The Annals of Statistics
1988, Vol. 16, No. 4, 1727–1740

OPTIMUM TESTS IN UNBALANCED TWO-WAY MODELS WITHOUT INTERACTION

BY THOMAS MATHEW AND BIMAL KUMAR SINHA[1]

University of Maryland Baltimore County

It is an *open* problem in the literature to derive optimum tests for the equality of treatment effects in an unbalanced two-way classification model. For such models without interaction, optimum tests are derived in the following cases: (i) the locally best invariant unbiased test for the random effects model corresponding to an equiblock and equireplicate design, (ii) the locally best invariant unbiased test for the mixed effects model with mixed treatment effects corresponding to a balanced incomplete block design and (iii) the uniformly most powerful invariant test or the locally best invariant test for the mixed effects model with random treatment effects.

Robustness of the optimum invariant tests against suitable deviations from normality is also indicated.

1. Introduction and summary. For ANOVA models with fixed effects, it is well known that the appropriate F-tests for testing the significance of fixed effects are optimum invariant tests under the assumption of normality and independence of the errors, see Lehmann [(1959), Chapter 7, Section 1]. Earlier work on similar optimality properties in the mixed and random effects models mainly dealt with the one-way classification model and the two-way classification model with or without interaction [see Thompson (1955a), Herbach (1959) and Spjøtvoll (1967)]. For a general ANOVA model with mixed effects, optimality properties of tests (for fixed effects or variance components) have been investigated only recently. For such models with balanced data, Seifert (1978, 1979) has shown that the usual F-tests for fixed effects are optimal (UMPU, UMPI). For variance components, however, only some exact tests are obtained in Seifert (1981, 1985) and no optimality is claimed or established. In a recent paper by Mathew and Sinha (1988), the UMPU and UMPIU character of the standard F-tests (for fixed effects as well as variance components) have been settled completely for a general balanced ANOVA model with mixed effects.

For a mixed effects ANOVA model with unbalanced data, the picture is entirely different. Even though exact tests are available in some cases [see Thompson (1955a, 1955b), Thomsen (1975), Pincus (1977), Seely and El-Bassiouni (1983) and Kleffe and Siefert (1988)], little is known by way of optimality of such tests, except for the one-way unbalanced random effects model [see Das and Sinha (1987) and Spjøtvoll (1967)]. Unlike in balanced

Received October 1987; revised February 1988.

[1]Research supported by Air Force Office of Scientific Research Grant F49620-85-C-0008. The United States Government is authorized to reproduce and distribute reprints for governmental purposes.

AMS 1980 *subject classifications*. Primary 62F03; secondary 62F04, 62K10.

Key words and phrases. Locally best invariant test, locally best invariant unbiased test, uniformly most powerful invariant test, balanced incomplete block design, variance balanced design.

models, the derivation of an optimum invariant test in unbalanced models presents considerable difficulty due to one or both of the following reasons: (i) the family of sufficient statistics under the null hypothesis is not complete and (ii) the group which leaves the underlying testing problem invariant is not large enough to guarantee a sufficiently small class of invariant tests. While it is impossible to do away with the above difficulties for a general unbalanced model, it is quite plausible to establish optimality results for some specific models by imposing suitable restrictions on the underlying designs in order to overcome the above hurdles. This is precisely what we accomplish here and in the sequel provide a solution to some *open* problems in this area.

In the present paper, we take up the problem of deriving optimum tests in the unbalanced two-way classification model without interaction. The model considered is the following:

$$y_{ijl} = \mu + \tau_i + \beta_j + e_{ijl}, \quad l = 1, 2, \dots, n_{ij},\ i = 1, 2, \dots, v,\ j = 1, 2, \dots, b. \tag{1.1}$$

Here μ is an unknown constant while τ_i (the treatment effect) and β_j (the block effect) are unknown parameters or random variables depending on whether these effects are fixed or random. Whenever τ_i and β_j are random variables we shall assume that $\tau_i \sim N(0, \sigma_\tau^2)$, $\sigma_\tau^2 \geq 0$, and $\beta_j \sim N(0, \sigma_\beta^2)$, $\sigma_\beta^2 \geq 0$. It is further assumed that all these random variables are independent and also independent of e_{ijl} which are i.i.d. $N(0, \sigma^2)$, $\sigma^2 > 0$. We have considered both the random effects model and the mixed effects model. In each case an optimum test is derived for testing the significance of the treatment effects (either fixed or random) by imposing suitable restrictions on the underlying block design.

In Section 2, for the completely random effects model, under a suitable group of transformations, we have derived the locally best invariant unbiased (LBIU) test for testing $\sigma_\tau^2 = 0$, assuming the design to be equiblock and equireplicate. The mixed effects model with τ_i fixed and β_j random is taken up in Section 3. By invoking invariance under a suitable group of transformations, we have derived the LBIU test for testing the equality of the τ_i's, assuming the design to be a balanced incomplete block design (BIBD). Section 4 deals with the mixed effects model with τ_i's random and β_j's fixed. For testing $\sigma_\tau^2 = 0$, the uniformly most powerful invariant (UMPI) test under a suitable group of transformations is derived for a variance balanced design. For a design that is not variance balanced, we have obtained the LBI test. The optimum tests in Sections 2 and 3 depend on the magnitude of the block variance σ_β^2 in relation to the error variance σ^2. The test comes out in terms of the adjusted or unadjusted treatment sum of squares (defined in Section 2), depending on whether σ_β^2/σ^2 is large or small. It is interesting to observe that our tests do not always coincide with the standard F-tests. The optimum tests in Sections 2 and 3 coincide with the standard F-test only when σ_β^2/σ^2 is large and the optimum test in Section 4 coincides with the F-test only for a variance balanced design. We have also indicated the robustness of the optimum invariant tests against suitable deviations from normality.

For a two-way model with interaction (with mixed or random effects), derivation of an optimum test for testing the significance of interaction presents considerable difficulties even with further restrictions, unless the design is balanced. This is because under the null hypothesis of no interaction, the family of sufficient statistics is not complete if the design is not balanced [this can be easily verified by checking the necessary and sufficient conditions in Seely (1977)]. It may be noted that for the two-way random effects model with interaction, an exact and simple test for interaction, based on an F-statistic, is given in Thomsen (1975). Moreover, for the same model, a simple test for the main effects variance (namely, for $\sigma_\tau^2 = 0$ and $\sigma_\beta^2 = 0$) appears in Thomsen (1975), assuming that the interaction term is absent, and in Khuri and Littel (1987) without this assumption. However, as noted before, the local optimality of Thomsen's test for $\sigma_\tau^2 = 0$ can be established only when the underlying design is both equireplicate and equiblock.

