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ASYMPTOTIC PROPERTIES OF NEYMAN–PEARSON TESTS FOR INFINITE KULLBACK–LEIBLER INFORMATION

BY ARNOLD JANSSEN

Universität Siegen

In the present paper we will improve the results concerning the rate of convergence of the error of second kind of the Neyman–Pearson test if the Kullback–Leibler information $K(P_0, P_1)$ is infinite. It is pointed out that in certain cases the sequence $\exp(-q_{\alpha,n})$ is the correct rate of convergence if $-q_{\alpha,n}$ denotes the logarithm of the critical value of the Neyman–Pearson test of level α and sample size n. Therefore we generalize the classical results of Stein, Chernoff, and Rao which deal with the error probability of second kind and state that $q_{\alpha,n} \sim nK(P_0, P_1)$ if the Kullback–Leibler information is finite. Moreover the relation between $q_{\alpha,n}$ and the local behavior of the Laplace transform of the log-likelihood distribution with respect to the hypothesis is studied. The results can be applied to one-sided test problems for exponential families if the hypothesis consists of a single point. In this case it may happen that $q_{\alpha,n}$ is of the order $n^{1/p}$ for some p, $0 < p < 1$.

1. Introduction. Let $E^n = (\Omega^n, \mathscr{A}^n, (P_0^n, P_1^n))$ be an n-fold binary experiment and suppose that $P_1^n(A) = \int_A (dP_1^n/dP_0^n)\, dP_0^n + P_1^n(A \cap N_n)$, $A \in \mathscr{A}^n$, is the Lebesgue decomposition of the nth product measure P_1^n with respect to P_0^n for some $N_n \in \mathscr{A}^n$, $n \in \mathbb{N}$. Suppose that the P_0^n density dP_1^n/dP_0^n of the absolutely continuous part of P_1^n is defined to be equal ∞ on N_n. Then we consider the Neyman–Pearson test $\varphi_{\alpha,n}$ of level $\alpha \in (0,1)$ for the test problem $H = \{P_0^n\}$ against $K = \{P_1^n\}$,

$$\varphi_{\alpha,n} = \begin{cases} 1 & \text{if } dP_1^n/dP_0^n > c_{\alpha,n} \\ \gamma_{\alpha,n} & \text{if } dP_1^n/dP_0^n = c_{\alpha,n}, \\ 0 & \text{if } dP_1^n/dP_0^n < c_{\alpha,n} \end{cases}$$

satisfying $E_{P_0^n}\varphi_{\alpha,n} = \alpha$. It is well known that the error probabilities of second kind $E_{P_1^n}(1 - \varphi_{\alpha,n})$ and the critical values $c_{\alpha,n}$ satisfy

$$\lim_{n\to\infty} \left[E_{P_1^n}(1 - \varphi_{\alpha,n})\right]^{1/n} = \lim_{n\to\infty} c_{\alpha,n}^{1/n} = \exp(-K(P_0, P_1)) \tag{1.1}$$

if $K(P_0, P_1) = \int \log(dP_0/dP_1)\, dP_0$ denotes the Kullback–Leibler information (compare with Chernoff (1956), who referred to Charles Stein; Rao (1962); Krafft and Plachky (1970)). Suppose that log is continuously extended on $[0, \infty]$. Note that (1.1) contains only little information if $K(P_0, P_1)$ is infinite. Therefore we are interested in the correct rate of convergence in the general case. Observe that for example the case $K(P_0, P_1) = \infty$ appears in connection with one-sided tests in exponential families and the local structure of exponential families; cf. Janssen

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(1986). To fix the idea of this paper let us first suppose that $K(P_0, P_1)$ is finite. If

$$q_{\alpha,n} := -\log c_{\alpha,n} \tag{1.2}$$

then (1.1) implies

$$\lim_{n\to\infty}\left[E_{P_1^n}(1-\varphi_{\alpha,n})\right]^{1/q_{\alpha,n}} = \exp(-1). \tag{1.3}$$

Since (1.3) is independent of $K(P_0, P_1)$ we ask whether (1.3) holds in a more general situation. It turns out that this result can be proved for a large class of interesting experiments having infinite Kullback–Leibler information. Thus $1/q_{\alpha,n}$ is the correct speed of convergence. Note that

$$n/q_{\alpha,n} \to 0 \quad \text{iff } K(P_0, P_1) = \infty. \tag{1.4}$$

On the other hand it is pointed out that there is some connection between $q_{\alpha,n}$ and the Laplace transform

$$\omega(t) = \int \exp(t \log dP_1/dP_0)\, dP_0 \tag{1.5}$$

of $\mathscr{L}(\log dP_1/dP_0|P_0)$, which exists for $t \in [0,1]$. Note that

$$d^+\log\omega(t)/dt|_{t=0} = -K(P_0, P_1) \tag{1.6}$$

and

$$n\log\omega(1/q_{\alpha,n}) \to -1 \quad \text{for } n\to\infty \text{ if } K(P_0, P_1) < \infty. \tag{1.7}$$

In order to prove the results we apply a convergence theorem for Laplace transforms. Only partial results are obtained by theorems for large deviations.

