF PREFERENCE AGGREGATION

In this section we describe some of the seemingly odd things that can happen with perfectly reasonable voting systems when voters attempt to maximize their individual utilities. The first topic, Condorcet's paradox, shows that under totally defensible assumptions, we can find no clear solution. This problem is addressed by Arrow's theorem, for which Kenneth Arrow won the Nobel Prize, showing that in fact there is *no perfect vote aggregation system* and, therefore, all implemented systems have a logical defect. Some solutions to this problem are better than others, and Black's median voter theorem elegantly demonstrates that single-peaked individual utility distributions lead to stable outcomes.

4.1 Condorcet's Paradox

Consider a small election situation such as a majority vote by a three-person city council. The principle discussed here applies in more general settings but the restriction to three voters makes the point more clearly. (The three-person election here is equivalent to a larger general election with the unrealistic assumption that the electorate could be divided into three groups of exactly the same size based on their vote preferences. Work that generalizes this can be significantly more complex.) The city council is faced with a funding decision for a youth center which has not yet shown to have a substantial positive effect. Council member A is a pessimist and believes that the program will never work at any funding level. Council member B believes that the current funding level is the problem and that the council should vote to increase the amount or cut the program (in that order). Council member C strongly supports the youth center at its current funding, but would accept more funding rather than elimination. These positions are given in Table 3 and Figure 1.

In this setup we apply Condorcet voting by first running the decrease alternative against the status quo

TABLE 2

	Number of voters					
	5	**4**	**2**	**4**	**2**	**4**
Preference	a	a	b	b	c	c
ordering	b	c	a	c	a	b
	c	b	c	a	b	a
	$X = (a, b, c), \ N = 21$					

TABLE 3
Condorcet preferences

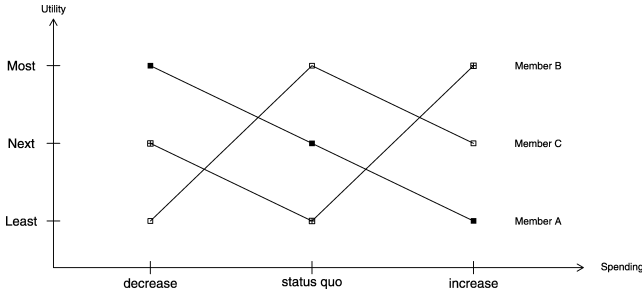| Position of preference	Council member		
	A	**B**	**C**
Most	Decrease	Increase	Status quo
Next	Status quo	Decrease	Increase
Least	Increase	Status quo	Decrease

FIG. 1. *Condorcet preferences.*

alternative. Member A prefers decrease, member B prefers decrease and member C prefers status quo, so decrease wins the first round. Now pair the decrease alternative against the increase alternative. Member A prefers decrease and members B and C prefer increase. Now increase wins. Finally, pair the status quo alternative against the increase alternative (the only pairwise contest we have not run so far). Member A prefers status quo, member B prefers increase and member C prefers status quo. Here status quo wins. Wait, this is alarming. We get a different winning alternative with each pairing and any order to Condorcet voting just sets up an endless cycle of futile comparisons. *There will never be any Condorcet winner with these three preferences.* Interestingly, of the systems described in Section 3, only Condorcet voting does not guarantee a winner.

4.2 Arrow's Theorem

Arrow (1951) showed that the Condorcet paradox is really a manifestation of a much more important phenomenon. In fact, he proved that unless one is willing to violate one of a set of reasonable democratic norms, the type of cycling seen with the Condorcet paradox is an inevitability.

The Arrow impossibility theorem directly addresses voter preference rather than the combination of turnout and preference. The question addressed is whether there is a *universal* mechanism for systematically collecting preferences to reflect an aggregate societal ordering of choice. That is, Arrow is concerned with determining whether the aggregation of voter preferences actually yields a democratic decision. He builds his theory around the previously discussed concept of a strict voter preference and the indifferent voter. It is assumed that each individual has a preference between any single pair of alternatives and that the preference among the alternatives $(x, y, z, \ldots)$ for each individ-

ual i has the following properties:

- $\forall x, y, x P_i y$ or $y P_i x$ (P_i is complete).
- $\forall x, y, z, x P_i y$ and $y P_i z \rightarrow x P_i z$ (P_i is transitive).
- $x I_i y$ and $y I_i z \rightarrow x I_i z$ (i.e., I_i is transitive).

These axioms must also be satisfied between the social conditions of each possible pair. If O denotes the order of social preferences, then the method of aggregation can be a function $O = f(O_1, O_2, \ldots, O_n)$ which specifies for each set of individual preferences a rank of the alternatives and this is used to create a *social welfare function* (Taylor, 1971). A social welfare function can be defined as a mapping of individual preference lists (without ties) to an aggregate social preference list. The relation P of strict preference and the relation I of indifference can be defined in terms of O:

$$(4.1) \quad x P_i y \leftrightarrow \neg y O_i x, \ x I_i y \leftrightarrow \neg x O_i y \text{ and } y O_i x.$$

Now that the assumed properties of the individual and social preferences have been established, the requirements to satisfy the social welfare function are then expressions of P and I, where the lack of indexing denotes group preference:

1. *Unrestricted domain.* Actors can hold any possible preference ordering over the outcomes.
2. *Independence of irrelevant alternatives.* The social decision between any two alternatives, x and y, depends only on the individual orderings of x and y.
3. *Pareto principle.* If $x P_i y \ \forall i$, then $x P y$.
4. *Nondictatorship.* There is no single i, such that, for all x and y, $x P_i y \rightarrow x P y$, regardless of the orderings of all individuals other than i.