In conclusion, we emphasize that for a two-way unbalanced model without interaction with mixed or random effects, the results we have obtained (under suitable restrictions on the block design) appear to be the best that can be established toward deriving optimum tests for the significance of treatment effects.

2. The random effects model. The hypothesis of interest is H_0: $\sigma_\tau^2 = 0$ against the alternative H_1: $\sigma_\tau^2 > 0$, when the τ_i's and β_j's are both random variables in the model (1.1). To derive an optimum test, we assume that $\sum_{i=1}^{v} n_{ij} = k$ for each j and $\sum_{j=1}^{b} n_{ij} = r$ for each i. In other words, the block design is taken to be equiblock and equireplicate. It will be clear from what follows that this assumption is indeed necessary to derive an optimum test for the random effects model. Let

$$\mathbf{y}_{\cdot j} = \left(y_{1j1}, y_{1j2}, \dots y_{1jn_{1j}}, \dots, y_{vj1}, \dots, y_{vjn_{vj}}\right)' \tag{2.1}$$

be the $k \times 1$ vector of observations from the jth block and write

$$\mathbf{y} = (\mathbf{y}_{\cdot 1}', \dots, \mathbf{y}_{\cdot b}')'. \tag{2.2}$$

If $\mathbf{1}_k$ denotes the k-component vector of 1's and $E_k = \mathbf{1}_k \mathbf{1}_k'$ (we will denote these quantities by $\mathbf{1}$ and E when the dimension is clear), then (1.1) can be written as

$$\mathbf{y} = \mu \mathbf{1}_{bk} + F\tau + (I_b \otimes \mathbf{1}_k)\beta + \mathbf{e}, \tag{2.3}$$

where F is the $bk \times v$ design matrix for the treatments, τ and β are vectors consisting of τ_i's and β_j's, respectively, and $\mathbf{e}$ is the vector of e_{ijl}'s. Thus

$$E(\mathbf{y}) = \mu \mathbf{1}_{bk}, \qquad \operatorname{cov}(\mathbf{y}) = \sigma^2 I + \sigma_\beta^2 E_0 + \sigma_\tau^2 FF', \tag{2.4}$$

where $E_0 = \operatorname{diag}(E, \dots, E)$, $E = \mathbf{1}_k \mathbf{1}_k'$. We shall write $\mathbf{B} = (B_1, B_2, \dots, B_b)'$, $\mathbf{T} = (T_1, \dots, T_v)'$ the vectors of block totals and treatment totals, respectively. N will denote the $v \times b$ incidence matrix whose ijth element is n_{ij}. Under H_0: $\sigma_\tau^2 = 0$, the model reduces to a balanced one-way classification model with random block effects and hence, the grand mean $\bar{y}_{\cdot\cdot} = \sum_{j=1}^{b} \mathbf{y}_{\cdot j}'\mathbf{1}/bk$, the sum of squares between blocks

$$\left(SS_B = \sum_{j=1}^{b} (\mathbf{y}_{\cdot j}'\mathbf{1}/k - \bar{y}_{\cdot\cdot})^2\right)$$

and the sum of squares within blocks

$$\left(SS_W = \sum_{j=1}^{b} \sum_{i=1}^{v} \sum_{l=1}^{n_{ij}} (y_{ijl} - \mathbf{y}_{\cdot j}'\mathbf{1}/k)^2\right)$$

jointly form a set of complete sufficient statistics (the reader may note that the assumption of equiblock design is used here to ensure this). Writing $\theta_1 = \sigma_\beta^2/\sigma^2$ and $\theta_2 = \sigma_\tau^2/\sigma^2$, the density of $\mathbf{y}$ is given by

$$\begin{aligned} &f(\mathbf{y}; \mu, \theta_1, \theta_2, \sigma^2) \\ &\quad = (2\pi\sigma^2)^{-bk/2}|I + \theta_1 E_0 + \theta_2 FF'|^{-1/2} \\ &\qquad \times \exp\left\{-\frac{1}{2\sigma^2}(\mathbf{y} - \mu\mathbf{1})'(I + \theta_1 E_0 + \theta_2 FF')^{-1}(\mathbf{y} - \mu\mathbf{1})\right\}. \end{aligned} \tag{2.5}$$

It turns out that there is no locally best or locally best unbiased test for testing H_0. Hence, in order to derive an optimum test, we reduce the problem through invariance.

Clearly the above testing problem remains invariant under the group G of transformations $\mathbf{y} \to c(\mathbf{y} + \alpha\mathbf{1})$ for $c > 0$ and α real. Noting that $d\alpha \cdot dc/c$ is a left invariant measure on G, applying the representation theorem due to Wijsman (1967), the ratio R of the nonnull and null distributions of a maximal invariant (induced by the group G) is given by

$$R = \frac{\int_G f(g\mathbf{y}/H_1)J^{-1}d\alpha \cdot dc/c}{\int_G f(g\mathbf{y}/H_0)J^{-1}d\alpha \cdot dc/c}. \tag{2.6}$$

Here $g = (c, \alpha)$ is an element in G, $g\mathbf{y} = c(\mathbf{y} + \alpha\mathbf{1})$, $f(\mathbf{y}/H_i)$ denotes the normal density of $\mathbf{y}$ under the hypothesis H_i $(i = 0, 1)$, and J is the Jacobian of the transformation $\mathbf{y} \to g\mathbf{y}$. We have simplified R in the Appendix. From (A.2) in the Appendix, it follows that there is no UMPI or UMPIU test for testing H_0. To derive a locally best invariant test, following Lehmann (1959) or Ferguson (1967), we expand R locally around $\theta_2 = 0$. It is proved in the Appendix [see (A.4) and (A.6)] that under the further assumption of equireplicate design,

$$R = 1 + \frac{bk-1}{2}\theta_2\left[\frac{\mathbf{Q}_\delta'\mathbf{Q}_\delta}{D_\delta} + \frac{h(\bar{y}_{\cdot\cdot}, \theta_1)}{D_\delta}\right] + o(\theta_2), \tag{2.7}$$

where

$$\mathbf{Q}_\delta = \mathbf{T} - \frac{1}{k}\delta N\mathbf{B}, \qquad \delta = \frac{k\theta_1}{1 + k\theta_1}, \qquad D_\delta = SS_W + (1 - \delta)SS_B, \tag{2.8}$$