2. Preliminaries. In connection with the tests $\varphi_{\alpha,n}$ the behavior of the log-likelihood distributions of product experiments is used. Therefore we first recall some facts for binary experiments $E = (\Omega, \mathscr{A}, (P_0, P_1))$. Let ν_0 be the log-likelihood distribution

$$\nu_0 := \mathscr{L}(\log dP_1/dP_0|P_0), \tag{2.1}$$

where $\mathscr{L}(Y|Q)$ denotes the distribution of a random variable Y with respect to Q. If μ is a finite measure on $[-\infty, \infty)$ then

$$\omega_\mu(t) = \int \exp(ty)\, d\mu(y) \tag{2.2}$$

defines the Laplace transform of μ where exp is continuously extended on $[-\infty, \infty]$. Let Y be a random variable. Then ω_Y denotes the Laplace transform of its distribution. Returning to E let us remark that ω_{ν_0} is bounded by 1 on $[0,1]$. Moreover

$$\nu_0(\{-\infty\}) = \lim_{\substack{t\to 0\\ t>0}}\left(1-\omega_{\nu_0}(t)\right) \tag{2.3}$$

and

$$\mathscr{L}(\log dP_1/dP_0|P_1)(A) = \int_A \exp(x)\, d\nu_0(x) + \left(1-\omega_{\nu_0}(1)\right)\varepsilon_\infty(A) \tag{2.4}$$

for each Borel subset A of $(-\infty, \infty]$. In addition

$$K(P_0, P_1) = -\int x \, d\nu_0(x) \quad \text{and} \quad 0 \le K(P_0, P_1) \le \infty. \tag{2.5}$$

Suppose that $E = \{P_0, P_1\}$ and $F = \{Q_0, Q_1\}$ are two binary experiments. Then

$$\begin{aligned} &\mathscr{L}(\log dP_i \otimes Q_i/dP_j \otimes Q_j | P_j \otimes Q_j) \\ &\quad = \mathscr{L}(\log dP_i/dP_j | P_j) * \mathscr{L}(\log dQ_i/dQ_j | Q_j), \end{aligned} \tag{2.6}$$

where $*$ denotes the convolution of measures on the topological semigroup $([-\infty, \infty), +)$ equipped with the usual topology; cf. Janssen (1985c), (9.4) and (9.21). Let P^n denote the n-fold product measure of P, ε_a the Dirac measure defined by a. Let E_P be the expectation with respect to P and suppose that ν^{*n} denotes the nth convolution product of ν. Two positive functions f, g are equivalent for $x \to 0$ or $\pm\infty$, $g(x) \sim f(x)$, if the ratio tends to 1.

In the sequel a continuity theorem for Laplace transforms is needed.

LEMMA 2.1. *Suppose that μ_n, $n \ge 0$, is a sequence of probability measures on $[-\infty, \infty)$ such that $\sup_{n \ge 0} \omega_{\mu_n}(1) \le K$ for some $K > 0$. Then the following statements are equivalent*:

$$\mu_n \to \mu_0 \quad \text{weakly}, \tag{2.7}$$

$$\omega_{\mu_n}(y) \to \omega_{\mu_0}(y) \quad \text{for all } y \in (0,1). \tag{2.8}$$

PROOF. Put $\rho_n = \mu_n * \varepsilon_{-\log K}$ and define $\rho'_n = \mathscr{L}(\exp(\cdot)|\rho_n)$. Then $\int x \, d\rho'_n(x) \le 1$ follows and $\omega_{\mu_n}(y) = \int x^y \, d\rho'_n(x)$ is the Mellin transform of ρ'_n. The continuity theorem for Mellin transforms shows that (2.8) is equivalent to the weak convergence of ρ'_n to ρ'_0; cf. Strasser (1985), Theorem (5.16). Thus the lemma is proved. □

Note that Lemma 2.1 does not hold in general if $\omega_{\mu_n}(1)$ is unbounded.

3. Main results. First of all we study the connection between the Laplace transform of ν_0 (2.1) and the logarithm $q_{\alpha, n} = -\log c_{\alpha, n}$ of the critical value $c_{\alpha, n}$ in order to generalize (1.6). Therefore we always assume that

$$P_0 \ne P_1 \quad \text{and} \quad P_0 \ll P_1. \tag{3.1}$$

If the second condition is not fulfilled then $c_{\alpha, n} = 0$ finally holds. Note that (3.1) implies that $\nu_0 \ne \varepsilon_0$ and ν_0 is concentrated on $\mathbb{R}$. Put

$$\varphi = \log \omega_{\nu_0}. \tag{3.2}$$

Then φ is a convex function such that $\varphi(t) < 0$ on $(0,1)$ and $\lim_{t \to 0} \varphi(t) = 0$. Thus for $n \ge (1 - \log \alpha)(\sup_{t \in (0,1)} |\varphi(t)|)^{-1} =: \gamma$ there exists an increasing sequence of reals $(k_{\alpha, n})_n$ such that

$$\varphi(1/k_{\alpha, n}) = -(1 - \log \alpha)/n. \tag{3.3}$$

LEMMA 3.1.