Arrow's theorem is aptly named the "impossibility theorem" because there is no social welfare *function* (aggregation scheme) that can satisfy conditions 1–4 simultaneously. It is logically impossible. Therefore, collective social decisions cannot yield a truly democratic system in this sense. This is not to imply that some form of oppression always results; high levels of agreement may mitigate the effects of violating these norms (see Sen, 1984, for an extended discussion). Many critics have altered this conclusion by relaxing conditions (Campbell, 1977; Plott, 1967). Tullock (1967) attempted to show that the cycles in Arrow's theorem are irrelevant for large groups of voters. Nonetheless, Arrow's theorem is remarkable in its simplicity and the surprising power of its conclusion.

4.3 The Median Voter Theorem

The simplest, most direct analysis of the aggregation of vote preferences in elections is the median voter theorem. Black's (1958) early work identified the role of a specific voter whose position in a single issue dimension is at the median of other voters' preferences. His theorem roughly states that if all of the voters' utility functions are unimodal on a single issue dimension, then the median voter will always be in the winning majority. Thus Black uses the *individual* unimodal assumption to escape the specter of Arrow's theorem and subsequent cycling. The unimodal assumption dispenses with Arrow's unrestricted domain requirement by mandating that each voter in the model have a single expressed preference.

The median voter theorem is displayed in Figure 2, which is a reproduction of Black's figure (1958, page 15). Shown are the tops of the utility functions for five hypothetical voters on an interval measured issue space (the *x* axis). It is assumed, but by convention not drawn, that the individual utility functions have support over the complete issue space with asymptotic tail behavior. In the case given here it is clear that the voter with the mode at O_3, his or her *ideal point*, is the median voter in this system.

The median voter theorem requires two primary restrictions. There must be a single issue dimension (unless the same person is the median voter in all relevant dimensions) and each voter must have a unimodal utility function. Also, for simplicity, the size of the voting population is often assumed to be odd. In the case of an even number (which is possible and important in the case of committee analysis), the winning position is the mean of the two modes that jointly comprise a median. We assume below that the voting population is odd to simplify the terminology. [The even versus odd distinction is actually much less important than it would seem. If we were to eliminate one of the voters in Figure 2, then in fact there would be two medians (presuming of course that we cannot
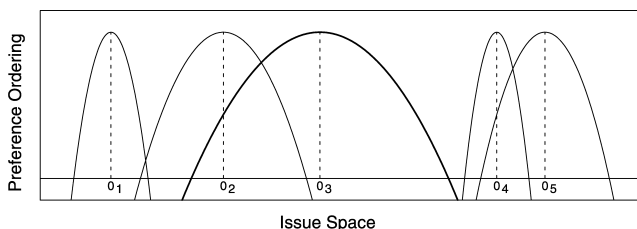

FIG. 2.   *Placing the median voter.*

cut voters in half or something). Black's theorem still works since there is no winning coalition without the approval of *both* of the median voters and, therefore, the median voter wins again, even though the definition of median voter now counts as two people. In such situations where ties are possible, which of course is much more likely with small voting bodies, it is typical to see detailed contingencies worked out in advance. Small wonder why there are nine Supreme Court justices in the United States.] There are also two other assumptions we glossed over above: all voters participate in the election and all voters express their true preferences (sincere voting). There is a substantial literature that evaluates the median voter theorem after altering these assumptions (see Dion, 1992, e.g.).

In Black's original notation, the median voter theorem is stated as follows:

THEOREM 2.   *When there are n members in a committee, all of whose curves are single peaked, and n is odd, the value $O_{(n+1)/2}$ can get at least a simple majority against every other and it is the only value which can do so.*

PROOF.   Suppose the median position, $O_{(n+1)/2}$, is placed against a lower value, $O_\ell < O_{(n+1)/2}$, in a vote. Because $\frac{1}{2}(n + 1)$ of the voters have modes above $O_{(n+1)/2}$ by definition, $\frac{1}{2}(n + 1)$ or more curves are up sloping to the right at O_ℓ. Therefore, the median position will get a number of votes in the interval $[\frac{1}{2}(n+1) : n]$, which is always a majority. A vote above $O_{(n+1)/2}$ loses as well by the symmetric argument.   □

Similar proofs were given by Enelow and Hinich (1984) and Mueller (1989). The core of this proof is the use of "up sloping" as an indicator of increasing utility received for one position over another for a specific individual. This is where the key assumption of unimodality is required.

