The Annals of Statistics
1980, Vol. 8, No. 5, 1177–1178

CORRECTION TO ON A CRITERION FOR SIMULTANEOUS EXTRAPOLATION IN NONFULL RANK NORMAL REGRESSION

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In O'Reilly (1976), an incomplete proof of Theorem 3.1 was given. No analysis was given of what is termed here the second case; namely that the matrix $I - U'(X'X)^{-1}U$ is positive semidefinite of rank greater than $n - p$. For ease of presentation, the theorem is stated again and the proof given.

THEOREM 3.1. *In the full rank normal regression model, extrapolation at U is valid iff $U \in S$.*

PROOF. Sufficiency is obvious. To show necessity assume $U \notin S$, and also assume $\exists \mathbf{h}(Y_1, Y_2, \cdots, Y_n) \sim N(U'\boldsymbol{\beta}, \sigma^2 I)$. Since $E(\mathbf{h}) = U'\boldsymbol{\beta}$ and T is complete, it follows that $E(\mathbf{h}|T) = U'\boldsymbol{\beta}$ a.s.. Because $U \notin S$, two cases need to be considered; the first one when $I - U'(X'X)^{-1}U$ is not positive semidefinite and the second one when it is positive semidefinite but with rank greater than $n - p$.

In the first case it suffices to observe that

$$V(\mathbf{h}) - V\{E(\mathbf{h}|T)\} = \sigma^2\big(I - U'(X'X)^{-1}U\big),$$

thus the nonpositive semidefiniteness contradicts the Rao-Blackwell theorem.

In the second case, let ν stand for the rank of $I - U'(X'X)^{-1}U$ and consider the difference $\mathbf{h} - E(\mathbf{h}|T)$ whose mean is $\mathbf{0}$ and whose variance covariance matrix is $\sigma^2(I - U'(X'X)^{-1}U)$. If it can be shown that the distribution of this difference is normal, then with a suitable diagonalization one can produce an unbiased estimator of σ^2. This estimator would be chi-square distributed with ν degrees of freedom and thus, with variance $2\sigma^4/\nu < 2\sigma^4/(n-p) = V(\hat{\sigma}^2)$, the variance of the MVUE of σ^2, which is absurd. So it suffices to show normality of $\mathbf{h} - E(\mathbf{h}|T)$.

Let $\boldsymbol{\lambda}'\{\mathbf{h} - E(\mathbf{h}|T)\}$ be an arbitrary linear combination. If $\boldsymbol{\lambda}'U'(X'X)^{-1}U\boldsymbol{\lambda} = 0$, it means that $\boldsymbol{\lambda}'E(\mathbf{h}|T) = 0$ and thus normality follows. If $\boldsymbol{\lambda}'(I - U'(X'X)^{-1}U)\boldsymbol{\lambda} = 0$ the distribution is degenerate, so the only case of interest is when $\boldsymbol{\lambda}'(I - U'(X'X)^{-1}U)\boldsymbol{\lambda} \in (0, 1)$. Without loss of generality let $\boldsymbol{\lambda}$ have norm 1 and denote $\boldsymbol{\lambda}'U'(X'X)^{-1}U\boldsymbol{\lambda}$ by ρ.

In order to show normality of $\boldsymbol{\lambda}'\mathbf{h} - \boldsymbol{\lambda}'E(\mathbf{h}|T)$ for $\boldsymbol{\lambda}$ selected as above, let $W = \boldsymbol{\lambda}'\mathbf{h}$, $Wo = \boldsymbol{\lambda}'E(\mathbf{h}|T)$ which are normally (but not independently) distributed with distributions $N(\mu, \sigma^2)$ and $N(\mu, \rho\sigma^2)$ respectively, where μ stands for $\boldsymbol{\lambda}'U'\boldsymbol{\beta}$. Denote by ϕ_{w-wo}, ϕ_w and ϕ_{wo} the characteristic functions of $W - Wo$, W and Wo and also let ψ stand for the characteristic function of a $N(0, \sigma^2(1-\rho))$. It will be shown that $\phi_{w-wo}(t) = \psi(t)$ for all t.

Received July 1979.

Since

$$\phi_w(t) = \exp\left(i\mu t - \tfrac{1}{2}t^2\sigma^2\right)$$

$$\phi_{wo}(t) = \exp\left(i\mu t - \tfrac{1}{2}t^2\rho\sigma^2\right)$$

and

$$\psi(t) = \exp\left(-\tfrac{1}{2}t^2(1-\rho)\sigma^2\right),$$

it follows that

$$\phi_{wo}(t) = \phi_w(t)\{\psi(t)\}^{-1}, \tag{1}$$

which means that $\exp(it\,Wo)$, which is a function of $\hat{\boldsymbol{\beta}}$, is an unbiased estimator of (1). On the other hand, observe that $\psi(t)$ is unbiasedly estimable independently of $\hat{\boldsymbol{\beta}}$, for let $\boldsymbol{\delta}$ be such that $\boldsymbol{\delta}'(I - X(X'X)^{-1}X')\boldsymbol{\delta} = 1 - \rho$, and define

$$\mathfrak{R}(\hat{\sigma}^2) = E\left\{\exp\left(it\boldsymbol{\delta}'(\mathbf{Y} - X\hat{\boldsymbol{\beta}})|\hat{\sigma}^2\right\}. \tag{2}$$

From (1) and (2), using the independence of $\hat{\boldsymbol{\beta}}$ and $\hat{\sigma}^2$, it follows that

$$E\left\{\mathfrak{R}(\hat{\sigma}^2)\exp(it\,Wo)\right\} = \phi_w(t). \tag{3}$$

In fact, due to the completeness of T, which is equivalent to $(\hat{\boldsymbol{\beta}}, \hat{\sigma}^2)$, $\mathfrak{R}(\hat{\sigma}^2)$ $\exp(it\,Wo)$ coincides a.s. with the MVUE of $\phi_w(t)$, i.e.,

$$E\left\{\exp(it\,W)|\hat{\boldsymbol{\beta}}, \hat{\sigma}^2\right\} = \mathfrak{R}(\hat{\sigma}^2)\exp(it\,Wo) \qquad \text{a.s.} \tag{4}$$

Taking conditional expectations given Wo in both sides of (4), one gets,

$$E\left\{\exp(it\,W)|Wo\right\} = \psi(t)\exp(it\,Wo) \qquad \text{a.s.} \tag{5}$$

which can be written as,

$$E\left\{\exp(it(W - Wo))|Wo\right\} = \psi(t) \qquad \text{a.s.} \tag{6}$$

Finally, taking expectations on (6), the desired result $\phi_{w-wo}(t) = \psi(t)$ is obtained.

REFERENCES

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