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ON THE YEH–BRADLEY CONJECTURE ON LINEAR TREND-FREE BLOCK DESIGNS

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Yeh and Bradley conjectured that every binary connected block design with blocks of size k and a constant replication number r for each treatment can be converted to a linear trend-free design by permuting the positions of treatments within blocks if and only if $r(k+1) \equiv 0 \pmod 2$. This conjecture is studied. Results include: (i) the conjecture is true whenever the block size is even and (ii) the conjecture is true for BIB designs.

1. Introduction. In many experiments where several treatments are compared in blocks, and within blocks the treatments are applied to the experimental units sequentially over time or space, there is a possibility that a systematic effect, or trend, influences the observations in addition to the block and the treatment effects. The analysis of these experiments will be different from the usual analysis of block designs, since trend effects have to be taken into account.

The problem of designing experiments in the presence of trends was first studied by Cox (1951, 1952). Bradley and Yeh (1980) studied block designs in the presence of trends and characterized block designs that are "trend-free." In essence, this means that the presence of trend does not affect the analysis of the treatment effects. The recent paper by Lin and Dean (1991) is an excellent source of literature in this area.

The focus of this paper is on construction of block designs in the presence of a *linear* trend. Yeh and Bradley (1983) derived a simple necessary condition for a design to be linear trend-free. This condition states that a block design in blocks of size k, with constant replication r for each treatment can be rearranged to become linear trend-free only if $r(k+1) \equiv 0 \pmod 2$. Yeh and Bradley [(1983), Conjecture 5.1] conjectured that this condition was also sufficient, that is, any block design satisfying this condition can be rearranged as a linear trend-free block design without altering the original assignment of treatments to blocks. We shall, for convenience, refer to the linear trend-free design obtained from a block design as the "linear trend-free version" of the block design. Yeh and Bradley (1983) established their conjecture for the case $k = 2$ and for complete block designs with two or more blocks.

A family of counterexamples given by Stufken (1988) showed that the conjecture is not true in general. These counterexamples are for classes of designs with certain properties, of which one is that k must be odd.

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The purpose of this paper is to examine the Yeh–Bradley conjecture. We want to determine when a design, that satisfies Yeh and Bradley's necessary condition, possesses a linear trend-free version. We shall prove several results that include: (i) the conjecture is true whenever k is even and (ii) the conjecture is true for balanced incomplete block (BIB) designs.

Indeed, (ii) will be generalized to balanced block designs (BBD's). A consequence of this result is that linear trend-free versions of BBD's are "universally optimal" for a model that includes a trend effect in addition to block and treatment effects, thus extending the celebrated result of Kiefer (1975) on the universal optimality of BBD's for models that consist of block and treatment effects only.

Some preliminary results and tools, notation and so forth are given in Section 2. The main results are in Section 3. Section 4 has some concluding remarks.

2. Notation and preliminary results. We assume that, within blocks there is a common polynomial trend of order p on the k periods, that can be expressed by the orthogonal polynomials $\phi_\alpha(l)$, $1 \le \alpha \le p$, on $l = 1, \dots, k$, where ϕ_α is a polynomial of degree α. The polynomials $\phi_1(l), \dots, \phi_p(l)$ satisfy

$$\sum_{l=1}^{k} \phi_\alpha(l) = 0, \qquad \sum_{l=1}^{k} \phi_\alpha(l)\phi_{\alpha'}(l) = \delta_{\alpha\alpha'},$$

where $\delta_{\alpha\alpha'}$ denotes the Kronecker delta, $\alpha, \alpha' = 1, \dots, p$. The model for an observation in period l of block j, $1 \le j \le b$, is

$$y_{jl} = \mu + \sum_{i=1}^{v} \delta_{jl}^{i}\tau_i + \beta_j + \sum_{\alpha=1}^{p} \phi_\alpha(l)\theta_\alpha + \varepsilon_{jl}. \tag{2.1}$$