It is also clear that along the interval-measured policy space shown in Figure 2, the winning proposal does not have to be *exactly* at the median voter's utility-maximizing point. Suppose that a proposal is offered to the voters that is just slightly off of the mode of O_3's utility curve ($O_3 + \delta$). Clearly for small δ this proposal would also win. So more generally we can define a *winset* as the *set of winning proposals over all other alternatives*. That is, for every point along the *x* axis it is possible to determine if this point wins on single vote if we know enough about the shape of each voter's utility distribution.

There is another interesting implication from Black's model. In basic settings, such as the one described

here, two candidates or parties are motivated to move toward the median voter from either end of the policy spectrum. This result is often used as an explanation for the similarity of the two dominant American political parties (at least in comparison with other industrialized democracies). While there are obvious exceptions and complications such as third party entries, non-single-member/single-district systems and multidimensional issues, this spatial convergence is a surprisingly robust phenomenon (Calvert, 1985; Enelow and Hinich, 1984, 1990; Shepsle, 1991).

5. SPATIAL MODELS OF VOTING

Black's median voter theorem is actually a simple case of what are now called spatial voting models. The general principle is that issue preference can be measured in Euclidean space: k-dimensional interval scales. So the further a given candidate or proposal is in issue dimension from a given voter's ideal position, the less utility that voter receives from voting positively. This is quite intuitive; if I am a member of Congress strongly supportive of the NRA's interpretation of the second amendment on gun ownership, then the more a bill restricts gun purchases, the less satisfaction I receive from its passage and therefore the less likely I am to vote in favor of it.

The study of spatial models started with Hotelling (1929), who showed why firms tend to cluster together geographically even though they are competitors (the classic modern example is the configuration of auto dealers). The field was initially slow to develop (important works were few: Smithies, 1941; Black, 1948, 1958; Luce and Raiffa, 1957; Downs, 1957; Davis and Hinich, 1966; Plott, 1967), but is now quite active as a research area. Riker and Ordeshook (1968), as well as Davis and Hinich (1967) and Davis, Hinich and Ordeshook (1970), provided the classic works on the underlying assumptions, and most work builds on these general ideas. In principle, there are two objectives in this literature: to find an equilibrium point derived from initial conditions that predict voter or candidate behavior and to explain empirically observed behavior from basic motivations. In both cases the fundamental ideas all revolve around the principle of issue space and issue distance as a negative utility factor.

Where spatial models get more interesting and more complicated is in the multidimensional case (more than one issue space). In fact, Hinich (1977) showed that the median voter is irrelevant in highly multidimensional settings because the probability that a single individual occupies the median position on every dimension is ridiculously small in such applications.

Consider a zero-sum voting game to divide money between two competing programs. In textbook economic parlance this is guns versus butter; slightly more realistically, Tullock (1967) compared appropriations for the Navy versus the Army. This is now a two-dimensional voting problem because for every point on the two-dimensional $x-y$ grid of spending, there is a utility level for each given voter. More realistically, evaluate public policy spending in terms of alternative energy sources. Imagine that a new administration needs to send Congress a budget for research and development spending that divides total possible spending between coal and petroleum.

A hypothetical member of Congress is assumed to have an ideal spending level for each project that trades off spending in one dimension against another, where the highest altitude ideal point, and therefore mode of the preference structure, is located at [petroleum = 0.65, coal = 0.35] on a standardized metric (dollars removed).

The first two panels of Figure 3 show the example representative's utility preference: a three-dimensional wire-frame drawing that reveals the dimensionality now present over the petroleum/coal grid and a contour plot which illustrates three specified levels of utility (U_1, U_2, U_3). The contour plot is actually a more useful heuristic because it allows us to see utility as part of the issue space. As indicated by the dashed lines, there is a modal point for our voter and utility decreases circularly (an assumption) as the potential spending tradeoff moves in any direction from this point. That is, this voter has a lower returned utility for a spending level at [0.2, 0.8] than at [0.6, 0.4]. Therefore, any point outside a given contour provides less utility than all the points inside this contour, no matter what the direction from the ideal point. Now unlike the simple one-dimensional case, the voter can be more or less satisfied by movement in two dimensions.

The third panel in Figure 3 generalizes the preference structure by showing a noncircular structure that indicates that the member has more sharply declining utility in the coal direction than in the petroleum direction. This means that he or she will be less amenable to compromise or negotiation in that direction. This is an important distinction because it is often unrealistic to assume that voters have the same declining rate of utility across multiple dimensions.

