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BOOTSTRAPPING UNSTABLE FIRST-ORDER AUTOREGRESSIVE PROCESSES

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Consider a first-order autoregressive process $X_t = \beta X_{t-1} + \varepsilon_t$, where $\{\varepsilon_t\}$ are independent and identically distributed random errors with mean 0 and variance 1. It is shown that when $\beta = 1$ the standard bootstrap least squares estimate of β is asymptotically *invalid*, even if the error distribution is assumed to be normal. The conditional limit distribution of the bootstrap estimate at $\beta = 1$ is shown to converge to a random distribution.

1. Introduction. Consider the first-order autoregressive process $\{X_t\}$, $t = 1, 2, \ldots,$ defined by

$$X_t = \beta X_{t-1} + \varepsilon_t, \qquad X_0 = 0, \tag{1.1}$$

where $\{\varepsilon_t\}$ are independent $N(0,1)$ random variables. The least squares estimator $\hat{\beta}_n$ of β, based on a sample of n observations $(X_1, \ldots, X_n)$, is given by

$$\hat{\beta}_n = \left\{ \sum_{t=1}^{n} X_t X_{t-1} \right\} \left\{ \sum_{t=1}^{n} X_{t-1}^2 \right\}^{-1}. \tag{1.2}$$

The asymptotic validity of the bootstrap estimator corresponding to $\hat{\beta}_n$ for the stationary case, viz., $|\beta| < 1$, follows from the work of Bose (1988), and the validity for the explosive case $|\beta| > 1$ has recently been established by Basawa, Mallik, McCormick and Taylor (1989). Both these papers consider the general case when the distribution of $\{\varepsilon_t\}$ is not necessarily known. The limit distribution of $\hat{\beta}_n$ in the unstable case $|\beta| = 1$ is known to be nonnormal with a complicated density. See, for instance, Rao (1978). It is therefore of special interest to consider the bootstrap approximation for the distribution of $\hat{\beta}_n$ for the unstable case. We shall show that the standard bootstrap fails in the unstable case, even if we assume the error distribution to be known (normal). We show that the conditional limit distribution of the bootstrap estimator converges to a random distribution when $\beta = 1$. The case $\beta = -1$ can be treated in a similar fashion since the distribution for $\beta = -1$ is a mirror image of that for $\beta = 1$.

In a different context involving estimation of the eigenvalues of a covariance matrix, Beran and Srivastava (1985) have noted that the standard bootstrap

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validity breaks down when the multiplicities of the eigenvalues exceed unity. Another instance of the invalidity of the naive bootstrap has been discussed by Athreya (1987) in the context of estimating the mean of a population when the variance is infinite.

2. Invalidity of the bootstrap estimator. Let $Z_n = (\sum_{j=1}^{n} X_{j-1}^2)^{1/2}(\hat{\beta}_n - \beta)$, where $\hat{\beta}_n$ is defined in (1.2). It is well known [see, for instance, Anderson (1959)] that when $\beta = 1$,

$$Z_n \to_d Z = \tfrac{1}{2}\{W^2(1) - 1\}\left\{\int_0^1 W^2(t)\,dt\right\}^{-1/2} \quad \text{as } n \to \infty, \tag{2.1}$$

where $\{W(t)\}$ is a standard Wiener process. The bootstrap sample $\{X_t^*\}$ is obtained recursively from the relation

$$X_t^* = \hat{\beta}_n X_{t-1}^* + \varepsilon_t^*, \qquad X_0^* = 0,$$

where $\{\varepsilon_t^*\}$ constitutes a random sample from $N(0,1)$. The bootstrap estimator $\hat{\beta}_n^*$ of β is then defined as in (1.2) with X's replaced by X^*'s. Let $Z_n^* = [\sum_{j=1}^{n} X_{j-1}^{*2}]^{1/2}(\hat{\beta}_n^* - \hat{\beta}_n)$ denote the bootstrap version of Z_n. It will be shown that Z_n and Z_n^* do not have the same limit distribution, thus invalidating the bootstrap. To that end consider a triangular array $\{X_{k,n},\ k \geq 1,\ n \geq 1\}$ satisfying

$$X_{k,n} = b_n X_{k-1,n} + \varepsilon_k, \qquad X_0 = 0, \tag{2.2}$$

with independent $\varepsilon_k \sim N(0,1)$ and where $\{b_n\}$ is a sequence of numbers such that $n(b_n - 1) \to \gamma$. Let