(a) $$k_{\alpha, n} \le q_{\alpha, n} \quad \text{for } n \ge \gamma;$$

(b) $$\log \alpha - 1 \le \liminf_{n \to \infty} n\varphi(1/q_{\alpha, n}) \le \limsup_{n \to \infty} n\varphi(1/q_{\alpha, n}) < 0.$$

PROOF. (a) Let $Y_1, \dots, Y_n$: $(\Omega, \mathscr{A}, P) \to \mathbb{R}$ be independent random variables with common distribution ν_0. For $u \in \mathbb{R}$ we apply the following well-known inequality of Chernoff (1952):

$$P(\{Y_1 + \cdots + Y_n \ge nu\}) \le P(\{\exp(t(Y_1 + \cdots + Y_n - nu)) \ge 1\}) \le \left(\exp(-tu)\omega_{\nu_0}(t)\right)^n \quad \text{for } t \ge 0.$$

Hence

$$\log \alpha \le \log P(\{Y_1 + \cdots + Y_n \ge -q_{\alpha, n}\}) \le n(\varphi(t) + tq_{\alpha, n}/n).$$

Inserting $t = 1/k_{\alpha, n}$ then the result follows.

In order to prove (b) we first remark that

$$\lim_{n \to \infty} n\varphi(1/k_{\alpha, n}) \le \liminf_{n \to \infty} n\varphi(1/q_{\alpha, n})$$

by (a) since φ is a convex function. Now assume that $\lim_{k \to \infty} n_k \varphi(t/q_{\alpha, n_k}) = 0$ for $t = 1$ and a subsequence n_k. Then the limit is equal to 0 for all $t \in (0, 1)$. Let $(Y_n)_{n \in \mathbb{N}}$ be a sequence of independent copies of random variables having the distribution ν_0. Then Lemma 2.1 implies that

$$q_{\alpha, n_k}^{-1} \sum_{j=1}^{n_k} Y_j \tag{3.4}$$

tends in distribution to 0. This yields the desired contradiction since

$$\alpha \ge P\left(\left\{ \sum_{j=1}^{n_k} Y_j > -\tfrac{1}{2} q_{\alpha, n_k} \right\}\right). \square \tag{3.5}$$

It should be remarked that (a) also follows from statement (14) of Krafft and Plachky (1970). The next theorem is well known for experiments having finite Kullback–Leibler information $K(P_0, P_1)$ without further restrictions concerning F.

THEOREM 3.1. *Let F be the distribution function of $\nu_0 = \mathscr{L}(\log dP_1/dP_0 | P_0)$. If F is positive and satisfies*

$$\limsup_{x \to -\infty} F(\lambda_0 x)/F(x) < 1 \quad \textit{for some } \lambda_0 > 1 \tag{3.6}$$

then

$$\limsup_{n \to \infty} q_{\alpha, n}/k_{\alpha, n} < \infty, \tag{3.7}$$

$$\left[E_{P_1^n}(1 - \varphi_{\alpha, n})\right]^{1/q_{\alpha, n}} \to \exp(-1). \tag{3.8}$$

$$\textit{If} \quad 0 < \alpha_1 \le \alpha_2 < 1 \quad \textit{then} \quad \liminf_{n \to \infty} q_{\alpha_1, n}/q_{\alpha_2, n} > 0. \tag{3.9}$$

PROOF. Note that (3.7) and (3.9) follow from (1.6), (1.7), and (1.1) in the case $K(P_0, P_1) < \infty$. Therefore we may assume that $K(P_0, P_1) = \infty$. Then by (2.5)

$$\int x\, d\nu_0(x) = -\infty. \tag{3.10}$$

Let $a(\lambda)$ denote $\limsup_{x \to -\infty} F(\lambda x)/F(x)$. Then $a(\lambda_0^n) \le a(\lambda_0)^n$ and

$$\lim_{\lambda \to \infty} a(\lambda) = 0 \tag{3.11}$$

since F is increasing. First of all we show that (3.6) implies

$$\lim_{s \to 0+} \limsup_{t \to 0+} \varphi(st)/\varphi(t) = 0. \tag{3.12}$$

Therefore we put $\omega = \omega_{\nu_0}$ and fix $1 > s > 0$. Since $\varphi(t) = \log \omega(t) = \omega(t) - 1 + o(\omega(t) - 1)$, we remark

$$\limsup_{t \to 0+} \frac{\varphi(st)}{\varphi(t)} = \limsup_{t \to 0+} \frac{\omega(st) - 1}{\omega(t) - 1}. \tag{3.13}$$