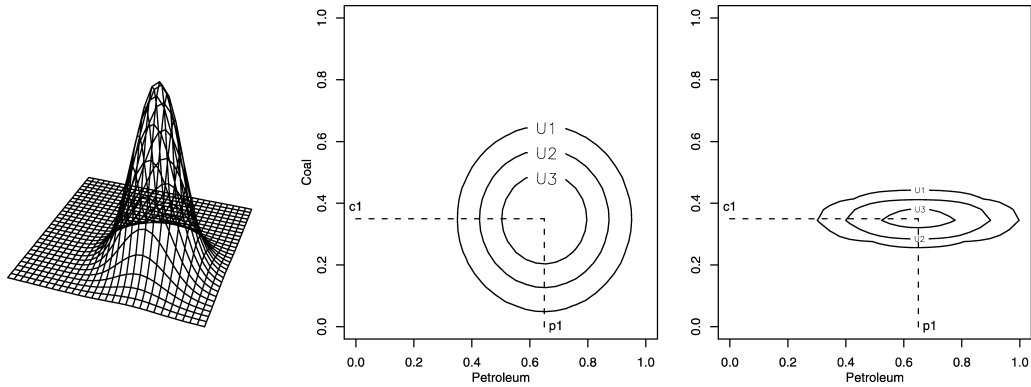Inasmuch as the single dimension assumption was a simplification, two dimensions also might not reflect

FIG. 3.   *Multidimensional issue preference.*

political reality. The setup described can be general-ized (although not always easily visualized) to higher dimensions wherein preference radiates outward in the downward direction from ideal points as spheres (three issues) or hyperspheres (four or more issues).

So far the spatial analysis has only shown one voter, which clearly is not very realistic. Now consider the utility preferences of three voters over the same issue space. Furthermore, instead of a detailed contour description of the utility structure for these three voters, we will consider only the contour level that immediately provokes an affirmative vote by that person. (We could also use a more rigid definition of approval here that says that the single contour represents a final, unyielding vote threshold. That is, the voter will always vote yes for proposals inside the threshold and always vote no for proposals outside the threshold.) For some this will be a tight contour,

indicating reluctance to deviate far from the ideal point; for others, it will be wider, indicating reasonable flexibility on an initial vote.

Figure 4 shows a hypothetical set of three voters with their immediate affirmative vote frontier given by a contour for each, with the ideal points marked in the middle. In the first panel there is an intersecting region between the contours of voter 1 and voter 2, meaning that a proposed spending level between the two policy alternatives that falls in this area will immediately get these two votes and therefore pass (a *winset*) even though the third voter's region does not also intersect (two voters constitute a majority here). The second panel shows a slightly more complicated situation. There are now three overlaps of interest that are all winsets. There remains the winset between voter 1 and voter 2, but there is also a winset between voter 2 and voter 3 that excludes voter 1, and a winset between
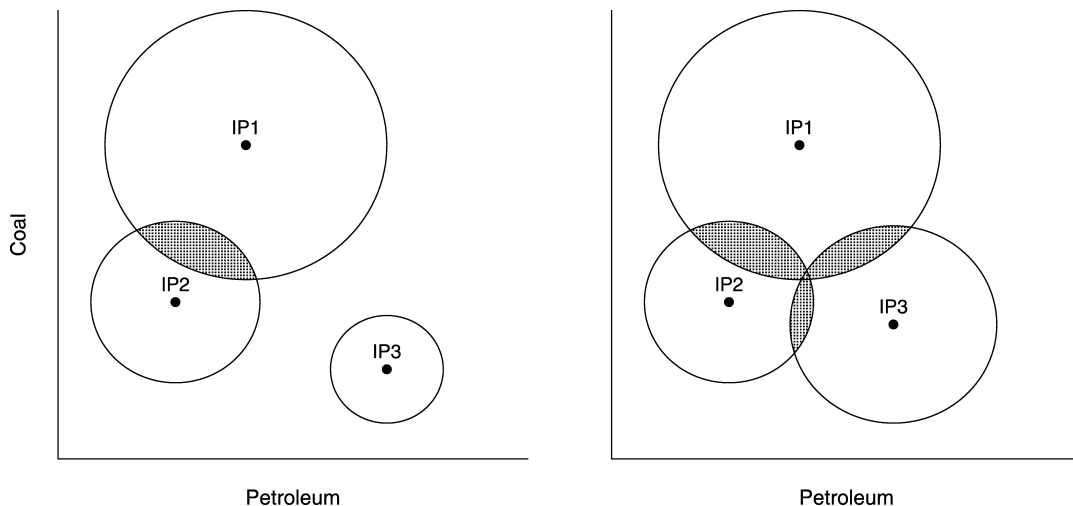

FIG. 4.   *Multidimensional issue preference.*

voter 1 and voter 3 that excludes voter 2. Note also that there is also a small winset produced by a unanimous vote.

The arrangement of ideal points is important. In Figure 4 they are spread out through the issue space without any special features. Interestingly, if the ideal points are perfectly linear in any direction, then this two-dimensional issue space is reduced to a one-dimensional (composite) issue space and Black's median voter theorem prevails. Plott (1967) showed that in general if there is "radial symmetry" around some point, meaning that the ideal points are distributed symmetrically and linearly in any given radial direction, then this center point functions like Black's median voter position, *even in high dimensions*. Unfortunately, this idea of radial symmetry is completely artificial and unrealistic, and McKelvey (1976) soon thereafter showed with his "chaos theorem" that no naturally occurring single-point winset will emerge. His theorem is named as such because in higher dimensions the voting is almost guaranteed to return to Arrow's configuration with no Condorcet winner and endless cycling (hence legislative chaos). This is where agenda setting matters. Consider again the second panel of Figure 4, except now stipulate that voter 2 has agenda control: he or she can decide which alternatives are put up for a vote. As long as he or she does not allow a vote on a proposition in the intersection between voter 1 and voter 3 that excludes his or her intersection, he or she will either get no bill passed or get a bill within his or her acceptance frontier.

