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A NOTE ON THE MINIMAX ESTIMATION OF THE POISSON INTENSITY FUNCTION[1]

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The empirical intensity function of the Poisson point process is shown to be a nonparametric minimax estimator for a weighted quadratic loss function.

1. Introduction and summary. In this article, we will consider the minimax estimation of a Poisson intensity function based on n independent realizations from the Poisson point process. The technique we use is the frequently applied criterion of constant risk given by Theorem 2.11.2 of Ferguson (1967). The applicability of this criterion depends on the existence of a Bayes solution. Our approach is nonparametric and the results here parallel the developments of Ferguson (1973) and Phadia (1973). We state their results briefly.

In Ferguson (1973), a nonparametric Bayes estimate of the cumulative distribution function with respect to the Dirichlet prior probability and a quadratic loss function was derived. Subsequently, using the constant risk criterion, several minimax estimators of the cumulative distribution function for different loss functions were obtained in Phadia (1973). Since estimation of the Poisson intensity function and estimation of the cumulative distribution function are similar problems, a nonparametric minimax estimator of the Poisson intensity function can be obtained analogously provided that the Bayes solution of the problem of estimating the Poisson intensity function is available. Exploiting the constant risk criterion, a nonparametric minimax estimator is derived based on the Bayes solution in Lo (1982).

In Section 2, we derive the minimax estimator of the Poisson intensity function for a class of loss functions.

In the sequel $N_i = \{N_i(t); t \in R\}$, $i = 1, \cdots, n$, denotes n independent realizations from a Poisson point process.

2. Nonparametric minimax estimation of the Poisson intensity function. The parameter space Ω is the space of nondecreasing right continuous functions $\mu(t)$, $t \in R$, such that $\mu(\infty) < \infty$. Let the action space be identical to the parameter space. The loss function is taken to be

$$L(\mu, \hat{\mu}) = \int_R \frac{\{\mu(t) - \hat{\mu}(t)\}^2}{\mu(t)} W(dt), \tag{2.1}$$

where W is a fixed member of Ω. We consider the following estimator of μ,

$$\hat{\hat{\mu}}(t) = \frac{1}{n} \sum_{i=1}^{n} N_i(t), \; t \in R. \tag{2.2}$$

We call $\hat{\hat{\mu}}(t)$ the empirical intensity function.

PROPOSITION 2.1.

$$E_\mu L(\mu, \hat{\hat{\mu}}) = \frac{1}{n} \int_R W(dt).$$

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PROOF. This is a simple consequence of the facts that for $1 \le i \ne j \le n$

$$E_\mu N_i(t) = \mu(t), \quad E_\mu N_i(t) N_j(t) = \mu^2(t), \quad E_\mu N_i^2(t) = \mu^2(t) + \mu(t).$$

The following theorem states the main result.

THEOREM 2.1. *If $N_1, N_2, \cdots, N_n$ is a sample of size n from a Poisson point process with intensity function μ, the empirical intensity function defined by*

$$\hat{\mu}(t) = \frac{1}{n} \sum_{i=1}^n N_i(t), \qquad t \in R,$$

is a minimax estimator of μ with respect to the loss function L in (2.1), *the parameter space Ω and the action space identical to Ω.*

PROOF. Using the notations in Lo (1982), let $\mathscr{P}_{\alpha_k, \beta}$ be a weighted Gamma probability on Ω with $\beta > 0$, and $\alpha_k = \bar{c}\delta_{-k}$ where k is a positive integer and $\bar{c} > 1$. We compute the Bayes risk with respect to the prior law $\mathscr{P}_{\alpha_k, \beta}$ on Ω and the loss function $L(\mu, \hat{\mu})$ as follows. Denote the Bayes risk by

$$EL(\mu, \hat{\mu}) = \int_R E \frac{\{\mu(t) - \hat{\mu}(t)\}^2}{\mu(t)} W(dt).$$

On the set of t such that $\alpha_k(t) > 1$, the above quantity can be reduced to

$$(2.3) \quad \int_R \int_{N^n} \left[\left(\frac{\beta}{1 + n\beta}\right)\{\alpha_k(t) - 1 + \Sigma N_i(t)\}\right]^{-1} \int_\Omega \{\mu(t) - \hat{\mu}(t)\}^2 \mathscr{P}_{\alpha_k - 1 + \Sigma N_i, \beta(1+n\beta)^{-1}}(d\mu) Q(d\mathbf{N}) W(dt),$$

where N^n is the n-fold product space of the Poisson point process and $Q(d\mathbf{N})$ is the marginal distribution of the n point processes; that $Q(d\mathbf{N})$ is the joint distribution of n negative binomial point processes is shown in Example 3.1 of Lo (1982). This means that the Bayes estimate $\hat{\mu}$ is given by the posterior mean, where α_k becomes $\alpha_k - 1$; i.e.

$$\hat{\mu}(t) = \begin{cases} 0 & \text{if } \alpha_k(t) = 0 \text{ (iff } t < -k), \\ [\alpha_k(t) - 1 + \Sigma N_i(t)]\left(\dfrac{\beta}{1 + n\beta}\right), & \text{if } \alpha_k(t) > 0 \text{ (iff } -k \le t). \end{cases}$$

Note also that the innermost integral of (2.3) becomes the posterior variance after α_k is updated to $\alpha_k - 1$. Thus, the innermost integral of (2.3) becomes, via Proposition 2.1 in Lo (1982),

$$\int_\Omega \{\mu(t) - \hat{\mu}(t)\}^2 \mathscr{P}_{\alpha_k - 1 + \Sigma N_i, \beta(1+n\beta)^{-1}}(d\mu) = \{\alpha_k(t) - 1 + \Sigma N_i(t)\}\left(\frac{\beta}{1 + n\beta}\right)^2, \quad -k \le t.$$

The next step is to evaluate the second integral of (2.3). Note that, if $-k \le t$, the second integral becomes

$$\int_{N^n} \left[\left(\frac{\beta}{1 + n\beta}\right)\{\alpha_k(t) - 1 + \Sigma N_i(t)\}\right]^{-1} \{\alpha_k(t) - 1 + \Sigma N_i(t)\}\left(\frac{\beta}{1 + n\beta}\right)^2 Q(d\mathbf{N}) = \frac{\beta}{1 + n\beta}.$$

Thus, the Bayes risk becomes

$$\frac{\beta}{1 + n\beta} \int_{-k}^{\infty} W(dt),$$

and with $\beta = k$, as $k \to \infty$ this converges upward to

$$\frac{1}{n}\int_R W(dt),$$

which is the constant risk of the empirical intensity function. We have proved the theorem.

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