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COMPARISON OF EXPERIMENTS VIA DEPENDENCE OF NORMAL VARIABLES WITH A COMMON MARGINAL DISTRIBUTION

BY MOSHE SHAKED[1] AND Y. L. TONG[2]

University of Arizona and Georgia Institute of Technology

In this note we study comparison of experiments via the positive dependence of normal variables with a common univariate marginal distribution. We show that positive dependence has an adverse effect on the information concerning the common mean θ, and give a partial ordering of the information via a majorization ordering of the correlation matrices. In the special case when the random variables are equally correlated, the main theorem yields a result for the comparison of experiments for permutation symmetric normal variables.

1. Introduction. For the motivation we first observe a standard definition and a known fact: Let $\mathbf{X} = (X_1, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be two n-dimensional random vectors with distributions F_θ and G_θ, respectively, where $\theta \in \Theta \subset \mathbb{R}^k$ is the parameter of concern ($n \geq 1$ and $k \geq 1$).

DEFINITION 1. The experiment associated with $\mathbf{Y}$ is said to be *at least as informative* as that associated with $\mathbf{X}$ for θ, in symbols $\mathbf{X} \leq_{(i)} \mathbf{Y}$ or $F_\theta \leq_{(i)} G_\theta$, if for every decision problem involving θ and every prior distribution on Θ, the expected Bayes risk from F_θ is not less than that from G_θ.

PROPOSITION 2. $\mathbf{X} \leq_{(i)} \mathbf{Y}$ *holds if there exists a function* ψ: $\mathbb{R}^{n+r} \to \mathbb{R}^n$ *and an r-dimensional random vector* $\mathbf{Z}$ ($r \geq 1$), *which is independent of* $\mathbf{Y}$ *and having a distribution which does not depend on* θ, *such that* $\mathbf{X} =_d \psi(\mathbf{Y}, \mathbf{Z})$ ($=_d$ *denotes equality in distribution*).

In a recent paper, Shaked and Tong (1990) provided the following monotonicity result: Let $\mathbf{Y}_1$ and $\mathbf{Y}_2$ be two n-dimensional normal random variables with means θ, a common known variance $\sigma^2 > 0$, and common correlation coefficients ρ_1, ρ_2, respectively; then

$$\mathbf{Y}_2 \leq_{(i)} \mathbf{Y}_1 \quad \text{for all } 0 \leq \rho_1 < \rho_2 \leq 1. \tag{1}$$

From (1) it follows that if permutation symmetric normal variables are more positively dependent, then the experiment is less informative. A question

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of interest is then what can be said for normal variables which are not permutation symmetric. In this note we provide an answer to this question, and show how a more general partial ordering of positive dependence yields a monotonicity result for normal variables with a common marginal distribution.

Note that the statistical inference problems for the common mean of normal random variables arise in various applicatioins. For example, an estimation problem and a hypothesis-testing problem were considered previously by Brown and Cohen (1974) and Cohen and Sacrowitz (1977), respectively.

2. The main result. To consider a partial ordering of positive dependence of multivariate normal variables we first consider an n-dimensional vector of nonnegative integers given by

$$(2)\qquad \mathbf{k} = (k_1, \dots, k_r, 0, \dots, 0), \qquad k_1 \geq \cdots \geq k_r \geq 1, \qquad \sum_{s=1}^{r} k_s = n$$

for some $r \leq n$. (The assumption of monotonicity of k_s in s is not an essential restriction. If it does not hold, then the random variables can always be relabelled, yielding the assumed monotonicity.) For arbitrary but fixed $0 \leq \rho_1 < \rho_2 \leq 1$, let us define a correlation matrix $\mathbf{R}(\mathbf{k})$ given by

$$\rho_{ij}(\mathbf{k}) = \begin{cases} 1, & \text{for } i = j, \\ \rho_2, & \text{for } i \neq j \text{ and } \sum_{s=0}^{m} k_s + 1 \leq i, j \leq \sum_{s=0}^{m+1} k_s \\ & \text{for any } m \in \{0, 1, \dots, r-1\}, \\ \rho_1, & \text{otherwise}, \end{cases}$$

where $k_0 \equiv 0$. If $\mathbf{X}$ has a correlation matrix $\mathbf{R}(\mathbf{k})$, then its components belong to r groups, with group sizes $k_1, \dots, k_r$, respectively, such that the correlations within groups are ρ_2 and the correlations between groups are ρ_1. For references on the applications of such a correlation matrix in an agricultural genetic selection problem see, for example, Tong [(1990), pages 129–130].