While it is true that in a multidimensional spatial model under majority rule there will almost certainly not be a single majority winner, it is clear from seeing elections and legislative action that there is a great deal of stability nonetheless (Fiorina and Plott, 1978; Tullock, 1981). This motivates a large literature that seeks to explain observed stability, the central focus of which is to specify *regions* of the multidimensional issue space that dominate others and therefore lead to equilibrium outcomes. Some of these are quite fundamental notions. The *core* is the set of points that beats all others under majority rule and is the "median of the induced ideal points on all lines containing it" (Cox, 1987). Related to the core, but weaker, is the *Copeland winner*, which is the point that is majority preferred to the largest proportion of all other alternative points (Grofman, Owen, Noviello and Glazer, 1987).

Sometimes the core does not even exist and so a generalization is needed. A point in the issue space, *a*, *covers* another, *b*, if the majority prefers *a* to *b*, and every other alternative, *c*, that the majority prefers over *a* is also preferred over *b*: *aCb* (McKelvey, 1986). An *uncovered* set, A', within a larger set (possibly all sets), A'', contains all points with the condition that there is no point *b* in A'' such that *bCa* (Miller, 1980). The uncovered set is equal to the core when the core exists and voters have spherical preferences (panel 2 of Figure 3, but not panel 3). It is also possible to define the *yolk* as the smallest sphere (circle for two dimensions) that intersects all median hyperplanes (lines for two dimensions; Ferejohn, McKelvey and Packel, 1984). A median line generalizes Black's theorem by cutting the plane such that half of the voter ideal points are on either side. The larger the yolk, the further the system is away from having a single core element (Grofman, 1989). There are many more related criteria with varying assumptions, but all are directed toward describing equilibrium regions of the multidimensional issue space distinct from the elusive single majority winner.

6. PROBABILISTIC VOTING MODELS

Probabilistic voting adds a new element of uncertainty to the voting process in which there is a random element to any voter's utility calculation that is expressly modeled with the mechanics of probability theory. Probabilistic voting models retain the spatial model description of preference, but replace the discrete and deterministic utility-maximizing decision with a continuously measured probabilistic calculation that sometimes produces a vote for alternatives with lower expected utility.

The motivation for assuming this probabilistic element in voting is to account for either inadvertent or deliberate uncertainty in the expected return for voting for a specific candidate or proposal. That is, candidates may have an incentive to be vague on certain issues (Franklin, 1991; Shepsle, 1972), issues may be complex or poorly posed, information may be limited and voters themselves may simply have a difficult time mapping alternatives to expected utility. In support of this point, Fiorina (1981) asserted that "In the real world choices are seldom so clean as those suggested by formal decision theory. Thus real decision makers are best analyzed in probabilistic terms rather than deterministic terms."

The idea that voting models should have a probabilistic component was primarily developed by a relatively small cohort of authors. (Key works include

Hinich, 1977; Hinich and Munger, 1994; Coughlin, 1982, 1992; Coughlin and Nitzan, 1981; Enelow, Endersby and Munger, 1993; Hinich, Ledyard and Ordeshook, 1972, 1973; Mueller, 1989; Enelow and Hinich, 1984; Ordeshook, 1986.) Burden (1997) provided an excellent overview of the differences between deterministic models and probabilistic models, and demonstrated empirically that the alternative assumptions can lead to differing conclusions with the same data. (See also Calvert, 1986, and Mueller, 1989, for contrasts between deterministic and probabilistic assumptions.)

Consider a two-candidate plurality election between c_1 and c_2, where voters can abstain due to alienation (the utility of the preferred candidate is below a minimum threshold) or indifference (the utility difference between the preferred candidate and the other is below a minimum threshold). Hinich, Ledyard and Ordeshook (1973) developed a model to account probabilistically for abstention by voter i from these two sources starting with the probability of voting for each candidate expressed proportionally with the respective utilities $p_i(c_1) = a_i[U(c_1)]$ and $p_i(c_2) = a_i[U(c_2)]$, where a_i is an individually determined constant. Here each candidate is evaluated separately and we assume that when $U(c_i) < U_{min}(c_i)$ for the preferred candidate (preference based on greater utility), the voter abstains due to alienation. The probability of voting for the preferred candidate j over the other candidate k is $p_i(c_j) = b_i[U(c_j) - U(c_k)]$, where b_i is a different individually determined constant. Here voter i now abstains if $[U(c_j) - U(c_k)] < \min_i[U(c_j) - U(c_k)]$, because the utility difference falls below an indifferent threshold. We can put these two calculations together by positing an importance weight such that ε_i is the importance of alienation to voter i, and $1 - \varepsilon_i$ is the complementary importance of indifference to voter i. Thus voter i votes for his or her preferred candidate j over k, versus abstaining, with probability $p_i(c_j) = (\varepsilon_i)a_i[U(c_j)] + (1 - \varepsilon_i)b_i[U(c_j) - U(c_k)]$. Now we have a probabilistic model that accounts for candidate utility preference, individualistic thresholds and weights on alienation and indifference, as well as a useful method for aggregating these effects across an electorate. Ironically, it has been demonstrated that this type of probabilistic model leads to a great degree of *certainty* in candidate strategy: there are identifiable strategies that produce higher probabilities of electoral success (Coughlin and Nitzan, 1981; Denzau and Kats, 1977).