$h(\bar{y}_{\cdot\cdot}, \theta_1)$ is a function of θ_1 and $\bar{y}_{\cdot\cdot}$ only, and $o(\theta_2)$ is uniformly so in $\mathbf{y}$. From (2.7), it follows that an invariant unbiased test for H_0 with maximum local power rejects H_0 for large values of $\mathbf{Q}_\delta'\mathbf{Q}_\delta$, conditional on the complete sufficient statistic $(\bar{y}_{..}, SS_B, SS_W)$. The dependence of the resultant test statistic $\mathbf{Q}_\delta'\mathbf{Q}_\delta$ on δ is rather unpleasant. However, meaningful tests can be derived in two special cases when δ approaches 1 or 0 according as σ_β^2/σ^2 becomes large or small. Clearly, $\mathbf{Q}_\delta \to \mathbf{Q} = \mathbf{T} - (1/k)N\mathbf{B}$, the vector of adjusted treatment totals as $\delta \to 1$ and $\mathbf{Q}_\delta \to \mathbf{T}$, the vector of unadjusted treatment totals as $\delta \to 0$. Moreover, in the present setup, it is easy to verify that, under H_0, $\mathbf{Q}$ and $\mathbf{B}$ are independent and $F_0 = (\mathbf{Q}'\mathbf{Q}/u)/(SS_B/(b-1))$ has central F-distribution with degrees of freedom $(u, b-1)$. Here u is the rank of the C-matrix. An application of Basu's theorem [Lehmann (1959), page 162] shows that under H_0, F_0 is independent of the complete sufficient statistic $(\bar{y}_{..}, SS_B, SS_W)$. Thus we have proved

THEOREM 2.1. *Suppose* (1.1) *is a random effects model corresponding to an equiblock and equireplicate block design. Let* $\mathbf{T}$ *and* F_0 *be as defined above. For testing* H_0: $\sigma_\tau^2 = 0$, (i) *the locally best invariant unbiased test rejects* H_0 *for large values of* F_0 *if* σ_β^2/σ^2 *is large and* (ii) *the locally best invariant unbiased test rejects* H_0 *for large value of* $\mathbf{T}'\mathbf{T}$, *conditional on* $(\bar{y}_{..}, SS_B, SS_W)$ *if* σ_β^2/σ^2 *is small.*

REMARK 2.1. In order to apply Theorem 2.1(ii), one needs to derive the conditional null distribution of $\mathbf{T}'\mathbf{T}$ given $(\bar{y}_{..}, SS_B, SS_W)$. Although this is difficult, it is easy to compute the first few conditional moments of $\mathbf{T}'\mathbf{T}$ given $(\bar{y}_{..}, SS_B, SS_W)$. However, this is not reported here. The same remark also applies to Theorem 3.1(ii) in the next section.

REMARK 2.2. From Kariya and Sinha [(1985), Section 2] we conclude that the above invariant tests are null, nonnull and optimality robust when the normal distribution is replaced by any spherically symmetric distribution and invariance is invoked.

3. The mixed effects model with τ_i fixed. Throughout this section it is assumed that β_j's in (1.1) are random variables and τ_i's are fixed unknown parameters satisfying $\sum_{i=1}^{v}\tau_i = 0$. It is decided to test H_0: $\tau = 0$. Here, due to the inherent nature of the multiparameter hypothesis H_0, reduction through invariance is essential in order to derive an optimum test. The problem is clearly invariant under the group of transformations which transform $\mathbf{y}$ to $c(\mathbf{y} + \alpha\mathbf{1})$, where α and c are scalars with $c > 0$. However, reducing the class of tests using this group alone appears to be inadequate to derive an optimum test. If we assume that the model (1.1) corresponds to a BIBD, then it is possible to have a subgroup of the permutation group leaving the problem invariant. It turns out that this further reduction is enough to guarantee the existence of an optimum test. *Hence, in this section, we consider the model* (1.1) *only for a BIBD with parameters* v, b, r, k, λ. We now proceed to describe the group of permutations which leaves the problem invariant. Following the notations in the previous section, we have

$$E(\mathbf{y}) = \mu \mathbf{1}_{bk} + F\tau, \qquad \operatorname{cov}(\mathbf{y}) = \sigma^2 I + \sigma_\beta^2 E_0 \tag{3.1}$$

[recall that $\mathbf{y}' = (\mathbf{y}_{\cdot 1}', \mathbf{y}_{\cdot 2}', \dots, \mathbf{y}_{\cdot b}')$, where $\mathbf{y}_{\cdot j}$ is the vector of observations from the jth block, F is the $bk \times v$ design matrix for the treatments and $E_0 = \operatorname{diag}(E, \dots, E)$]. Let $\mathscr{G}$ be the group of $bk \times bk$ permutation matrices which, when applied to the bk plots of the BIBD, permutes the blocks and the plots within the blocks in such a way that for $\Gamma \in \mathscr{G}$, $\Gamma F = FP$ for some $v \times v$ permutation matrix P. Note that Γ does not permute the plots from two different blocks. It is easy to check that $\mathscr{G}$ is a group with $k!b$ elements (using the symmetry of the BIBD). For such a Γ,

$$\begin{aligned} E(\Gamma \mathbf{y}) &= \Gamma(\mu \mathbf{1}_{bk} + F\tau) = \mu \mathbf{1}_{bk} + F\tau^*, \qquad \text{where } \tau^* = P\tau, \\ \operatorname{cov}(\Gamma \mathbf{y}) &= \Gamma\left(\sigma^2 I + \sigma_\beta^2 E_0\right)\Gamma' = \sigma^2 I + \sigma_\beta^2 E_0. \end{aligned} \tag{3.2}$$

[Such a group has been considered earlier by Sinha (1982) for the fixed effects model in connection with invariant estimation of treatment contrasts.]