Integration by parts (Feller (1971), page 150) yields

$$1 - \omega(t) = t\int_{-\infty}^{0} F(x)\exp(tx)\, dx - tg(t), \tag{3.14}$$

where

$$g(t) = \int_0^{\infty} (1 - F(x))\exp(tx)\, dx$$

is continuous for $t \to 0+$, $g(0) < \infty$. Thus

$$\frac{1 - \omega(st)}{1 - \omega(t)} = \frac{\int_{-\infty}^{0} F(x/s)\exp(tx)\, dx - sg(st)}{\int_{-\infty}^{0} F(x)\exp(tx)\, dx - g(t)}. \tag{3.15}$$

Note that the denominator of the right-hand side tends to infinity for $t \to 0+$ because of (3.10); see Feller (1971), page 151. Therefore it is sufficient to prove

$$\limsup_{t \to 0+} \frac{\int_{-\infty}^{0} F(x/s)\exp(tx)\, dx}{\int_{-\infty}^{0} F(x)\exp(tx)\, dx} \le a\left(\frac{1}{s}\right). \tag{3.16}$$

Choose $x_0 < 0$ such that $F(x/s) \le (a(1/s) + \varepsilon)F(x)$ for $x \le x_0$. Then

$$\int_{-\infty}^{0} F\left(\frac{x}{s}\right)\exp(tx)\, dx \le |x_0| + \left(a\left(\frac{1}{s}\right) + \varepsilon\right)\int_{-\infty}^{0} F(x)\exp(tx)\, dx.$$

Hence (3.16) follows since the denominator tends to infinity for $t \to 0+$. In the sequel we omit the index α if possible and write c_n, φ_n, q_n, k_n. Now it will be pointed out that (3.12) implies (3.7).

Suppose that there exists a subsequence such that $\lim_{j \to \infty} q_{n_j}/k_{n_j} = \infty$. Then

$$n_j \varphi\left(t/q_{n_j}\right) = (-1 + \log \alpha) \frac{\varphi\left((1/k_{n_j})(tk_{n_j}/q_{n_j})\right)}{\varphi\left(1/k_{n_j}\right)} \tag{3.17}$$

if we take (3.3) into account and observe that φ is decreasing in a neighbourhood of 0.

In view of (3.4) and (3.5) statement (3.17) is impossible. In order to prove (3.8) we remark that

$$\int (1-\varphi_n)\,dP_1^n \le \int_{\{x\le -q_n\}} \exp(x)\,d\nu_0^{*n}(x) \le \exp(-q_n) \tag{3.18}$$

if we observe (2.6), (2.4), and note that the absolutely continuous part of $\mathscr{L}(\log dP_1^n/dP_0^n|P_1^n)$ has the ν_0^{*n} density $\exp(x)$. Next we prove the converse inequality. Let $Y_i\colon (\Omega, \mathscr{A}, P) \to \mathbb{R}$ be a sequence of independent random variables having common distribution ν_0. Then for $d > 1$ and $W_n = \Sigma_{i=1}^n Y_i$

$$\begin{aligned} \int (1-\varphi_n)\,dP_1^n \ge \int_{\{-q_n > W_n \ge -dq_n\}} &\exp(W_n)\,dP \\ &+ (1-\gamma_n)\exp(-q_n)P(\{W_n = q_n\}). \end{aligned} \tag{3.19}$$

Suppose that n_j is a subsequence such that

$$\left[\int (1-\varphi_{n_j})\,dP_1^{n_j}\right]^{1/q_{n_j}} \to a \tag{3.20}$$

is convergent. Then we prove that either

$$a = \exp(-1) \quad \text{or} \quad \liminf_{j\to\infty} P\Big(\big\{q_{n_j}^{-1}W_{n_j} \in (-d,-1)\big\}\Big) = \delta > 0. \tag{3.21}$$

Therefore we introduce the distributions $\mu_n = \mathscr{L}(q_n^{-1}W_n|P)$. Since $\log \omega_{\mu_n}(1) = n\varphi(q_n^{-1})$ is uniformly bounded (compare with Lemma 3.1) we note that $(\mu_n)_n$ is a tight sequence of probability measures on $[-\infty,\infty)$. Therefore each subsequence has a weak accumulation point in the set of probability measures on $[-\infty,\infty)$.

In order to prove (3.21) let us assume that

$$P\Big(\big\{q_{n_i}^{-1}W_{n_i} \in (-d,-1)\big\}\Big) \to 0 \tag{3.22}$$

for some subsequence of n_i. Passing to a further subsequence we may assume that

$$\mu_{n_i} \to Q \tag{3.23}$$

tends weakly to some probability measure Q on $[-\infty,\infty)$.

Let us first assume that Q is a Dirac measure. Since -1 is a $(1-\alpha)$th quantile of μ_n we conclude $Q = \varepsilon_{-1}$. Together with (3.22) we note that $\mu_{n_i}((-\infty,-1)) \to 0$. Thus $(1-\gamma_{n_i})P(\{W_{n_i} = -q_{n_i}\}) \to 1-\alpha$ since $(1-\alpha) = \mu_n((-\infty,-1) + (1-\gamma_n)\mu_n(\{-1\})$. Hence

$$\Big[(1-\gamma_{n_i})\exp(-q_{n_i})P\big(\{W_{n_i} = -q_{n_i}\}\big)\Big]^{1/q_{n_i}} \to \exp(-1) \tag{3.24}$$

and (3.19) proves (3.21).