Here μ is a general effect, $\tau_1, \dots, \tau_v$ the treatment effects, $\beta_1, \dots, \beta_b$ the block effects and $\theta_1, \dots, \theta_p$ the trend effects. Moreover,

$$\delta_{jl}^{i} = \begin{cases} 1, & \text{if treatment } i \text{ is applied in period } l \text{ of block } j, \\ 0, & \text{otherwise,} \end{cases}$$

with $\sum_{i=1}^{v}\delta_{jl}^{i} = 1$. If the trend is linear then $p = 1$ in (2.1) and we have only one trend parameter θ_1 in the model. Let

$$\tau = (\tau_1, \dots, \tau_v)', \qquad \beta = (\beta_1, \dots, \beta_b)' \quad \text{and} \quad \theta = (\theta_1, \dots, \theta_k)'.$$

A design d will be represented by a $k \times b$ array of symbols $1, \dots, v$, with columns denoting blocks and rows periods. Thus, if the entry in cell (l, j) of d is i, it means that under d treatment i has to be applied in period l of block j. Let $D(v, b, k)$ be all connected designs in b blocks, k periods based on v treatments. To avoid trivialities we consider henceforth only classes $D(v, b, k)$ with $k \ge 2$. For $d \in D(v, b, k)$, let n_{dij} denote the number of times treatment i appears in column (block) j and s_{dil} denote the number of times treatment i appears in row (period) l. Let $r_{di} = \sum_j n_{dij} = \sum_l s_{dil}$ denote the replication of treatment i. We shall use the notation

$$D(v,b,k;r_1,\dots,r_v) = \{d \in D(v,b,k)\colon r_{di} = r_i,\, i = 1,\dots,v\},$$

$$D_e(v,b,k;r) = \{d \in D(v,b,k;r_1,\dots,r_v)\colon r_i = r,\, i = 1,\dots,v\}.$$

For $d \in D(v,b,k)$, let $R_d(\tau|\mu,\beta,\theta)$ denote the adjusted treatment sum of squares under model (2.1) and $R_d(\tau|\mu,\beta)$ denote $R_d(\tau|\mu,\beta,\theta)$ when $\theta = 0$ in (2.1). Bradley and Yeh (1980) defined a design d to be trend-free if

$$R_d(\tau|\mu,\beta,\theta) = R_d(\tau|\mu,\beta).$$

It can be shown that a design is linear trend-free if and only if

$$\sum_{l=1}^{k} s_{dil}\phi_1(l) = 0, \qquad i = 1,\dots,v. \tag{2.2}$$

This characterization holds for binary as well as non-binary designs, and also irrespective of whether k is larger or smaller than v [Lin and Dean, (1991)].

The polynomial, $\phi_1(l)$ satisfies $\phi_1(l) = -\phi_1(k-l+1)$. In addition, $\phi(\frac{1}{2}(k+1)) = 0$ when k is odd. It follows, therefore, that (2.2) is true whenever

$$s_{dil} = s_{di(k-l+1)}, \qquad l = 1,\dots,[(k+1)/2], \quad i = 1,\dots,v, \tag{2.3}$$

with $[\cdot]$ denoting the largest integer function. Note that, when k is odd (2.3) does not impose any restriction on $s_{di(k+1)/2}$. It is not difficult to see that condition (2.2) does not, in general, imply condition (2.3). Condition (2.3) is, in fact, necessary and sufficient for a design d to be "odd-degree trend-free" [Lin and Dean, (1991), Corollary 2.1.2]. This means that the design is trend-free in the presence of polynomial trends of the form $\phi_1(l)\theta_1 + \phi_3(l)\theta_3 + \cdots + \phi_e(l)\theta_e$ where $e \in \{k-2, k-1\}$ is odd.

If $d \in D(v,b,k;r_1,\dots,r_v)$, then it can be seen from the results of Yeh and Bradley (1983), that a necessary condition for d to be linear trend-free is

$$r_i(k+1) \equiv 0 \pmod 2,\ i = 1,\dots,v. \tag{2.4}$$

If $d \in D_e(v,b,k;r)$, then (2.4) reduces to

$$r(k+1) \equiv 0 \pmod 2,$$

the Yeh–Bradley necessary condition mentioned in the introduction.