We are now ready to derive an optimum invariant test. Clearly the testing problem is invariant under the group G of transformations

$$y \to c(\Gamma \mathbf{y} + \alpha \mathbf{1}_{bk}), \qquad c > 0, \Gamma \in \mathscr{G}, \tag{3.3}$$

α equal to an arbitrary scalar. Then $d\Gamma \cdot d\alpha \cdot dc/c$ is a left invariant measure on G, where $d\Gamma$ denotes the discrete uniform probability measure with mass $1/k!b$ at each point in $\mathscr{G}$. Applying the representation theorem due to Wijsman (1967), the ratio R of the nonnull and null distributions of a maximal invariant (induced by the group G) is given by

$$R = \frac{\int_G f(g\mathbf{y}/H_1) J^{-1} d\Gamma \, d\alpha \, dc/c}{\int_G f(g\mathbf{y}/H_0) J^{-1} d\Gamma \, d\alpha \, dc/c}. \tag{3.4}$$

Here $g = (c, \Gamma, \alpha)$ is an element in G, $g\mathbf{y} = c(\Gamma \mathbf{y} + \alpha \mathbf{1}_{bk})$, $f(\mathbf{y}/H_i)$ denotes the normal density of $\mathbf{y}$ under the hypothesis H_i $(i = 0, 1)$ and J is the Jacobian of the transformation $\mathbf{y} \to g\mathbf{y}$. We have simplified R in the Appendix. From (A.12) in the Appendix it follows that there is no UMPI or UMPIU test for testing H_0. To derive a locally best invariant unbiased test, we need to expand R locally around $\tau = \mathbf{0}$. This is done in (A.21) which reads

$$\begin{aligned} R = 1 &+ \frac{a_0}{D_\delta^2}\left(\mathbf{Q}_\delta' \mathbf{Q}_\delta + h(\bar{y}_{\cdot\cdot})\right)\tau'\tau \\ &- \frac{bk-1}{2} \frac{1}{D_\delta} \tau' \left(rI - \frac{1}{k} \delta N'N \right) \tau + o(\tau'\tau), \end{aligned} \tag{3.5}$$

where a_0 is a positive constant and $\mathbf{Q}_\delta$ and D_δ are as defined in the previous section. It is clear from (3.5) and the fact that $(\bar{y}_{\cdot\cdot}, SS_B, SS_W)$ is complete sufficient under H_0 that an invariant unbiased test with maximum local power rejects H_0 for large value of $\mathbf{Q}_\delta' \mathbf{Q}_\delta$, conditional on $(\bar{y}_{\cdot\cdot}, SS_B, SS_W)$. Noting that

$F_0 = [\mathbf{Q}'\mathbf{Q}/(v-1)]/[SS_B/(b-1)]$ is distributed as central F with $(v-1, b-1)$ d.f. under H_0, and arguing as in the previous section, we establish

THEOREM 3.1. *Suppose* (1.1) *is a mixed effects model corresponding to a BIBD with τ_i fixed. Let F_0 be as defined above. For testing H_0: $\tau = \mathbf{0}$,* (i) *the locally best invariant unbiased test rejects H_0 for large values of F_0 if σ_β^2/σ^2 is large and* (ii) *the locally best invariant unbiased test rejects H_0 for large values of $\mathbf{T}'\mathbf{T}$, conditional on $(\bar{y}_{..}, SS_B, SS_W)$ if σ_β^2/σ^2 is small.*

REMARK 3.1. It is clear from the preceding analysis and Kariya and Sinha [(1985), Section 3] that the above locally best invariant unbiased tests as $\delta \to 1$ and 0 are optimality robust when normality is replaced by appropriate spherical symmetry. The tests are not nonnull robust although they are null robust under any spherically symmetric distribution [Kariya (1981)].

4. The mixed effects model with τ_i random. Here we assume the model (1.1) with β_j's fixed and τ_i's random. It is decided to test H_0: $\sigma_\tau^2 = 0$. To derive an optimum test, we shall use some of the results from Thompson (1955a, b), who considered the above set up for a binary design. Let C denote the associated C-matrix defined as $C = \operatorname{diag}(r_1, \dots, r_v) - N\operatorname{diag}(1/k_1, \dots, 1/k_b)\, N'$, where r_i is the replication of the ith treatment, k_j is the jth block size and N is the incidence matrix and suppose $\operatorname{rank}(C) = u$ $(u \le v-1)$. Let the matrix M: $v \times v$ be such that $M'M = I_v$ and $M'CM = \operatorname{diag}(D, 0)$, where the diagonal elements $d_1, \dots, d_u$ of the $u \times u$ diagonal matrix D are the nonzero eigenvalues of C. Let $M'\mathbf{Q} = \mathbf{Z} = (\mathbf{Z}_1', \mathbf{Z}_2')'$, where $\mathbf{Z}_1$ is $u \times 1$. Recall that $\mathbf{Q}$ is the vector of adjusted treatment totals defined in Section 2. Following the arguments in Thompson (1955a, b), we conclude that

(i) the statistics $\mathbf{B}$, $\mathbf{Z}_1$ and SS_W jointly form a sufficient statistic for the normal family of distributions of $\mathbf{y}$;

(ii) the testing problem above is invariant under the group G of transformations $B_j \to cB_j + k_j c_j$, $\mathbf{Z}_1 \to c\mathbf{Z}_1$ and $SS_W \to c^2 SS_W$, where the c_j's are real numbers, $c > 0$ and k_j is the jth block size. A maximal invariant under G is easily seen to be $\mathbf{S} = (1/\sqrt{SS_W})D^{-1/2}\mathbf{Z}_1$, whose density is given by [see Thompson (1955a), page 327]

$$\text{constant}\left(1 + \sum_{i=1}^{u} \frac{S_i^2}{1 + d_i\theta}\right)^{-(n+u)/2}, \tag{4.1}$$

where $\mathbf{S} = (S_1, \dots, S_u)'$ and $\theta = \sigma_\tau^2/\sigma^2$. Hence for testing H_0: $\theta = 0$, the ratio R of the nonnull to null densities of the maximal invariant boils down to

$$R = \left[\frac{1 + \sum_{i=1}^{u} S_i^2/(1 + d_i\theta)}{1 + \sum_{i=1}^{u} S_i^2}\right]^{-(n+u)/2}. \tag{4.2}$$

If the d_i's are all equal (if $u = v - 1$, such a design is said to be variance-balanced), then R in (4.2) is monotonically increasing in $\sum_{i=1}^{u} S_i^2$. Hence the test which rejects H_0 for large values of $\sum_{i=1}^{u} S_i^2$ is UMPI. Note that

$$\sum_{i=1}^{u} S_i^2 = d^{-1}\mathbf{Z}_1'\mathbf{Z}_1/SS_W = d^{-1}\mathbf{Q}'\mathbf{Q}/SS_W,$$

where d is the common value of the d_i's. Hence, the UMPI test rejects H_0 for large values of $F_0 = (\mathbf{Q}'\mathbf{Q}/u)/(SS_W/(n-b-u))$ which is distributed as central F (under H_0) with degrees of freedom $(u, n-b-u)$. Here n is the total number of observations. If the d_i's are not all equal, then a UMPI test does not exist while an LBI test is easily derived by expanding R around $\theta = 0$. This immediately gives

$$R = 1 + \theta\frac{(n+u)}{2}\sum_{i=1}^{u} d_i S_i^2 \Big/ \left(1 + \sum_{i=1}^{u} S_i^2\right) + o(\theta). \tag{4.3}$$