Suppose in the second case that Q is not a Dirac measure. Then we first show that Q is concentrated on $\mathbb{R}$. Therefore we apply Lemma 2.1 which proves

$$\log \omega_Q(t) = \lim_{i\to\infty} n_i\varphi(t/q_{n_i}) \quad \text{for } t \in (0,1).$$

Using the technique of (3.17) we conclude that

$$\lim_{t \to 0+} \log \omega_Q(t) = 0 \tag{3.25}$$

by (3.12). The assertion follows from (2.3).

Next we shall prove that

$$(-d, -1) \quad \text{is contained in the support of } Q. \tag{3.26}$$

Since Q is the limit distribution of an infinitesimal triangular (or null) array it is infinitely divisible. If Q has a normal factor then the claim is obvious. Otherwise the Lévy measure η of Q is nontrivial and

$$n_i F(q_{n_i} x) \to \eta((-\infty, x]), \qquad x < 0, \tag{3.27}$$

$$n_i\big(1 - F(q_{n_i} x)\big) \to \eta([x, \infty)), \qquad x > 0, \tag{3.28}$$

for all continuity points x of the distribution function of η; see Feller (1971), page 585. Let us prove that η vanishes on $(0, \infty)$. Therefore consider $\int_0^\infty \exp(x)\, d\nu_0(x) < \infty$ which implies $n(1 - F(nx)) \to 0$ for all $x > 0$. By (1.4) the inequality $1 - F(q_n x) \le 1 - F(nx)$ follows for sufficiently large n. Moreover let us show that

$$\eta((-\varepsilon, 0)) > 0 \quad \text{for each } \varepsilon > 0. \tag{3.29}$$

Since Q is not degenerate there exists a continuity point $x_0 \in (-\varepsilon, 0)$ such that $\eta((-\infty, x_0]) > 0$. Choose $\lambda \ge \lambda_0$ such that $\lambda^{-1} x_0$ is a continuity point of $x \to \eta((-\infty, x])$. Then by (3.6) and (3.27)

$$1 > \lim_{i \to \infty} \frac{F(q_{n_i} x_0)}{F\big(q_{n_i} \lambda^{-1} x_0\big)} = \frac{\eta((-\infty, x_0])}{\eta\big((-\infty, \lambda^{-1} x_0]\big)}. \tag{3.30}$$

It is well known that the support of Q is equal to $\mathbb{R}$ or $(-\infty, c]$ for some $c \in \mathbb{R}$ in view of (3.29); compare with Tucker (1975). Note that the case $[c, \infty)$ can be excluded since Q contains a factor which is a compound Poisson distribution having the support $(-\infty, 0]$. Since -1 is a $(1 - \alpha)$th quantile of μ_n the inequality $c \ge -1$ holds and (3.26) follows. The Portmanteau theorem yields

$$\liminf_{i \to \infty} P\Big(\big\{q_{n_i}^{-1} W_{n_i} \in (-d, -1)\big\}\Big) \ge Q((-d, -1)) \tag{3.31}$$

and (3.21) is proved. Returning to (3.19) we compute

$$\begin{aligned} &\liminf_{i \to \infty} \left[\int \big(1 - \varphi_{n_i}\big)\, dP_1^{n_i}\right]^{1/q_{n_i}} \\ &\quad \ge \lim_{i \to \infty} \Big[\exp(-d q_{n_i}) P\Big(\big\{q_{n_i}^{-1} W_{n_i}(-d, -1)\big\}\Big)\Big]^{1/q_{n_i}} \\ &\quad = \exp(-d). \end{aligned} \tag{3.32}$$

If d tends to 1 then the result follows since all accumulation points a of (3.20) are equal to $\exp(-1)$.

The proof of (3.9): Suppose that $\liminf q_{\alpha_1, n}/q_{\alpha_2, n} = 0$. Passing to subsequences there are probability measures Q_1, Q_2 on $\mathbb{R}$ such that $\mu_{\alpha_j, n_i} :=$

$\mathscr{L}(q_{\alpha_j, n_i}^{-1} W_{n_i} | P) \to Q_j$ weakly for $j = 1, 2$ and

$$\lim_{i \to \infty} q_{\alpha_1, n_i} / q_{\alpha_2, n_i} = 0. \tag{3.33}$$

Note that -1 is a $(1 - \alpha_j)$th quantile of μ_{α_j, n_i}. But (3.33) implies that $(q_{\alpha_1, n_i}/q_{\alpha_2, n_i})(W_{n_i}/q_{\alpha_1, n_i})$ tends to 0 in distribution, which is impossible. □

EXAMPLE. Suppose that the distribution function F satisfies the condition

$$F(x) \sim |x|^{-\rho} L(x) B(x) \quad \text{for } x \to -\infty, \tag{3.34}$$

where the index ρ is positive, B is bounded away from zero and infinity:

$$0 < m \le B(x) \le M \quad \text{for } x \le x_0. \tag{3.35}$$

Let L be a positive function varying slowly at $-\infty$, i.e., $L(x\lambda)/L(x) \to 1$ if $x \to -\infty$ for all $\lambda > 0$. Then (3.6) is satisfied. The condition (3.34) implies that $K(P_0, P_1) = \infty$ if $\rho < 1$ and $K(P_0, P_1) < \infty$ for $\rho > 1$ (use integration by parts).