Our main tools are the *system of distinct representatives* (SDR) which was defined by Hall (1935), the generalization of SDR due to Agrawal (1966) and some theorems on SDR's. For the sake of completeness, we reproduce a definition and a basic result on the topic.

DEFINITION 2.1 [Agrawal (1966)]. If $S_1, S_2, \dots, S_n$ are subsets of a finite set S, then $(O_1, O_2, \dots, O_n)$ will be called a $(m_1, m_2, \dots, m_n)$ SDR if (i) $O_i \subseteq S_i$, (ii) $|O_i| = m_i$ and (iii) $O_i \cap O_j = \varnothing$, the empty set, for $i \neq j = 1,\dots,n$. ($|A|$ denotes the cardinality of set A).

THEOREM 2.1 [Agrawal (1966)]. *A necessary and sufficient condition that* $S_1, S_2, \ldots, S_n$ *possess a* $(m_1, m_2, \ldots, m_n)$ *SDR is that*

$$|S_{i_1} \cup S_{i_2} \cup \cdots \cup S_{i_t}| \geq \sum_{j=1}^{t} m_{i_j},$$

for $1 \leq i_1 < i_2 < \cdots < i_t \leq n$, $1 \leq t \leq n$.

3. Main results. Given a block design $d \in D(v, b, k)$ we wish to determine whether or not we can transform it, by permuting symbols within columns, which are the blocks, to a design that is linear trend-free, that is, an array that satisfies (2.2) or (2.3).

DEFINITION 3.1. An array d_{tf}, derived from a design d by permuting symbols within columns, will be called a linear trend-free version of d if it satisfies (2.2). The array d_{tf} will be called a strongly linear trend-free version of d if it satisfies (2.3).

Since (2.3) implies (2.2), a strongly linear trend-free version is also a linear trend-free version. Also recall that, as mentioned in Section 2, a strongly linear trend-free version of d is odd-degree trend-free [Lin and Dean (1991)]. When k is even, it follows from (2.4) that a necessary condition for d_{tf} to exist is: r_i is even for each $i = 1, \ldots, v$; on the other hand when k is odd, (2.4) imposes no restriction on the replication numbers.

LEMMA 3.1. *Each design in* $D_e(v, b, 2m; 2)$ *has a strongly linear trend-free version.*

PROOF. Clearly $v = mb$. Let $S = \{1, 2, \ldots, mb\}$, $d \in D_e(v, b, 2m; 2)$. Let C_j denote the jth column and S_j denote the set of symbols in column j of d, $j = 1, \ldots, b$. The column C_j coincides with the set S_j for each $j = 1, \ldots, b$ when $n_{dij} \in \{0, 1\}$, that is, the block design is binary. Whether binary or not, it is easy to see that d satisfies

$$|S_{j_1} \cup S_{j_2} \cup \cdots \cup S_{j_t}| \geq mt,$$

for $1 \leq j_1 < \cdots < j_t \leq b$, $1 \leq t \leq b$. Thus, by Theorem 2.1, $S_1, S_2, \ldots, S_b$ possesses a $(m, m, \ldots, m)$ SDR, $(A_1, A_2, \ldots, A_b)$, say. Let $B_j = C_j \setminus A_j$, where the notation $M \setminus N$ denotes the collection of elements of M that are not in N. Clearly $m = |B_j| = |A_j|$, $j = 1, 2, \ldots, b$. $\{A_1, A_2, \ldots, A_b\}$ forms a partition of S; so does $\{B_1, B_2, \ldots, B_b\}$. Hence, by a theorem of König (1950) [cf. Raghavarao (1971), Corollary 6.2.6.1] there is a permutation π of $\{1, 2, \ldots, b\}$ such that $A_j \cap B_{\pi(j)} \neq \varnothing$. Let $a_j \in S$, $a_j \in A_j \cap B_{\pi(j)}$ for $j = 1, 2, \ldots, b$. Obviously, $a_1, a_2, \ldots, a_b$ are all distinct. Let us rearrange the columns of d such that $a_1, a_2, \ldots, a_b$ appears in the first and the last row of d, by putting a_j in the first row of column C_j and last row of column $C_{\pi(j)}$, $j = 1, 2, \ldots, b$. Call this new array d_1.