It now follows that the LBI test rejects H_0 for large values of

$$\begin{aligned}\sum_{i=1}^{u} d_i S_i^2 \Big/ \left(1 + \sum_{i=1}^{u} S_i^2\right) &= \mathbf{Z}_1'\mathbf{Z}_1/\left(SS_W + \mathbf{Z}_1'D^{-1}\mathbf{Z}_1\right) \\ &= \mathbf{Q}'\mathbf{Q}/\left(SS_W + \mathbf{Q}'C^{-}\mathbf{Q}\right),\end{aligned} \tag{4.4}$$

where C^{-} is a generalized inverse of C. Thus we have proved

THEOREM 4.1. *Suppose* (1.1) *is a mixed effects model with* τ_i*'s random and* β_j*'s fixed. For testing* H_0: $\sigma_\tau^2 = 0$ *against* H_1: $\sigma_\tau^2 > 0$, (i) *if the nonzero eigenvalues of the C-matrix are all equal, then the UMPI test rejects* H_0 *for large values of* $F_0 = (\mathbf{Q}'\mathbf{Q}/u)/(SS_W/(n-b-u))$ *and* (ii) *if the nonzero eigenvalues of the C-matrix are not all equal, then the LBI test rejects* H_0 *for large values of* $\mathbf{Q}'\mathbf{Q}/(SS_W + \mathbf{Q}'C^{-}\mathbf{Q})$.

It is interesting to observe that Theorem 4.1 is valid without any restrictions on the model whatsoever.

REMARK 4.1. From Kariya and Sinha [(1985), Section 2] we conclude that the above UMPI and LBI tests are null, nonnull and optimality robust when normality is replaced by any spherically symmetric distribution.

REMARK 4.2. It is not difficult to check that an UMPU or an LBU test of H_0: $\sigma_\tau^2 = 0$ (without invariance) exists only when the design is balanced.

5. Concluding remarks. Tests for the significance of treatment effects in a block design without interaction are typically based on the ratio of adjusted treatment mean squares to the error mean squares [Montgomery (1984)]. The optimality of such a test under the fixed effects model has long been well known. Theorems 2.1, 3.1 and 4.1 as well as the Remarks 2.1, 3.1 and 4.1 bring out clearly what kind of optimalities one can expect of such a test under the random and mixed effects models and under what restrictions on the design and what conditions on the parameters.

APPENDIX

Derivation of (2.7). In (2.6), the Jacobian of the transformation $\mathbf{y} \to g\mathbf{y}$ is $J = c^{-bk}$. Also, for evaluating R in (2.6), we may assume without loss of generality, $\mu = 0$ and $\sigma^2 = 1$, in view of invariance. Using the density of $\mathbf{y}$ given in (2.5) (with $\mu = 0$, $\sigma^2 = 1$) and writing $V_1 = I + \theta_1 E_0 + \theta_2 FF'$, we get

$$\text{Numerator of (2.6)} = \text{constant}|V_1|^{-1/2}$$

$$\times \int_0^\infty \int_{-\infty}^\infty \exp\left\{-\frac{c^2}{2}(\mathbf{y} + \alpha\mathbf{1})'V_1^{-1}(\mathbf{y} + \alpha\mathbf{1})\right\} c^{bk-1}\, d\alpha\, dc$$

$$= \text{constant}|V_1|^{-1/2}\left(\mathbf{1}'V_1^{-1}\mathbf{1}\right)^{-1/2}$$

$$\text{(A.1)} \qquad \times \int_0^\infty \exp\left\{-\frac{c^2}{2}\left[\mathbf{y}'V_1^{-1}\mathbf{y} - \frac{\left(\mathbf{1}'V_1^{-1}\mathbf{y}\right)^2}{\left(\mathbf{1}'V_1^{-1}\mathbf{1}\right)}\right]\right\} c^{bk-2}\, dc$$

$$= \text{constant}|V_1|^{-1/2}\left(\mathbf{1}'V_1^{-1}\mathbf{1}\right)^{-1/2}$$

$$\times \left[\mathbf{y}'V_1^{-1}\mathbf{y} - \frac{\left(\mathbf{1}'V_1^{-1}\mathbf{y}\right)^2}{\left(\mathbf{1}'V_1^{-1}\mathbf{1}\right)}\right]^{-(bk-1)/2} \int_0^\infty e^{-u/2}u^{(bk-3)/2}\, du.$$

Noting that the denominator of (2.6) is the above expression with $\theta_2 = 0$ and writing $V_0 = I + \theta_1 E_0$, we get

$$\text{(A.2)} \qquad R = \left(\frac{|V_1|}{|V_0|}\right)^{-1/2}\left(\frac{\mathbf{1}'V_1^{-1}\mathbf{1}}{\mathbf{1}'V_0^{-1}\mathbf{1}}\right)^{-1/2}\left[\mathbf{y}'V_1^{-1}\mathbf{y} - \frac{\left(\mathbf{1}'V_1^{-1}\mathbf{y}\right)^2}{\left(\mathbf{1}'V_1^{-1}\mathbf{1}\right)}\right]^{-(bk-1)/2}$$

$$\div \left[\mathbf{y}'V_0^{-1}\mathbf{y} - \frac{\left(\mathbf{1}'V_0^{-1}\mathbf{y}\right)^2}{\left(\mathbf{1}'V_0^{-1}\mathbf{1}\right)}\right]^{-(bk-1)/2}.$$