It should be remarked that (3.6) is not satisfied if F is slowly varying at $-\infty$. In this case the rate of convergence of $q_{\alpha, n}$ can be computed and (3.9) is no longer true.

THEOREM 3.2. *Suppose that the distribution function F of ν_0 is positive and slowly varying at $-\infty$. Then*

$$nF(-q_{\alpha, n}) \to -\log \alpha, \qquad n\varphi(q_{\alpha, n}^{-1}) \to -\log \alpha, \tag{3.36}$$

$$\lim_{n \to \infty} \frac{q_{\alpha_1, n}}{q_{\alpha_2, n}} = 0 \quad \textit{if } 0 < \alpha_1 < \alpha_2 < 1. \tag{3.37}$$

PROOF. We use the notation of the proof of the preceding theorem. Integration by parts yields

$$\omega(t) = \int_{-\infty}^{\infty} (1 - F(x)) t \exp(tx)\, dx \tag{3.38}$$

and

$$t^{-1}(1 - \omega(t)) \sim \int_{-\infty}^{0} F(x) \exp(tx)\, dx - \int_0^{\infty} (1 - F(x)) \exp(tx)\, dx. \tag{3.39}$$

The second term on the right-hand side is continuous for $t \to 0+$. Moreover (5.22) of Feller (1971), page 447, can be applied to $F(0-)^{-1} \nu_{0|(-\infty, 0)}$. Thus

$$-\varphi(t) = -\log \omega(t) \sim 1 - \omega(t) \sim F(-1/t) \quad \text{for } t \to 0+. \tag{3.40}$$

According to Lemma 3.1 let us choose a subsequence n_i such that $n_i \varphi(q_{n_i}^{-1}) \to c < 0$ is convergent. By (3.40)

$$\lim_{i \to \infty} \frac{n_i \log \omega(s q_{n_i}^{-1})}{n_i \log \omega(q_{n_i}^{-1})} = \lim_{i \to \infty} \frac{F(-s^{-1} q_{n_i})}{F(-q_{n_i})} = 1 \tag{3.41}$$

for each $s \in (0,1)$. Thus

$$\lim_{i\to\infty} n_i\varphi\left(sq_{n_i}^{-1}\right) = c. \tag{3.42}$$

Let $(X_j)_{j\in\mathbb{N}}$ be a sequence of i.i.d. random variables having the distribution ν_0. Then $n_i\varphi(sq_{n_i}^{-1})$ is the logarithm of the Laplace transform of $Y_{n_i} = q_{n_i}^{-1}\sum_{j=1}^{n_i} X_j$. By Lemma 2.1 Y_{n_i} tends in distribution to Q where $\omega_Q(s) = \exp(c)$ for $s \in (0,1)$. Hence

$$Q = (1-a)\varepsilon_{-\infty} + a\varepsilon_0, \qquad a = \exp(c). \tag{3.43}$$

Moreover the value -1 is a $(1-\alpha)$th quantile of the distribution of Y_{n_i}. Therefore $a = \alpha$ and (3.36) is proved if we take (3.40) into account and remark that $c = \log\alpha$ does not depend on the special choice of the subsequence.

Note that by Seneta (1976), Theorem (1.1),

$$\lim_{x\to-\infty} \frac{F(xs)}{F(x)} = 1 \tag{3.44}$$

uniformly in s for $s\in[a,b]$, $0<a<b<\infty$. If we assume that $s_{n_i} = q_{\alpha_1,n_i}q_{\alpha_2,n_i}^{-1} \ge \delta > 0$ is bounded away from 0 then (3.44) contradicts

$$\lim_{i\to\infty} \frac{F(-q_{\alpha_2,n_i}s_{n_i})}{F(-q_{\alpha_2,n_i})} = \frac{\log\alpha_1}{\log\alpha_2} \ne 1. \tag{3.45}$$

□

4. Applications to one-sided test problems in exponential families. In contrast to the classical results (1.1) and (1.6), the rate of convergence of the error of second kind and the critical value depend on the level α if $K(P_0, P_1) = \infty$. Therefore let us study some examples and applications.