7. COST–BENEFIT MODELS OF VOTING

One area of study focuses on the individual's decision to *participate* in a given election. Essentially the question condenses to, Why does anyone vote? Clearly individuals experience personal costs in the process of voting: time, transportation and inconvenience (without any real hope of personally affecting the election). So one would therefore surmise that there must be countervailing benefits. It seems very simple at first.

7.1 The Downsian Model

The modern study of voter turnout starts in 1957 with Anthony Downs, who calculated that voting is typically an irrational act. Downs (1957) argued that a rational, utility-maximizing voter would weigh the cost of voting against the expected utility of a preferred candidate winning times the likelihood that the vote will make a difference in the election. The expected utility of the preferred candidate winning may differ dramatically depending on the voter's attitude toward the opposing candidate and the expected benefits from the preferred candidate's programs. However, since the likelihood of any one person's vote being the critical decider of an election, especially a national one, is infinitesimally small (see Gelman, King and Boscardin, 1998), the rational voter will abstain so as to not incur the costs. Unfortunately for the theory, approximately half of the eligible U.S. voters show up at the polls and are, therefore, Downsian irrationals. This henceforth has been referred to as *the paradox of not voting* in the voting literature. Downs wrestled with the paradox that some people do vote and concluded that there must be a significant benefit to the voter: the long run participation value of supporting the democratic system.

7.2 The Free-Rider Problem

Olson (1965) damaged Downs' last argument by pointing out that in large collective action problems, like elections, there is an incentive for people to simply let others decide since the probability that their own vote is a determinant is near zero and the cost of voting is likely to supersede the Downsian social or psychological benefit of voting. He called these nonparticipating utility maximizers *free-riders* because they ride on the system, collecting the same benefits (or costs) as every other citizen, but they do not contribute by participating in the decision.

Olson's observation leads to two questions. First, will the expected noneconomic (direct) benefits ever

exceed the costs for voters? Second, can incentive systems be designed such that the free-rider problem is eliminated or reduced? The answer to the first question is an obvious "yes," since people actually do show up and vote in large elections. The answer to the second question is actually "sometimes," and where we see direct evidence of this effect is in the behavior of interest groups. Interest groups (pressure groups, lobbying groups, special interests, etc.) confront an extreme example of the free-rider problem because it is commonly perceived by individuals that these groups will continue to pursue their public policy interests regardless of a single individual's membership or not. Suppose that a citizen believes strongly in protecting the natural environment. It is also unlikely that this person feels that their individual annual membership cost of $25 (in 2002) will alter the effectiveness of the Sierra Club as a national advocate for natural protection, and therefore there is an incentive to be a free rider rather than a dues-paying member. Interests groups subsequently circumvent this problem by offering *particularistic benefits* to members only, in addition to their public advocacy work. For instance, the Sierra Club currently offers (in 2002) new members "a free Sierra Club Expedition Pack (limited time only), one-year subscription to Sierra magazine, members-only eco-travel opportunities, automatic membership in your local chapter and discounts on Sierra Club calendars, books and other merchandise."

7.3 Formalization of the Cost–Benefits Model

Riker and Ordeshook (1968) codified the standard Downs model into mathematical symbology:

> R = the satisfaction in utiles of voting,
> P = the probability that the voter will affect the outcome with his or her particular vote,
> B = the difference in benefits between the two candidates measured in utiles, $B1 - B2$,
> C = the cost of voting in utiles (i.e., time, effort, money).

Thus the Downsian model is represented as $R = PB - C$, where the voter will abstain if $R < 0$. Also note that if $R > 0$, the voter still may not vote because there may be other competing activities that produce a higher R for that given point in time. Assuming that $C > 0$, then PB must be greater than C for the voter to vote, and since we know that P is very small (he or she is very unlikely to be the *pivotal voter*), then B must be very large for a vote to occur. Obviously it would take an unrealistically large value for B to overcome this small value of P. Riker and Ordeshook addressed this problem by formalizing the additional Downsian satisfaction parameter for each voter, D. This D is added to the right-hand side of the equation and represents the personal satisfaction/utility that a citizen receives from the act of voting regardless of the actual outcome of the election, thus producing a new Downsian model: $R = PB - C + D$. Therefore, some people will vote and some will not, depending on whether D_i sufficiently overcomes C_i for individual i. The parameter D is said to consist of various social and psychological subfactors such as citizen duty, prestige, guilt relief and a sense of continuing the political system (although this term strikes some as a rather ad hoc repair to the model).