We now simplify R when θ_2 is near 0. We first show that the denominator in (A.2) is a function of the complete sufficient statistic, namely, $\bar{y}_{..}$, SS_B and SS_W. Recalling that $E_0 = \operatorname{diag}(E, E, \dots, E)$, we get

$$V_0^{-1} = (I + \theta_1 E_0)^{-1} = I - \frac{1}{k}\delta E_0$$

[where $\delta = k\theta_1/(1 + k\theta_1)$] and $\mathbf{1}'(I + \theta_1 E_0)^{-1} = (1 - \delta)\mathbf{1}'$. Hence, the denominator D of (A.2) simplifies to

$$D = \left[\mathbf{y}'\left(I - \frac{1}{k}\delta E_0\right)\mathbf{y} - bk(1 - \delta)\bar{y}_{..}^2\right]^{-(bk-1)/2}$$

$$= \left[\mathbf{y}'\mathbf{y} - k\sum_{j=1}^{b}\bar{y}_{.j}^2 + (1 - \delta)k\sum_{j=1}^{b}\bar{y}_{.j}^2 - bk(1 - \delta)\bar{y}_{..}^2\right]^{-(bk-1)/2},$$

where $\bar{y}_{.j}$ denotes the average of the observations in the jth block. Hence,

$$D = \left[SS_W + (1-\delta)SS_B\right]^{-(bk-1)/2} = D_\delta^{-(bk-1)/2}, \tag{A.3}$$

where $D_\delta = SS_W + (1-\delta)SS_B$. Thus D is a function of SS_B and SS_W only. Using (A.3) and assuming the design to be equireplicate, R in (A.2) is easily expanded as

$$R = 1 + \frac{bk-1}{2}\theta_2\left[\frac{\mathbf{y}'V_0^{-1}FF'V_0^{-1}\mathbf{y}}{D_\delta} + \frac{h(\bar{y}_{..},\theta_1)}{D_\delta}\right] + o(\theta_2), \tag{A.4}$$

uniformly in $\mathbf{y}$, around $\theta_2 = 0$. Here $h(\bar{y}_{..},\theta_1)$ is a function of $\bar{y}_{..}$ and θ_1 only. This follows from the assumption that the design is equireplicate along with the following observations (A.5).

If N, $\mathbf{T}$, $\mathbf{B}$ and r, respectively, denote the incidence matrix, vector of treatment totals, vector of block totals and the replication number for the treatments, then

$$\begin{gathered} F'\operatorname{diag}(\mathbf{1},\mathbf{1},\dots,\mathbf{1}) = N, \qquad F'\mathbf{1} = r\mathbf{1}, \\ F'\mathbf{y} = \mathbf{T} \quad \text{and} \quad \operatorname{diag}(\mathbf{1}',\mathbf{1}',\dots,\mathbf{1}')\mathbf{y} = \mathbf{B}. \end{gathered} \tag{A.5}$$

Using (A.5) and the fact that $V_0^{-1} = I - (1/k)\,\delta E_0$, we also get

$$\begin{aligned} \mathbf{y}'V_0^{-1}FF'V_0^{-1}\mathbf{y} &= \mathbf{T}'\mathbf{T} - \frac{2}{k}\delta\mathbf{T}'N\mathbf{B} + \frac{1}{k^2}\delta^2\mathbf{B}'N'N\mathbf{B} \\ &= \mathbf{Q}_\delta'\mathbf{Q}_\delta, \end{aligned} \tag{A.6}$$

where $\mathbf{Q}_\delta = \mathbf{T} - (1/k)\,\delta N\mathbf{B}$. (A.4) along with (A.6) gives (2.7).

Derivation of (3.5). We note that, in (3.4), the Jacobian of the transformation $\mathbf{y} \to g\mathbf{y}$ is $J = c^{-bk}$. To evaluate R in (3.4), we may assume, in view of invariance, $\mu = 0$ and $\sigma^2 = 1$. Following the derivation of (A.1) above, integrating out α and c immediately gives

numerator of (3.4)

$$\begin{aligned} &= \text{constant} \sum_{\Gamma} \int_0^\infty \int_{-\infty}^\infty \exp\left\{-\frac{c^2}{2}(\Gamma\mathbf{y} - \alpha\mathbf{1} - F\tau)'(I + \theta_1 E_0)^{-1}\right. \\ &\qquad\qquad \left.\times(\Gamma\mathbf{y} - \alpha\mathbf{1} - F\tau)\right\} c^{bk-1}\, d\alpha\, dc \\ &= \text{constant} \sum_{\Gamma} \left[(\Gamma\mathbf{y} - F\tau)'(I + \theta_1 E_0)^{-1}(\Gamma\mathbf{y} - F\tau)\right. \\ &\qquad\qquad \left. - \frac{\left(\mathbf{1}'(I + \theta_1 E_0)^{-1}(\Gamma\mathbf{y} - F\tau)\right)^2}{\mathbf{1}'(I + \theta_1 E_0)^{-1}\mathbf{1}}\right]^{-(bk-1)/2} \\ &\quad \times \int_0^\infty e^{-u/2} u^{(bk-1)/2-1}\, du. \end{aligned} \tag{A.7}$$

The denominator of (3.4) is the above expression with $\tau = \mathbf{0}$. Hence (3.4) simplifies to

$$R = \sum_{\Gamma}\left[(\Gamma\mathbf{y} - F\tau)'(I + \theta_1 E_0)^{-1}(\Gamma\mathbf{y} - F\tau) - \frac{\left(\mathbf{1}'(I + \theta_1 E_0)^{-1}(\Gamma\mathbf{y} - F\tau)\right)^2}{\mathbf{1}'(I + \theta_1 E_0)^{-1}\mathbf{1}}\right]^{-(bk-1)/2} \div \sum_{\Gamma}\left[(\Gamma\mathbf{y})'(I + \theta_1 E_0)^{-1}(\Gamma\mathbf{y}) - \frac{\left(\mathbf{1}'(I + \theta_1 E_0)^{-1}\Gamma\mathbf{y}\right)^2}{\mathbf{1}'(I + \theta_1 E_0)^{-1}\mathbf{1}}\right]^{-(bk-1)/2} \tag{A.8}$$

The crux of the problem now is to simplify R when $\tau'\tau$ is near 0. Toward this end, note that, under H_0, the grand mean $\bar{y}_{..}$, the sum of squares between blocks (SS_B) and the sum of squares within blocks (SS_W) jointly form a set of complete sufficient statistics. As in the derivation of (A.3), we note that the denominator D of (A.8) simplifies to

$$\begin{aligned} D &= k!b[SS_W + (1 - \delta)SS_B]^{-(bk-1)/2} \\ &= k!bD_\delta^{-(bk-1)/2}, \end{aligned} \tag{A.9}$$

where $D_\delta = SS_W + (1 - \delta)SS_B$. Thus D is a function of the complete sufficient statistic. In order to simplify the numerator of R in (A.8), we note that

$$\mathbf{1}'(I + \theta_1 E_0)^{-1}F\tau = (1 - \delta)\mathbf{1}'F\tau = (1 - \delta)r\mathbf{1}'\tau = 0 \tag{A.10}$$