Now let $\mathbf{k}^*$ be another vector of nonnegative integers such that

$$(3)\qquad \mathbf{k}^* = (k_1^*, \dots, k_{r^*}^*, 0, \dots, 0), \qquad k_1^* \geq \cdots \geq k_{r^*}^* \geq 1, \qquad \sum_{s=1}^{r^*} k_s^* = n$$

for some $r^* \leq n$; and let $\mathbf{R}(\mathbf{k}^*)$ be defined similarly. Let $\mathbf{X}$ and $\mathbf{Y}$ have multivariate normal distributions such that

$$(4)\qquad \mathbf{X} \sim \mathcal{N}_n(\theta\mathbf{1}, \sigma^2\mathbf{R}(\mathbf{k})) \quad \text{and} \quad \mathbf{Y} \sim \mathcal{N}_n(\theta\mathbf{1}, \sigma^2\mathbf{R}(\mathbf{k}^*))$$

for some $\mathbf{k}$ and $\mathbf{k}^*$ satisfying (2) and (3), respectively, where $\theta \in \mathbb{R}$ is the common mean, $\sigma^2 > 0$ is the common known variance and $\mathbf{1} = (1, \dots, 1)$. Clearly the X_i's and Y_i's defined in (4) have a common univariate $\mathcal{N}(\theta, \sigma^2)$ distribution. In the special case $\mathbf{k} = (n, 0, \dots, 0)$ and $\mathbf{k}^* = (1, 1, \dots, 1)$, both $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ are permutation symmetric normal variables with correlation coefficients ρ_2, ρ_1, respectively. However, they are not permutation symmetric otherwise. A result of Tong (1989) states that if $\mathbf{k} \succ \mathbf{k}^*$, where $\succ$ denotes the majorization ordering, then the X_i's tend to hang together more than the Y_i's, hence are more positively dependent, in the sense that

$$(5) \qquad E\prod_{i=1}^{n}\phi(X_i) \geq E\prod_{i=1}^{n}\phi(Y_i) \quad \text{for all } \phi\colon \mathbb{R} \to [0,\infty)$$

such that the expectations exist. The question of interest is whether this partial ordering of positive dependence also provides a partial ordering for information on θ in the sense of Definition 1. This question is answered in the following theorem.

THEOREM 3. *Assume that* $\mathbf{X}$ *and* $\mathbf{Y}$ *satisfy* (4) *where* $\theta \in \mathbb{R}$ *is the unknown parameter,* $\sigma^2 > 0$ *is the common known variance and* $0 \leq \rho_1 < \rho_2 < 1$ *are arbitrary but fixed. If* $\mathbf{k} \succ \mathbf{k}^*$, *then* $\mathbf{X} \leq_{(i)} \mathbf{Y}$.

PROOF. (a) We first prove the special case in which $\rho_1 = 0$. If $\rho_2 \in (0,1)$, then, for fixed $\mathbf{k}$, $\mathbf{R}(\mathbf{k})$ reduces to

$$(6) \qquad \begin{pmatrix} \Lambda_1(\mathbf{k}) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \Lambda_2(\mathbf{k}) & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \Lambda_r(\mathbf{k}) \end{pmatrix} \equiv \Lambda(\mathbf{k}),$$

where $\Lambda_s(\mathbf{k})$ is a $k_s \times k_s$ correlation matrix with off-diagonal elements ρ_2. Let $\mathbf{T}(\mathbf{k}) = (t_{ij}(\mathbf{k}))$ and $\mathbf{T}_s(\mathbf{k}) = (t_{ij}^{(s)}(\mathbf{k}))$ denotes the inverses of $\Lambda(\mathbf{k})$ and $\Lambda_s(\mathbf{k})$, respectively $(s = 1,\dots,r)$. Then it follows from Tong [(1990), pages 105–106] and a simple calculation that $\sum_{i=1}^{k_s}\sum_{j=1}^{k_s}t_{ij}^{(s)}(\mathbf{k}) = k_s/[1 + (k_s - 1)\rho_2]$. Thus we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n}t_{ij}(\mathbf{k}) = \sum_{s=1}^{r}\frac{k_s}{1 + (k_s - 1)\rho_2} \equiv h(\mathbf{k}).$$

Applying Theorem 3.A.4 in Marshall and Olkin [(1979), page 57] it can be shown that $h(\mathbf{k})$ is a Schur-concave function of $\mathbf{k}$ for $\mathbf{k} \in [0,\infty)^n$. Thus $\mathbf{k} \succ \mathbf{k}^*$ implies

$$\frac{1}{\mathbf{1}(\Lambda(\mathbf{k}))^{-1}\mathbf{1}'} = \frac{1}{h(\mathbf{k})} \geq \frac{1}{h(\mathbf{k}^*)} = \frac{1}{\mathbf{1}(\Lambda(\mathbf{k}^*))^{-1}\mathbf{1}'}.$$

Taking $\mathbf{A} = \mathbf{B} = \mathbf{1}$ in Torgersen [(1984), page 14], we have $\mathbf{X} \leq_{(i)} \mathbf{Y}$ when $\rho_1 = 0$.