The array d_1 with the first and last rows deleted is in $D_e((m-1)b, b, 2(m-1); 2)$. We can repeat the argument to this $2(m-1) \times b$ array. Continuing in this fashion we obtain in m steps an array that satisfies the relation (2.3). This is a strongly linear trend-free version of d. □

THEOREM 3.1. *Let $k, r_1, r_2, \dots, r_v$ be even numbers. For each design in $D(v,b,k; r_1, r_2, \dots, r_v)$ there exists a strongly linear trend-free version.*

PROOF. Suppose $k = 2m$ and $r_i = 2w_i$, $i = 1, 2, \dots, v$. Let $d \in D(v, b, k; r_1, r_2, \dots, r_v)$. Let us derive an array d^* from d in the following fashion. Select any two cells of d that have the symbol i and replace i in these two cells by the ordered pair $(i, 1)$. Choose two other cells that have the symbol i from the $2(w_i - 1)$ remaining cells and replace by the ordered pair $(i, 2)$. Continue in this fashion until all i's have been replaced by $(i, 1)$, $(i, 2), \dots, (i, w_i)$. Do this for each $i = 1, 2, \dots, v$. Call the resulting design d^*. Clearly $d^* \in D_e(\Sigma_{i=1}^{v} w_i, b, 2m; 2)$; d^* is based on symbols represented by (i, g), $g = 1, 2, \dots, w_i$; $i = 1, 2, \dots, v$. Now, use Lemma 3.1 to obtain d_{tf}^*, a strongly linear trend-free version of d^*. Finally, in d_{tf}^*, replace each pair (i, g) by the symbol i, to obtain the array d_{tf}. Clearly, d_{tf} is a strongly linear trend-free version of d. □

Theorem 3.1 shows that the Yeh–Bradley conjecture is true whenever k is even. Now let us turn our attention to the case k odd. As has been mentioned in the introduction, Stufken (1988) gave an infinite family of designs with odd k, no member of which possess a linear trend-free version. Therefore, the Yeh–Bradley conjecture is not true for Stufken's families. There are, however, many designs with odd k which are not covered by Stufken's results. We shall show that some of them do possess linear trend-free versions. In Theorems 3.2, 3.3 and 3.4 our method is to identify one symbol per column so that after removing these symbols, reducing the column size by 1, we are left with a design with even replications and even block size where we can apply Theorem 3.1; the symbols that were removed will be inserted as the $[(k+1)/2]$th row of the linear trend-free version.

THEOREM 3.2. *Suppose k is odd and for $0 \le t \le v$, $r_1, r_2, \dots, r_t$ are odd, $r_{t+1}, \dots, r_v$ are even.* (*If $t = 0$, then there are no odd r_i's and if $t = v$, there are no even r_i's*). *Let $d \in D(v, b, k; r_1, r_2, \dots, r_t, r_{t+1}, \dots, r_v)$ be such that the columns of d can be partitioned into sets of columns $P_1, P_2, \dots, P_t, Q$, which satisfy*: (a) *For each $i = 1, 2, \dots, t$, the number of columns in P_i, $|P_i|$, is odd and the symbol i is common to each block in P_i; and* (b) *$|Q|$ is even and the columns in Q can be divided into $|Q|/2$ pairs of columns such that any two columns that form a pair have at least one symbol in common. Then d has a strongly linear trend-free version.*

PROOF. It follows from the structure of the array d that after permuting within columns, if necessary, we can write

$$d = \begin{bmatrix} \rho \\ d_0 \end{bmatrix} \tag{3.1}$$

where ρ is a $1 \times b$ row vector which is a permutation of

$$(1, \dots, 1, 2, \dots, 2, \dots, t, \dots, t, a_1, a_2, \dots, a_h),$$

where $h = b - \Sigma_{i=1}^{t} |P_i|$, such that, for each $i = 1, \dots, t$, the symbol i appears $|P_i|$ (odd) number of times, while the symbols $t+1, \dots, v$ either do not appear or appear an even number of times each in ρ. Now applying Theorem 3.1 it is clear that d_0 has a strongly linear trend-free version d_{otf} that satisfies (2.3). Let us write