(since $\mathbf{1}'\tau = 0$). Also

$$F'E_0F = N'N \quad \text{and} \quad F'F = rI. \tag{A.11}$$

Using these observations and (A.9), we get

$$R = (k!b)^{-1}\sum_{\Gamma}\left[1 - \frac{1}{D_\delta}2\mathbf{y}'\Gamma'\left(I - \frac{1}{k}\delta E_0\right)F\tau + \frac{1}{D_\delta}\tau'\left(rI - \frac{1}{k}\delta N'N\right)\tau\right]^{-(bk-1)/2} \tag{A.12}$$

Expanding the above expression for R around $\tau = \mathbf{0}$ yields

$$\begin{aligned} R = 1 + \frac{1}{k!b}\sum_{\Gamma}\Bigg[&\frac{bk-1}{2}\frac{1}{D_\delta}2\left\{\mathbf{y}'\Gamma'\left(I - \frac{1}{k}\delta E_0\right)F\tau\right\} \\ &- \frac{bk-1}{2}\frac{1}{D_\delta}\left\{\tau'\left(rI - \frac{1}{k}\delta N'N\right)\tau\right\} + \frac{1}{2!}\frac{bk-1}{2}\frac{bk+1}{2}\frac{1}{D_\delta^2} \\ &\times 4\left\{\tau'F'\left(I - \frac{1}{k}\delta E_0\right)\Gamma\mathbf{y}\mathbf{y}'\Gamma'\left(I - \frac{1}{k}\delta E_0\right)F\tau\right\} \\ &- \frac{1}{2!}2\frac{bk-1}{2}\frac{bk+1}{2}\frac{1}{D_\delta^2}2\left\{\mathbf{y}'\left(I - \frac{1}{k}\delta E_0\right)F\tau\right\} \\ &\times\left\{\tau'\left(rI - \frac{1}{k}\delta N'N\right)\tau\right\} + \text{remainder term}\Bigg]. \end{aligned} \tag{A.13}$$

Clearly the remainder term is $o(\tau'\tau)$ uniformly in $\mathbf{y}$. We now simplify the various terms in (A.13). We first show that the first and the fourth terms within the square brackets simplify to 0. This follows from the fact that $\Sigma_\Gamma \Gamma = (k-1)!\mathbf{1}\mathbf{1}'$ and hence $\Sigma_\Gamma y'\Gamma'(I - (1/k)\delta E_0)F\tau = (k-1)!\mathbf{y}'\mathbf{1}(1-\delta)r\mathbf{1}'\tau = 0$, since $\mathbf{1}'\tau = 0$. To simplify the most crucial third term within the square brackets in (A.13), we denote by Y the $bk \times k!b$ matrix whose columns are $\Gamma\mathbf{y}$ for $\Gamma \in \mathscr{G}$. Then

$$\sum_\Gamma \tau' F'\left(I - \frac{1}{k}\delta E_0\right)\Gamma\mathbf{y}\mathbf{y}'\Gamma'\left(I - \frac{1}{k}\delta E_0\right)F\tau = \tau'F'\left(I - \frac{1}{k}\delta E_0\right)YY'\left(I - \frac{1}{k}\delta E_0\right)F\tau. \tag{A.14}$$

To further simplify the right-hand side (RHS) expression in (A.14), we need to observe the following properties of the $v \times k!b$ matrix $F'Y$:

(i) each column of $F'Y$ is a permutation of $\mathbf{T}$, the vector of treatment totals and

(ii) each row of $F'Y$ has $k!b$ elements and hence each T_i occurs in a row $k!b/v$ times, i.e., $(k-1)!r$ times. Hence any diagonal element of $F'YY'F$ is $(k-1)!r\Sigma_{i=1}^v T_i^2$. Similarly any off-diagonal element of $F'YY'F$ is $((k-1)!r/(v-1))\Sigma_{i\neq i'}T_iT_{i'} = (k-2)!\lambda\{(\Sigma_{i=1}^v T_i)^2 - \Sigma_{i=1}^v T_i^2\}$.

Thus

$$\begin{aligned} F'YY'F = {} & \{(k-1)!r + (k-2)!\lambda\}\left(\sum_{i=1}^v T_i^2\right)I_v \\ & -(k-2)!\lambda\left(\sum_{i=1}^v T_i\right)^2 I_v \\ & +(k-2)!\lambda\left\{\left(\sum_{i=1}^v T_i\right)^2 - \sum_{i=1}^v T_i^2\right\}\mathbf{1}_v\mathbf{1}_v'. \end{aligned} \tag{A.15}$$

Consequently

$$\begin{aligned} \tau'F'YY'F\tau = {} & \{(k-1)!r + (k-2)!\lambda\}\left(\sum_{i=1}^v T_i^2\right)(\tau'\tau) \\ & -(k-2)!\lambda(bk)^2\bar{y}_{..}^2(\tau'\tau). \end{aligned} \tag{A.16}$$

Here we have used the fact that $\mathbf{1}'\tau = 0$. To simplify the other terms in the RHS of (A.14), note that

$$\begin{aligned} F'E_0Y &= F'\mathrm{diag}(\mathbf{1},\dots,\mathbf{1})\mathrm{diag}(\mathbf{1}',\dots,\mathbf{1}')Y \\ &= N\mathrm{diag}(\mathbf{1}',\dots,\mathbf{1}')Y. \end{aligned} \tag{A.17}$$

Again, clearly each column of $\mathrm{diag}(\mathbf{1}',\dots,\mathbf{1}')Y$ is a permutation of $\mathbf{B} = (B_1, B_2, \dots, B_b)'$, the vector of block totals and each row of $\mathrm{diag}(\mathbf{1}',\dots,\mathbf{1}')Y$ has every B_j repeated $k!b/b = k!$ times. Hence using (A.17) and the observations (i) and (ii) made before, $F'E_0YY'F$ has equal diagonal elements and equal off-diagonal elements given, respectively, by $(k-1)!r\mathbf{T}'N\mathbf{B}$ and

$$
\begin{aligned}
&(k-2)!\lambda \sum_{i\neq i'} T_i \sum_{j=1}^{b} n_{i'j}B_j \\
&\quad = (k-2)!\lambda\left\{\left(\sum_{i=1}^{v} T_i\right)\left(\sum_{i=1}^{v}\sum_{j=1}^{b} n_{ij}B_j\right) - \mathbf{T}'N\mathbf{B}\right\} \\
&\quad = (k-2)!\lambda\left\{(bk)^2 k\bar{y}_{..}^2 - \mathbf{T}'N\mathbf{B}\right\}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\tau'\{F'E_0YY'F + F'YY'E_0F\}\tau \\
&\quad = 2\{(k-1)!r + (k-2)!\lambda\}(\mathbf{T}'N\mathbf{B})(\tau'\tau) \\
&\qquad -2(k-2)!\lambda(bk)^2 k\bar{y}_{..}^2(\tau'\tau).
\end{aligned}
\tag{A.18}
$$