(b) We now use the result established in (a) to prove the general case $0 < \rho_1 < \rho_2 < 1$. Let $\mathbf{U}$ and $\mathbf{V}$ be such that

$$\mathbf{U} \sim \mathscr{N}_n\left(\frac{\theta}{\sqrt{1-\rho_1}}\mathbf{1}, \sigma^2\mathbf{R_U}\right) \quad \text{and} \quad \mathbf{V} \sim \mathscr{N}_n\left(\frac{\theta}{\sqrt{1-\rho_1}}\mathbf{1}, \sigma^2\mathbf{R_V}\right),$$

where $\mathbf{R_U} = \Lambda(\mathbf{k})$, $\mathbf{R_V} = \Lambda(\mathbf{k}^*)$ and $\Lambda(\mathbf{k})$ is defined as in (6). Form (a) we have $\mathbf{k} \succ \mathbf{k}^* \Rightarrow \mathbf{U} \leq_{(i)} \mathbf{V}$. By the remark that follows Corollary 2.4 of Torgersen (1984), there exist an $n \times n$ real matrix $\mathbf{C} = (c_{ij})$ and an n-dimensional normal variable $\mathbf{Z}$ (which is independent of $\mathbf{U}$ and $\mathbf{V}$ and whose distribution does not depend on θ) such that

$$\mathbf{U} =_d \mathbf{Z} + \mathbf{CV}. \tag{7}$$

Since

$$\frac{\theta}{\sqrt{1-\rho_1}}\mathbf{1} = E_\theta \mathbf{U} = E\mathbf{Z} + \mathbf{C}E_\theta\mathbf{V} = E\mathbf{Z} + \frac{\theta}{\sqrt{1-\rho_1}}\mathbf{C1}$$

for every $\theta \in \mathbb{R}$, we must have $E\mathbf{Z} = 0$ and

$$\sum_{j=1}^{n} c_{ij} = 1 \quad \text{for } i = 1, \dots, n. \tag{8}$$

On the other hand, by letting W be an $\mathscr{N}(0, \sigma^2)$ random variable that is independent of $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{Z}$, we have

$$\mathbf{X} =_d \sqrt{1-\rho_1}\,\mathbf{U} + \sqrt{\rho_1}\,W\mathbf{1} \quad \text{and} \quad \mathbf{Y} =_d \sqrt{1-\rho_1}\,\mathbf{V} + \sqrt{\rho_1}\,W\mathbf{1}, \tag{9}$$

where $\mathbf{X}$ and $\mathbf{Y}$ are given in (4). Combining (7), (8) and (9) we have

$$\begin{aligned}\mathbf{X} &=_d \sqrt{1-\rho_1}\,(\mathbf{Z} + \mathbf{CV}) + \sqrt{\rho_1}\,W\mathbf{1} \\ &= \sqrt{1-\rho_1}\,\mathbf{Z} + \mathbf{C}\left(\sqrt{1-\rho_1}\,\mathbf{V} + \sqrt{\rho_1}\,W\mathbf{1}\right) \\ &=_d \sqrt{1-\rho_1}\,\mathbf{Z} + \mathbf{CY};\end{aligned}$$

and the proof is complete by applying Proposition 2. □

REMARK 4. Note that if $\mathbf{X}$ and $\mathbf{Y}$ satisfy (4) with known σ^2, then by Torgersen (1984), $\mathbf{X} \leq_{(i)} \mathbf{Y}$ holds if and only if $\mathbf{1}(\mathbf{R}(\mathbf{k}))^{-1}\mathbf{1}' \leq \mathbf{1}(\mathbf{R}(\mathbf{k}^*))^{-1}\mathbf{1}'$ for fixed $0 \leq \rho_1 \leq \rho_2 < 1$. When $\rho_1 > 0$, we are unable to find a proof for Theorem 3 by a direct verification of this condition because in general the inverses of $\mathbf{R}(\mathbf{k})$ and of $\mathbf{R}(\mathbf{k}^*)$ are quite complicated.

3. Some concluding remarks.

We have given a result for the comparison of experiments for the multivariate normal distribution with a common marginal distribution, and it depends on a partial ordering of the positive dependence of normal variables via a majorization ordering. We observe that when combining with existing results, other useful results can be obtained. For example, if $\mathbf{X} \sim \mathscr{N}_n(\theta\mathbf{1}, \sigma^2\mathbf{R}(\mathbf{k}))$ and $\mathbf{Z} \sim \mathscr{N}_n(\theta\mathbf{1}, \Sigma_\mathbf{Z})$ and if there exists a correlation matrix $\mathbf{R}(\mathbf{k}^*)$ such that (i) $\mathbf{k} \succ \mathbf{k}^*$ and (ii) $\sigma^2\mathbf{R}(\mathbf{k}^*) - \boldsymbol{\Sigma}_{\mathbf{Z}}$ is either positive definite or positive semidefinite, then $\mathbf{X} \leq_{(i)} \mathbf{Z}$ holds. Furthermore, we observe that when $\mathbf{k} = (n, 0, \ldots, 0)$ and $\mathbf{k}^* = (1, 1, \ldots, 1)$, then Theorem 3 reduces to Theorem 4.1 of Shaked and Tong (1990).

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ARIZONA
TUCSON, ARIZONA 85721

SCHOOL OF MATHEMATICS
GEORGIA INSTITUTE OF TECHNOLOGY
ATLANTA, GEORGIA 30332