$$d_{otf} = \begin{bmatrix} d_{otf}^u \\ d_{otf}^l \end{bmatrix},$$

where d_{otf}^u is a $((k-1)/2) \times b$ array and so is d_{otf}^l. Then

$$\begin{bmatrix} d_{otf}^u \\ \rho \\ d_{otf}^l \end{bmatrix}$$

satisfies (2.3) and is therefore a strongly linear trend-free version of d. □

COROLLARY 3.1. *Suppose k is odd, $r_1, r_2, \dots, r_v$ are even, $2k \geq v$ and $b \geq 2$. Then every binary design d in $D(v, b, k; r_1, r_2, \dots, r_v)$ has a strongly linear trend-free version.*

PROOF. If $2k > v$, then any two columns of d have at least one symbol in common. Hence the result follows from Theorem 3.2. If $2k = v$, then we can divide d into two subarrays d_1 and d_2 such that d_1 consists of all the columns of d that contain symbol 1 and d_2 consists of the remaining columns of d. Notice that any two columns of d_i $i = 1, 2$, have at least one symbol in common. Hence the result follows from Theorem 3.2. □

THEOREM 3.3. *Suppose k is odd, $r_1, r_2, \dots, r_v$ are even. Let $d \in D(v, b, k; r_1, r_2, \dots, r_v)$ be a binary design such that for one symbol (say symbol 1), the union of all columns that contain symbol 1 contains at least one copy of each of the other symbols $2, \dots, v$. Then d has a strongly linear trend-free version.*

PROOF. Suppose $C_1, C_2, \dots, C_{r_1}, C_{r_1+1}, \dots, C_b$ are the columns of the array d of which $C_1, C_2, \dots, C_{r_1}$ contain symbol 1. Let $X = \{C_1, C_2, \dots, C_b\}$, $X_1 = \{C_1, C_2, \dots, C_{r_1}\}$, $X_2 \subset X \setminus X_1$ be a collection of columns which can be divided into pairs of columns such that the columns in each pair have at least one symbol in common. Furthermore X_2 has the property that no two columns of $X \setminus (X_1 \cup X_2)$ have a symbol in common. Call $X \setminus (X_1 \cup X_2) = X_3$. Suppose $|X_3| = q$. Clearly $q \le [(v - 1)/k]$. Also, q is even, since the quantities b, $|X_1| = r_1$ and $|X_2|$ are all even. Let $X_2 = \{C_{r_1+1}, \dots, C_{b-q}\}$, $X_3 = \{C_{b-q+1}, \dots, C_b\}$, and let S^* be the set consisting of all symbols used in $C_{b-q+1}, \dots, C_b$.

If $S^* = \varnothing$, then the result follows from Theorem 3.2; hence assume $S^* \ne \varnothing$. Let $A_0 = \varnothing$, $A_i = (C_i \cap S^*) \setminus (A_0 \cup \cdots \cup A_{i-1})$, for $i = 1, 2, \dots, r_1$. Suppose that $\{A_{i_1}, A_{i_2}, \dots, A_{i_n}\}$ are all the nonempty sets in the collection $\{A_0, A_1, \dots, A_{r_1}\}$. Clearly, $n > q$ and $\{A_{i_1}, A_{i_2}, \dots, A_{i_n}\}$ forms a partition of S^*. It follows from Theorem 2 of Hall (1935), [cf. Raghavarao (1971), Theorem 6.2.6] that there is a subset $\{j_1, j_2, \dots, j_q\}$ of $\{i_1, i_2, \dots, i_n\}$ with the property: $A_{j_t} \cap C_{b-t+1} \ne \varnothing$ for $t = 1, 2, \dots, q$.