In connection with local behavior of one-sided test problems for a simple hypothesis the critical value of the test can be estimated if the hypothesis belongs to the domain of attraction of a stable law; see Janssen (1985b), (1986). Therefore suppose that Q is a probability measure on $\mathbb{R}$ such that ω_Q is finite on some interval $[0,b)$ for $b>0$. Let $E = (\mathbb{R}, \mathscr{L}, (Q_\vartheta)_{\vartheta\in[0,b)})$ be the exponential family generated by Q such that

$$\frac{dQ_\vartheta}{dQ}(x) = \frac{1}{\omega_Q(\vartheta)}\exp(\vartheta x) \quad \text{for } 0\le\vartheta<b. \tag{4.1}$$

Let us consider the test problem $H:\{Q_0\}$ against $K:\{Q_\vartheta\colon \vartheta\in(0,b)\}$ of sample size n.

The most powerful α test is equal to

$$\tilde\varphi_{\alpha,n}(x_1,\dots,x_n) = \begin{cases} 1 & \text{if } \sum_{i=1}^n x_i > -\tilde q_{\alpha,n} \\ \tilde\gamma_n & \text{if } \sum_{i=1}^n x_i = -\tilde q_{\alpha,n} \\ 0 & \text{if } \sum_{i=1}^n x_i < -\tilde q_{\alpha,n} \end{cases}$$

if $E_{Q^n}\tilde{\varphi}_{\alpha,n} = \alpha$. Note that

$$\exp(-\vartheta \tilde{q}_{\alpha,n} - n \log \omega_Q(\vartheta)), \qquad 0 < \vartheta, \tag{4.2}$$

is a critical value of the level α test for $\{Q^n\}$ against $\{Q_\vartheta^n\}$. Suppose in the sequel that Q belongs to the domain of attraction of a nondegenerate stable law P with the index $p > 0$ of stability on $\mathbb{R}$, i.e.,

$$\mathscr{L}(T_{\delta_n}|Q)^{*n} * \varepsilon_{a_n} \to P \tag{4.3}$$

weakly for some $a_n \in \mathbb{R}$ and a sequence $(\delta_n)_n$ of positive numbers tending to 0. Define $T_{\delta_n}(x) = \delta_n x$. As pointed out in Janssen (1986)

$$\delta_n(-\tilde{q}_{\alpha,n}) + a_n \to u_\alpha \tag{4.4}$$

follows where u_α is the $(1-\alpha)$th quantile of P. If $p > 1$, the first moment of Q exists and $K(Q, Q_\vartheta)$ is finite. The classical results imply

$$\left[E_{Q_\vartheta^n}(1 - \tilde{\varphi}_{\alpha,n})\right]^{1/n} \to \exp(-K(Q, Q_\vartheta)), \qquad \vartheta > 0, \tag{4.5}$$

where $K(Q, Q_\vartheta) = -\vartheta \int x\, dQ(x) + \log \omega_Q(\vartheta)$. Note that (4.5) does not depend on α. If $p \le 1$, the first moment of Q no longer exists and hence $K(Q, Q_\vartheta) = \infty$ for $\vartheta > 0$.

Let us first study the case $p < 1$. Following the arguments used in Janssen (1985b), (1986) let us remark that P is one-sided stable on $\mathbb{R}$ having a support

$$\operatorname{supp} P = (-\infty, a] \quad \text{for some } a \in \mathbb{R}. \tag{4.6}$$

The centering procedure of Feller (1971), page 580, implies

$$a_n \to a \quad \text{since} \quad \mathscr{L}(T_{\delta_n}|Q)^{*n} * \varepsilon_{a_n - a} \to P * \varepsilon_{-a} \quad \text{weakly}. \tag{4.7}$$

In order to estimate the rate of convergence we recall some known results (Janssen (1985b), (1986)):

$$\log \omega_P(y) = -c_1 y^p + ay \quad \text{for some } c_1 > 0 \text{ and all } y \ge 0, \tag{4.8}$$

$$\log \omega_Q(y) \sim \Gamma(2-p)(p-1)^{-1} c_2 y^p L(1/y), \qquad y \to 0+, \tag{4.9}$$

$$Q((-\infty, t]) \sim c_2 |t|^{-p} L(|t|), \qquad t \to -\infty, \tag{4.10}$$

where $c_2 > 0$ and L is a positive function varying slowly at infinity. Suppose that $\tilde{k}_{\alpha,n}$ is increasing defined by

$$\log \omega_Q(\tilde{k}_{\alpha,n}^{-1}) = -(1 - \log \alpha) n^{-1} \tag{4.11}$$

for sufficiently large n. If $\alpha_1, \alpha_2 \in (0,1)$ then

$$\tilde{q}_{\alpha_1,n} \tilde{q}_{\alpha_2,n}^{-1} \to (a - u_{\alpha_1})(a - u_{\alpha_2})^{-1} \tag{4.12}$$

if we observe (4.4) and (4.7).