A similar argument yields

$$
\begin{aligned}
\tau'F'E_0YY'E_0F\tau &= \{(k-1)!r + (k-2)!\lambda\}(\mathbf{B}'N'N\mathbf{B})(\tau'\tau) \\
&\quad -(k-2)!\lambda(bk)^2k^2\bar{y}_{..}^2(\tau'\tau) \\
&= (k-2)!\lambda\left[v(\mathbf{B}'N'N\mathbf{B})(\tau'\tau) - (bk)^2k^2\bar{y}_{..}^2(\tau'\tau)\right],
\end{aligned}
\tag{A.19}
$$

since $(k-1)!r + (k-2)!\lambda = (k-2)!\lambda v$. Hence using (A.16), (A.18) and (A.19), the RHS of (A.14) simplifies to

$$
\begin{aligned}
&(k-2)!\lambda v\left\{\mathbf{T}'\mathbf{T} - 2\frac{1}{k}\delta\mathbf{T}'N\mathbf{B} + \frac{1}{k^2}\delta^2\mathbf{B}'N'N\mathbf{B} + h(\bar{y}_{..})\right\}(\tau'\tau) \\
&\quad = (k-2)!\lambda v(\mathbf{Q}_\delta'\mathbf{Q}_\delta + h(\bar{y}_{..}))(\tau'\tau),
\end{aligned}
\tag{A.20}
$$

where $\mathbf{Q}_\delta = \mathbf{T} - (1/k)\delta N\mathbf{B}$. Thus (A.13) can be written as

$$
\begin{aligned}
R &= 1 + \frac{a_0}{D_\delta^2}(\mathbf{Q}_\delta'\mathbf{Q}_\delta + h(\bar{y}_{..}))(\tau'\tau) \\
&\quad - \frac{bk-1}{2}\frac{1}{D_\delta}\tau'\left(rI - \frac{1}{k}\delta N'N\right)\tau + o(\tau'\tau),
\end{aligned}
\tag{A.21}
$$

where a_0 is a positive constant.

Acknowledgments. Our sincere thanks are due an Associate Editor and two referees for their constructive suggestions which resulted in the elimination of an error and improved presentation.

REFERENCES

DAS, R. and SINHA, B. K. (1987). Robust optimum invariant unbiased tests for variance components. In *Proc. of the Second International Tampere Conference in Statistics* (T. Pukkila and S. Puntanen, eds.) 317–342. University of Tampere, Tampere, Finland.

FERGUSON, T. S. (1967). *Mathematical Statistics, A Decision Theoretic Approach.* Academic, New York.

HERBACH, L. H. (1959). Properties of Model II-type analysis of variance tests, A: Optimum nature of the F-test for Model II in the balanced case. *Ann. Math. Statist.* **30** 939–959.

KARIYA, T. (1981). Robustness of multivariate tests. *Ann. Statist.* **9** 1267–1275.

KARIYA, T. and SINHA, B. K. (1985). Nonnull and optimality robustness of some tests. *Ann. Statist.* **13** 1182–1197.

KHURI, A. I. and LITTELL, R. C. (1987). Exact tests for the main effects variance components in an unbalanced random two-way model. *Biometrics* **43** 545–560.

KLEFFE, J. and SEIFERT, B. (1988). On the role of MINQUE in testing hypotheses under mixed linear models. *Comm. Statist. A—Theory Methods* **17** 1287–1309.

LEHMANN, E. L. (1959). *Testing Statistical Hypotheses*. Wiley, New York.

MATHEW, T. and SINHA, B. K. (1988). Optimum tests for fixed effects and variance components in balanced models. *J. Amer. Statist. Assoc.* **83** 133–135.

MONTGOMERY, D. C. (1984). *Design and Analysis of Experiments*. Wiley, New York.

PINCUS, R. (1977). On tests in variance components models. *Math. Operationsforsch. Statist. Ser. Statist.* **8** 251–255.

SEELY, J. F. (1977). Minimal sufficient statistics and completeness for multivariate normal families. *Sankhyā Ser. A* **39** 170–185.

SEELY, J. F. and EL-BASSIOUNI, Y. (1983). Applying Wald's variance component test. *Ann. Statist.* **11** 197–201.

SEIFERT, B. (1978). A note on the UMPU character of a test for the mean in balanced randomized nested classification. *Math. Operationsforsch. Statist. Ser. Statist.* **9** 185–189.

SEIFERT, B. (1979). Optimal testing for fixed effects in general balanced mixed classification models. *Math. Operationsforsch. Statist. Ser. Statist.* **10** 237–256.

SEIFERT, B. (1981). Explicit formulae of exact tests in mixed balanced ANOVA models. *Biometrical J.* **23** 535–550.

SEIFERT, B. (1985). Estimation and test of variance components using the MINQUE method. *Statistics* **16** 621–635.

SINHA, B. K. (1982). On complete classes of experiments for certain invariant problems of linear inference. *J. Statist. Plann. Inference* **7** 171–180.

SPJØTVOLL, E. (1967). Optimum invariant tests in unbalanced variance components models. *Ann. Math. Statist.* **38** 422–429.

THOMPSON, W. A. (1955a). The ratio of variances in a variance components model. *Ann. Math. Statist.* **26** 325–329.

THOMPSON, W. A. (1955b). On the ratio of variances in the mixed incomplete block model. *Ann. Math. Statist.* **26** 721–733.

THOMSEN, I. (1975). Testing hypotheses in unbalanced variance components models for two-way layouts. *Ann. Statist.* **3** 257–265.

WIJSMAN, R. A. (1967). Cross-sections of orbits and their application to densities of maximal invariants. *Proc. Fifth. Berkeley Symp. Math. Statist. Prob.* **1** 389–400. Univ. California Press.

DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MARYLAND
BALTIMORE COUNTY CAMPUS
CATONSVILLE, MARYLAND 21228