Hence $C_{j_t} \cap C_{b-t+1} \ne \varnothing$ for $t = 1, 2, \dots, q$, where $\{C_{j_1}, \dots, C_{j_q}\} \subset X_1$. Note that $|X_1| - |\{C_{j_1}, \dots, C_{j_q}\}| = r_1 - q$ is even. Thus we can construct a $(1 \times b)$ row vector $\rho = (\rho_1, \dots, \rho_{r_1}, \rho_{r_1+1}, \dots, \rho_{b-q}, \rho_{b-q+1}, \dots, \rho_b)$ with the properties:

(i) $\rho_j \in C_j$, $j = 1, 2, \dots, b$,
(ii) $\rho_{b-t+1} = \rho_{j_t}$, $t = 1, 2, \dots, q$, (recall $\{j_1, j_2, \dots, j_q\} \subseteq \{1, 2, \dots, r_1\}$),
(iii) $\rho_j = 1$, $j \in \{1, 2, \dots, r_1\} \setminus \{j_1, j_2, \dots, j_q\}$,
(iv) $\{\rho_{r_1+1}, \dots, \rho_{b-q}\}$ is formed by selecting a common symbol in any two columns that form one of the $(b - q - r_1)/2$ pairs of columns in X_2.

Clearly ρ consists of some symbols from the set $\{1, 2, \dots, v\}$, repeated an even number of times each. Therefore d has the structure (3.1) and the proof follows as in the proof of Theorem 3.2. □

THEOREM 3.4. *Suppose $k, r_1, r_2, \dots, r_v$ are odd. Let $d \in D(v, b, k; r_1, r_2, \dots, r_v)$ be a binary design such that for one symbol (say symbol 1), the union of all columns that contain symbol 1 contains at least one copy of each of the other symbols $2, \dots, v$. Furthermore, suppose that the collection of columns not containing symbol 1 has a subcollection of $v - 1$ columns such that the ith column among these $v - 1$ columns contains symbol $i + 1$, $i = 1, 2, \dots, v - 1$. Then d has a strongly linear trend-free version.*

PROOF. Let X, X_1 be as in the proof of Theorem 3.3.

X_2 = the collection of the $v - 1$ columns from $X \setminus X_1$ such that the ith column contains symbol $i + 1$,

X_3 = the collection of columns from $X \setminus (X_1 \cup X_2)$ which can be divided into $|X_3|/2$ pairs of columns such that in each pair the two columns have at least one symbol in common,

$X_4 = X \setminus (X_1 \cup X_2 \cup X_3)$.

Using the same technique as in the proof of Theorem 3.3, we can express d as in (3.1) with ρ consisting of all symbols $1, 2, \dots, v$ each repeated an odd number of times. Hence the proof follows as in the proof of Theorem 3.2. □

Observe that Theorems 3.3 and 3.4 can be extended to $D(v, b, k; r_1, r_2, \dots, r_v)$ with some r_i's odd and the rest even. The statement of such a theorem, though not difficult, is quite long and involved and hence it is omitted.

We shall use the standard notation BIB(v, b, r, k, λ) to denote a balanced incomplete block design in b blocks of size k each based on v symbols; r denotes the replication of each symbol and λ the number of blocks that contain a pair of symbols.

THEOREM 3.5. *Any BIB(v, b, r, k, λ) satisfying $r(k+1) \equiv 0 \pmod 2$ has a strongly linear trend-free version.*

PROOF. We divide the proof into four cases:

CASE 1. k is even (hence r is even). The desired result follows from Theorem 3.1.

CASE 2. k is odd, r is even. Since the symbols $2, \dots, v$ occur λ times each in the columns that contain 1, the result follows from Theorem 3.3.

CASE 3. k is odd, r is odd and $b \geq v + r - 1$. With no loss of generality, the design d can be viewed as the array

$$d = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & \\ & & & & & d_2 \\ & & d_1 & & & \end{bmatrix},$$

where $d_1((k-1) \times r)$ and $d_2(k \times (b-r))$ are arrays based on the symbols $2, \dots, v$ only. The array d_2, with columns as blocks, is a binary equireplicate design in $v-1$ symbols. On applying Theorem 3.1 or 3.2 of Agrawal (1966) to d_2, and rearranging the order of the columns of d_2, it follows that column i of d_2 contains symbol $i+1$, $i = 1, \dots, v-1$. The result now follows from Theorem 3.4.