THEOREM 4.1. *If $p < 1$ then*

$$\delta_n \tilde{q}_{\alpha,n} \to a - u_\alpha, \tag{4.13}$$

$$n \log \omega_Q(\tilde{q}_{\alpha,n}^{-1}) \to -c_1(a - u_\alpha)^{-p}, \tag{4.14}$$

$$\frac{\tilde{q}_{\alpha,n}}{\tilde{k}_{\alpha,n}} \to \left(\frac{1 - \log \alpha}{c_1}\right)^{1/p} (a - u_\alpha). \tag{4.15}$$

If $\alpha_n \to \alpha \in [0,1]$ *then* (*for* $u_0 := a,\ u_1 := -\infty$)

$$\lim_{n\to\infty}\left[E_{Q_\vartheta^n}\left(1-\tilde{\varphi}_{\alpha_n,n}\right)\right]^{\delta_n} = \exp(\vartheta(u_\alpha - a)). \tag{4.16}$$

PROOF. Let $q_{\alpha,n}$ be defined by (1.2) for $\{Q_0^n\}$ against $\{Q_\vartheta^n\}$. Then

$$q_{\alpha,n} = \vartheta\tilde{q}_{\alpha,n} + n\log\omega_Q(\vartheta). \tag{4.17}$$

According to (1.4) the result follows from (3.8) and (4.4) since $\tilde{q}_{\alpha,n}/q_{\alpha,n} \to \vartheta$ for fixed α. In general the result follows from the monotonicity of the error function for α and the continuity of the right-hand side of (4.16).

The proof of (4.14) and (4.15): Put $b = ((1-\log\alpha)/c_1)^{1/p}(a-u_\alpha)$. Then by (4.9)

$$\frac{\log\omega_Q\left(b\tilde{q}_{\alpha,n}^{-1}\right)}{\log\omega_Q\left(\tilde{k}_{\alpha,n}^{-1}\right)} \to 1 \tag{4.18}$$

since $n\log\omega_Q(\delta_n) \to \log\omega_P(1) = -c_1$ because $\mathscr{L}(T_{\delta_n}|Q)^{*n} \to P * \varepsilon_{-a}$ weakly; compare with Janssen (1985b), (1986).

It is well known that (4.18) implies $\tilde{k}_{\alpha,n} b\tilde{q}_{\alpha,n}^{-1} \to 1$ since $\log\omega_Q$ is regularly varying; compare with Janssen (1985a), Lemma 7c. □

REMARK 1. It is well known that δ_n is of the order

$$\delta_n = n^{-1/p}\tilde{L}(n), \tag{4.19}$$

where $\tilde{L}$ denotes a positive function varying slowly at infinity. If the index of stability p is less than 1 then (4.13), (4.16), and (4.17) show that (4.19) contains the correct rate of convergence of the interesting quantities for $\alpha \in (0,1)$ which is faster than the classical speed of convergence $1/n$.

REMARK 2. A straightforward calculation proves that, in view of (4.16), $\tilde{\psi}_{\alpha,n}$ can be substituted by the test sequence $\psi_{n,\alpha}$ for fixed level α which was proposed in Janssen (1986), Theorem 13. Note that $\psi_{n,\alpha}$ is an asymptotically most powerful test sequence for $\{Q_0^n\}$ against $\{Q_{\delta_n\vartheta}^n\}$.

REMARK 3. For $p=1$ we only have a partial result. If $(\omega_Q(\delta_n y))^n \exp(ya_n)$ tends to $\omega_P(y) = \exp(cy\log y)$ for $y \in [0,1]$ and some $c > 0$, then Q belongs to the domain of the one-sided stable distribution P with index 1. In general a_n does not converge and Theorem 4.1 no longer holds. For example, if $P = Q$ for $c = 1$ then Q is not strictly stable and

$$\tilde{q}_{\alpha,n} = -nu_\alpha - n\log\frac{1}{n}, \qquad a_n = -\log\frac{1}{n}, \qquad \delta_n = \frac{1}{n}, \qquad n \ge 2,$$

and (4.17) yields the connection between $\tilde{q}_{\alpha,n}$ and $q_{\alpha,n}$, where u_α is the $(1-\alpha)$th quantile of P. In this special case we obtain that for fixed $\vartheta > 0$ the exponent

$1/q_{\alpha,n}$ of (1.3) and (3.8) is equal to

$$\frac{1}{q_{\alpha,n}} = \frac{1}{n}\left[\frac{1}{\log \omega_Q(\vartheta) - u_\alpha - \log\dfrac{1}{n}}\right],$$

which is a faster rate of convergence than $\delta_n = 1/n$.

REMARK 4. In the case of Theorem 3.1 statement (3.9) proves that the order of the rate of convergence of

$$\log E_{P_1^n}(1 - \varphi_{\alpha,n}) \tag{4.20}$$

is the same for all $\alpha \in (0,1)$. But the exact rate of convergence of (4.20) may depend on α; cf. (4.16). On the other hand, (4.16) shows that a different order of the rate of convergence occurs if $\alpha_n \to 1$. This phenomenon is also new compared with the results of Krafft and Plachky (1970) who showed that (1.1) still holds if $(1 - \alpha_n)^{1/n} \to 1$.

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UNIVERSITÄT SIEGEN GHS
FACHBEREICH 6 MATHEMATIK
HÖLDERLINSTRASSE 3
5900 SIEGEN 21
WEST GERMANY