To include uncertainty about the election outcome, begin with a hypothetical citizen facing a two-candidate race where candidate K_1 is preferred to candidate K_2. There are five mutually exclusive and exhaustive election states based on the number of votes for the first candidate, n_1, and the number of votes for the second candidate, n_2, *not including our hypothetical voter's participation*. These are listed in Table 4.

Using the simple Downsian model we can construct a payoff matrix that cross-tabulates the utility to the citizen for each voting action, voting for candidate 1, voting for candidate 2 and abstaining (V_1, V_2, Abs), by the five possible election outcomes. Essentially this internalizes and elaborates the P term across each of the possibilities in Table 4. Here the citizen's utility for K_1 winning is B, K_2 winning is 0 and, in the event of a tie, there will be a coin flip, thus giving utility $\frac{1}{2}U(K_1) + \frac{1}{2}U(K_2) = \frac{1}{2}B$. This structure is elaborated in Table 5.

The citizen is assumed to be a Downsian rational calculator of the value of voting given the cost of voting, C. In the standard setup, this person will make a priori subjective calculations about the probability of the S_i events, and these probabilities are labelled p_i, $i = \{1, 2, 3, 4, 5\}$, corresponding to each S_i, where

TABLE 4
The five possible election states

State	Formula	Description
S_1	$n_1 > n_2 + 1$	K_1 wins by more than one vote
S_2	$n_1 = n_2 + 1$	K_1 wins by only one vote
S_3	$n_1 = n_2$	K_1 and K_2 tie
S_4	$n_1 = n_2 - 1$	K_1 loses by only one vote
S_5	$n_1 < n_2 - 1$	K_1 loses by more than one vote

TABLE 5
Payoff matrix for the Downsian voter

	S_1	S_2	S_3	S_4	S_5
V_1	$B - C + D$	$B - C + D$	$B - C + D$	$\frac{1}{2}B - C + D$	$-C + D$
V_2	$B - C + D$	$\frac{1}{2}B - C + D$	$-C + D$	$-C + D$	$-C + D$
Abs	B	B	$\frac{1}{2}B$	0	0

$\sum p_i = 1$. Since there are now defined outcome utilities and associated event probabilities, we can calculate the expected value of each voting alternative to this citizen:

$$E[U(V_1)] = p_1(B - C + D) + p_2(B - C + D)$$
$$+ p_3(B - C + D) + p_4(\tfrac{1}{2}B - C + D)$$
$$+ (1 - p_1 - p_2 - p_3 - p_4)(-C + D)$$
$$= B(p_1 + p_2 + p_3 + \tfrac{1}{2}p_4) - C + D,$$

$$E[U(V_2)] = p_1(B - C + D) + p_2(\tfrac{1}{2}B - C + D)$$
$$(7.1) \qquad + p_3(-C + D) + p_4(-C + D)$$
$$+ (1 - p_1 - p_2 - p_3 - p_4)(-C + D)$$
$$= B(p_1 + \tfrac{1}{2}p_2) - C + D,$$

$$E[U(\text{Abs})] = p_1(B) + p_2(B) + p_3(\tfrac{1}{2}B) + p_4(0)$$
$$+ (1 - p_1 - p_2 - p_3 - p_4)(0)$$
$$= B(p_1 + p_2 + \tfrac{1}{2}p_3).$$

We can see immediately that the second option (V_2) is fully dominated by the first option (V_1) since the terms inside the B parentheses will always be greater in the former case. Therefore, the real decision is whether to vote for the preferred candidate or to abstain. This, of course, makes intuitive sense since any model that provided for a rational individual voting for the least-preferred candidate would not be internally consistent (in the absense of strategic voting). The individual will vote here if the expected utility of voting exceeds the expected utility of abstaining. This is given by $E[U(V_1)] > E[U(\text{Abs})]$, and substituting in the definitions

$$B(p_1 + p_2 + p_3 + \tfrac{1}{2}p_4) - C + D$$
$$> B(p_1 + p_2 + \tfrac{1}{2}p_3)$$

and then simplifying gives

$$(7.2) \qquad \tfrac{1}{2}Bp_3 + \tfrac{1}{2}Bp_4 - C + D > 0.$$

This looks quite reasonable until we remember that p_3 is the subjective probability of a tie and p_4 is

the subjective probability of a one-vote margin for K_2 (before the citizen of interest votes). What this means is that, except for very small elections, $-C + D$ dominates, and voting in elections is therefore not about affecting the elections, but instead about costs and personal satisfaction.

8. THE OUTCOME OF THE 2002 U.S. PRESIDENTIAL ELECTION UNDER DIFFERING SYSTEMS

The outcome of the 2000 presidential election centered on events in Florida, including ballot confusion, allegations of voter fraud, apparent racial bias and of course the subsequent attempt at a recount along with its ordered cessation by the U.S. Supreme Court. Suppose that we could replay the vote in Florida between the four main candidates, George Bush, Al Gore, Ralph Nader and Pat Buchanan, under different voting systems as described in this review. Would the final result be different if voters were allowed to express their preferences through alternative mechanisms?