CASE 4. k is odd, r is odd and $v \leq b \leq v + r - 2$. We can write, using Theorem 3.1 or 3.2 of Agrawal (1966),

$$d = \begin{bmatrix} 1 & 2 & \cdots & v-1 & v & \\ & & & & & d_2 \\ & & d_1 & & & \end{bmatrix},$$

where d_1 is a $((k-1) \times v)$ array and d_2 is a $(k \times (b-v))$ array. Note that

$b - v$ is even. Let c_{ju} denote the number of symbols common to columns j and u of d_2. From Corollary 3.1 of Connor (1952) it follows that for all $j \neq u$,

$$c_{ju} \geq k + \lambda - r.$$

Using $b \leq v + r - 2$ it is easy to show that $k + \lambda - r > 0$. Hence the integer $c_{ju} \geq 1$. Therefore the columns of d_2 can be divided into $(b - v)/2$ pairs of columns such that each pair has at least one symbol in common. The result now follows from Theorem 3.2, if we set $t = v$. □

Finally we turn our attention to balanced block designs (BBD's). It is known that if $d \in D_e(v, b, k; r)$ is a BBD, then

$$d = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_h \\ d_0 \end{bmatrix},$$

where for each $i = 1, 2, \ldots, h$, d_i is a complete block design (CBD) and d_0 is a BIB(v, b, r', k', λ) where $k = hv + k'$ and $bk' = vr'$. When $k' = 0$, there is no design d_0. When $k' = 1$, d_0 will be a row consisting of symbols $\{1, 2, \ldots, v\}$ each replicated b/v times.

Theorem 3.6. *Let $d \in D_e(v, b, k; r)$ be a BBD with $r(k + 1) \equiv 0 \pmod 2$ and $b \geq 3$. Then d has a linear trend-free version.*

Proof. The linear trend-free version is obtained by combining a strongly linear trend-free version of the BIB design d_0 with suitably structured versions of the CBD's $d_1, \ldots, d_h$. □

Remark. With the possible exception of the case k odd, $k'(\equiv k \pmod v)$ even and b odd, strongly linear trend-free versions of BBD's can be constructed.

Theorems 3.5 and 3.6 show that the Yeh–Bradley conjecture is true for two important families of designs—BIB's and BBD's. This fact leads us to a strong optimality result. For a definition of universal optimality—the criterion that we shall use—the reader is referred to Kiefer (1975).

Theorem 3.7. *Trend-free versions of balanced incomplete block designs and balanced block designs are universally optimal within their respective classes $D(v, b, k)$ under model (2.1) with $p = 1$.*

Proof. The result follows on applying Proposition 1 of Kiefer (1975) to the C-matrix of a $d \in D(v, b, k)$. The C-matrix is given in equation (3) of Yeh, Bradley and Notz (1985). □

We conclude this section by an example of a strongly linear trend-free version of a BIB(9, 18, 8, 4, 3) expressed as a 4×18 array with columns representing blocks and rows periods:

4	2	7	8	3	6	9	1	2	8	6	2	5	9	7	9	8	7
3	1	5	1	9	1	6	7	5	4	2	8	3	4	3	5	4	6
1	9	1	3	1	4	5	5	6	2	8	7	4	3	5	4	6	3
2	4	2	6	8	7	1	8	3	5	9	9	8	7	9	6	7	2

4. Concluding remarks. The initial results of Yeh and Bradley (1983), the results of Stufken (1988) and the results of this paper have settled the Yeh–Bradley conjecture for a large number of classes of designs. The remaining cases are currently under investigation.

We can drop the requirement that the designs are connected, if we adopt (2.2) as the definition of a linear trend-free block design.

In this paper we gave existence results. The next step is to devise algorithms to construct linear trend-free versions of designs. Bradley and Odeh (1988) has one such algorithm. Hall (1956) and Ash (1981) have algorithms to find SDR's. Recently Stufken (personal communication) has sharpened Bradley and Odeh's algorithm. Chai (1992) has a computer program to derive strongly linear trend-free versions of designs with k even and r_i even.

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