$$\Psi(\gamma) = \frac{\int_0^1 (1 - t + te^{-2\gamma})^{-1} W(t)\,dW(t)}{\left\{\int_0^1 (1 - t + te^{-2\gamma})^{-2} W^2(t)\,dt\right\}^{1/2}}, \tag{2.3}$$

where $\{W(t)\colon 0 \leq t \leq 1\}$ is a standard Brownian motion and

$$H(\gamma, x) = P(\Psi(\gamma) \leq x). \tag{2.4}$$

Then by Theorem 1 of Chan and Wei (1987) we have

$$\lim_{n\to\infty} P_{b_n}(\tau_n \leq x) = H(\gamma, x), \tag{2.5}$$

where

$$\tau_n = \left(\sum_{k=1}^{n} X_{k-1,n}^2\right)^{1/2}\left(\frac{\sum_{k=1}^{n} X_{k,n} X_{k-1,n}}{\sum_{k=1}^{n} X_{k-1,n}^2} - b_n\right)$$

and where P_{b_n} signifies the distribution induced by the model in (2.2).

Define

$$H_n(\hat{\beta}_n, x) = P\left\{Z_n^* \leq x \middle| X_1, \dots, X_n\right\} \tag{2.6}$$

which is taken to be a regular conditional probability distribution function.

Therefore we may define a random measure

$$\eta_n(A) = \int_A H_n(\hat{\beta}_n, dx).$$

Since $H(\gamma, x)$ given in (2.4) is continuous in γ for each fixed x, we have

$$\eta(A) = \int_A H(Z', dx),$$

where Z' is the random variable defined in (2.1) with the exponent $-\frac{1}{2}$ for the second bracket replaced by -1, also defines a random measure. We refer to Kallenberg (1975) as a basic reference on random measures and particularly, for criterion for weak convergence of random measures.

If the bootstrap approximation were valid then along almost all paths H_n given in (2.6) would converge in distribution to the distribution of Z given in (2.1). However, we have in fact that

$$\eta_n \Rightarrow \eta \quad \text{as } n \to \infty \tag{2.7}$$

in $M_p(R)$, the space of probability measures on $\mathbb{R}$ topologized by weak convergence. Indeed, by almost sure representations of convergent laws, it is possible to define $\tilde{\beta}_n$, $n \geq 1$, and $\tilde{Z}$ with $\hat{\beta}_n =_d \tilde{\beta}_n$, $Z' =_d \tilde{Z}$ and

$$n(\tilde{\beta}_n - 1) \to \tilde{Z} \quad \text{a.s. as } n \to \infty. \tag{2.8}$$

See, for example, Billingsley (1971), Theorem 3.3, page 7, and recall [Anderson (1959)] that

$$n(\hat{\beta}_n - 1) \to_d Z' \quad \text{as } n \to \infty.$$

Therefore by (2.5) and (2.8) we have

$$H_n(\tilde{\beta}_n, x) \to H(\tilde{Z}, x) \quad \text{a.s. as } n \to \infty. \tag{2.9}$$

Hence for sets of the form

$$A_j = \bigcup_{i=1}^{m_j} (x_{ij}, y_{ij})$$

representing a disjoint union of intervals, (2.9) implies that

$$\left(H_n(\tilde{\beta}_n, A_1), \dots, H_n(\tilde{\beta}_n, A_k)\right) \to \left(H(\tilde{Z}, A_1), \dots, H(\tilde{Z}, A_k)\right) \quad \text{a.s. as } n \to \infty. \tag{2.10}$$

Since

$$(\eta_n(A_1), \dots, \eta_n(A_k)) =_d \left(H_n(\tilde{\beta}_n, A_1), \dots, H_n(\tilde{\beta}_n, A_k)\right)$$

and

$$(\eta(A_1), \dots, \eta(A_k)) =_d \left(H(\tilde{Z}, A_1), \dots, H(\tilde{Z}, A_k)\right),$$

(2.7) follows from (2.10).

REMARK. A similar invalidity of the bootstrap procedure occurs when considering the distribution of $n(\hat{\beta}_n - \beta)$. An analysis similar to that in Chan and Wei (1987) determines the limit corresponding to (2.5) of $n(\sum_{k=1}^{n} X_{k,n} X_{k-1,n} / \sum_{k=1}^{n} X_{k-1,n}^2 - b_n)$. Details may be found in Basawa, Mallik, McCormick, Reeves and Taylor (1990).

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