Bush (apparently but not certainly) won the plurality election over Gore by 537 votes out of approximately 5.82M cast in Florida. The closest alternative to the plurality system is the majority system, but none of the candidates met the 50% plus one vote standard: Bush received 49.84% (2,912,790 votes), Gore received 49.83% (2,912,253 votes), Nader received 0.28% (16,415 votes) and Buchanan received 0.04% (2,281 votes).

If we hypothetically mandated a runoff, then Bush would have faced Gore in Florida in the second part of the contest. Given that the same set of voters participated in the runoff (actually not a very realistic assumption, recall the French example in Section 1), and voters expressed sincere preferences, then it is likely that Gore would have beaten Bush since Nader voters are closer to Gore voters and Buchanan voters are closer to Bush voters in multidimensional issue space. In fact, Gore would only have needed 17% of the Nader voters to surpass the combined total for Bush

TABLE 6
Hypothetical approval votes

Approval for	Buchanan	Bush	Gore	Nader
Semi-independents	0	582,504	582,504	0
Supporters	2,281	2,281	0	0
Nader supporters	0	0	16,415	16,415
Remaining Bush	262,151	2,621,511	0	0
Remaining Gore	0	0	2,621,028	262,103
Total	264,432	3,206,296	3,219,947	278,518

and Buchanan (assuming that the remaining Nader voters did not turn out and vote for Bush).

Is there a Condorcet winner here? The existence of a Condorcet winner is conditional on the behavior of the Nader and Buchanan voters. Given spatial issue preference, full turnout and sincerity among voters, Gore would presumably pick up more Nader voters than Bush would Buchanan voters and so would be the Condorcet winner.

To evaluate the hypothetical results under an approval voting system we make the reasonably realistic assumptions that 10% of Gore and Bush voters are basically independent and also somewhat ambivalent between the two, all Buchanan voters approve of Bush as well, all Nader voters approve of Gore and 10% of the 90% nonambivalent Bush and Gore voters also approve of Buchanan and Nader, respectively. Applying these assumptions to the observed vote totals and calculating the approval totals in this order produces the totals shown in Table 6, where Gore wins.

This is interesting because our assumptions are just mild reinterpretations of two common admonitions heard during the campaign: "a vote for Nader is a vote for Bush because these are votes being taken away from Gore, not Bush," which was heard from many Gore supporters, and "there's not a dime's bit of difference between Gore and Bush," which was heard from many Nader supporters.

It is obviously very difficult to conjecture about the potential results had a cumulative voting system been implemented because it would be necessary to fully understand the relative utility produced by each candidate for each voter. It is likely that Buchanan supporters generally felt more positive about Bush than Nader voters did about Gore, but the distribution of their votes is hard to predict. Furthermore, the ambivalent group identified above is also difficult to predict. Nonetheless, the results from cumulative voting are likely to be similar to the hypothetical approval results in Table 6, with the important caveat that the closeness of Bush and Gore makes things less predictable.

It is somewhat easier to consider what might have occurred under a Borda count system. Suppose that every observed vote was relisted such that the $K - 1 = 3$ ordered choices were given by the closest in issue space with a preference away from the middle. This would give the results in Table 7, where again Gore wins.

Of course this analysis assumes sincere voting and this is highly unlikely to occur in great measure here: Bush and Gore voters would certainly be tempted to list Nader and Buchanan (respectively) higher than their preferred candidate's closest competitor as perhaps third on their list, even though this choice is further in issue distance than their sincere third pick.

TABLE 7
Hypothetical cumulative votes

Counts for	Buchanan	Bush	Gore	Nader
Buchanan voters	$2,281 \times 3$	$2,281 \times 2$	$2,281 \times 1$	0
Bush voters	$2,912,790 \times 2$	$2,912,790 \times 3$	$2,912,790 \times 1$	0
Gore voters	0	$2,912,253 \times 1$	$2,912,253 \times 3$	$2,912,253 \times 2$
Nader voters	0	$16,415 \times 1$	$16,415 \times 2$	$16,415 \times 3$
Total	5,832,423	11,671,600	11,684,660	5,873,751

Other voting systems discussed do not really lend themselves to this type of analysis because they are more focused on fashioning legislative representation rather than selecting an executive. Various aggregation methods such as the Hare and Coombs procedures require some form of proportional representation and apply to parliamentary government. If we could develop some hypothetical parliamentary outcome for this election, it would most likely favor Bush, since the congressional delegation from Florida has quite a few more Republicans than Democrats.

Although we have had to make a reasonable number of assumptions in analyzing the 2000 Florida presidential vote, it is clear that the outcome could have differed under alternative voting and aggregation schemes. This is a somewhat unusual case study because the vote was so close and because the Florida outcome became the deciding factor in the national presidential race. However, it still makes our point that *the system in which votes are counted matters in determining elections.* The intention of this review essay has been to introduce these particular theories and systems of voting as well as to demonstrate their current relevance.

ACKNOWLEDGMENTS

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