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(k − 1)- MEAN SIGNIFICANCE LEVELS OF NONPARAMETRIC MULTIPLE COMPARISONS PROCEDURES

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We consider the nonparametric pairwise comparisons procedures derived from the Kruskal-Wallis k-sample test and from Friedman's test. For large samples the $(k-1)$-mean significance level is determined, i.e., the probability of concluding incorrectly that some of the first $k-1$ samples are unequal. We show that in general this probability may be larger than the simultaneous significance level α. Even when the kth sample is a shift of the other $k-1$ samples, it may exceed α, if the distributions are very skew. Here skewness is defined with Van Zwet's c-ordering of distribution functions.

1. Introduction. Consider k samples of size n with continuous distribution functions $F_1, \cdots, F_k$. The projection argument, by which the Scheffé simultaneous confidence intervals are derived from the F statistic, can also be applied to the Kruskal-Wallis statistic (see Miller (1966), page 165–172). This leads to the following pairwise comparisons procedure, proposed by Nemenyi (1963): conclude $F_i \neq F_j$ for large values of $|\bar{R}_i - \bar{R}_j|$, where $\bar{R}_i$ is the mean of the ranks of the ith sample. Throughout this paper we shall assume n to be large (except for Section 8, where finite sample studies are treated) and under the null hypothesis H_0: $F_1 = \cdots = F_k$ we have for $n \to \infty$

$$(1.1)\qquad P\Big[\max_{1 \leqslant i,j \leqslant k} |\bar{R}_i - \bar{R}_j| < q_k^\alpha \{k(kn+1)/12\}^{\frac{1}{2}}\Big] = 1 - \alpha,$$

where q_k^α is the upper α point of the distribution of the range of k independent standard normal variables. So for large n the procedure prescribes

$$(1.2)\qquad \text{conclude } F_i \neq F_j \quad \text{if} \quad |\bar{R}_i - \bar{R}_j| \geqslant q_k^\alpha \{k(kn+1)/12\}^{\frac{1}{2}}$$

and the simultaneous significance level (sometimes called experimentwise error rate) is approximately equal to α.

We shall be concerned with the following problem: if H_0 is not valid, but $F_1 = \cdots = F_{k-1} = F$ and $F_k = G$, what will be the value of $\alpha(F, G)$, defined by

$$(1.3)\quad \alpha(F,G) = \lim_{n\to\infty} P\Big[\max_{1 \leqslant i,j \leqslant k-1} |\bar{R}_i - \bar{R}_j| \geqslant q_k^\alpha \{k(kn+1)/12\}^{\frac{1}{2}}\Big],$$

i.e., what is the probability of concluding incorrectly that some of $F_1, \cdots, F_{k-1}$ are different? Usually this probability is called the $(k-1)$-*mean significance level*. It is clear that it depends also on G, as the distributions of $\bar{R}_i$ and $\bar{R}_j$ $(1 \leqslant i, j \leqslant k-1)$ depend on F_k. Dunn (1964) computed $\alpha(F, G)$ for the situation that the

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distributions F and G have no overlap at all. We shall compare $\alpha(F, G)$ with α, as we require that probabilities on type I errors should not exceed the overall significance level α.

In Sections 3 and 4 we shall see that there exist pairs (F, G) such that $\alpha(F, G)$ is larger than α, even when G is a shift of F. In Section 4 and later sections only shift alternatives are regarded and it turns out that $\alpha(F)$, defined by $\alpha(F) = \sup_{a \in \mathbf{R}} \alpha(F, F(. - a))$, is larger than α only if F is very skewed. Here skewness will be defined with the c-comparison of distribution functions, introduced by Van Zwet (1964). If F is less skewed than the exponential distribution, that is, $\log F$ and $\log(1 - F)$ both concave, then $\alpha(F) \leqslant \alpha$ (Section 6).

If block effects are present, a similar multiple comparisons procedure can be derived from Friedman's test (see Miller (1966), page 172-178). Here the situation is quite similar to the previous one; the $(k - 1)$-mean significance level may be larger than α, and more specifically, $\alpha^*(F)$ is larger as F is more skewed (Section 7).

An auxiliary result which we shall prove is the following one (see Section 5). Let X have distribution function F and define

$$v(F) = \sup_{a \in \mathbf{R}} \operatorname{Var} F(X - a) \tag{1.4}$$
$$c(F) = \sup_{a \in \mathbf{R}} \operatorname{Cov}(F(X), F(X - a)).$$

Then we have:

If F_2 is more skewed than F_1, then $v(F_2) \geqslant v(F_1)$ and $c(F_2) \geqslant c(F_1)$.

The problem, that the distribution of $\bar{R}_i - \bar{R}_j$ is affected by the distributions of the other samples, has already been noticed by Miller (1966, page 168) and also by Gabriel (1969, Example 2.3). For that reason Miller recommends the use of the alternative nonparametric method, proposed by Steel (1960). Here the pairwise comparisons are based on pairwise ranking, so that the whole problem disappears and thus the $(k - 1)$-mean significance level is automatically smaller than α.

A recent paper of Koziol and Reid (1977) deals with the methods of Nemenyi and Steel in another context.

2. Another expression for $\alpha(F, G)$. Up to and including Section 6 we shall consider the case where no blocks are present, so let $X_{11}, \cdots, X_{1n}; \cdots; X_{k1}, \cdots, X_{kn}$ be independent random variables ($k \geqslant 3$), where X_{ij} has a continuous distribution function F_i. Let R_{ij} denote the rank of X_{ij} among all observations and define $\bar{R}_i$ by $\bar{R}_i = n^{-1}\Sigma_{j=1}^n R_{ij}$.

In order to determine $\alpha(F, G)$, we first must know the asymptotic distribution of the range of $\bar{R}_1, \cdots, \bar{R}_{k-1}$ for the case $F_1 = \cdots F_{k-1} = F$ and $F_k = G$. Using Theorem 2.1 of Hájek (1968), one can easily prove the asymptotic normality of the vector $(\bar{R}_1, \cdots, \bar{R}_{k-1})$ under this alternative (the proof is omitted here).

If we define p, q and r by

$$p = \int G dF$$
$$q = \int G^2 dF \tag{2.1}$$
$$r = \int FG dF,$$

then, after a tedious computation, the following relationships can be found for $1 \leqslant i, j \leqslant k - 1$:

$$\mathcal{E}\bar{R}_i = \tfrac{1}{2}(kn + 1) + \left(p - \tfrac{1}{2}\right)n \tag{2.2}$$

$$\begin{aligned} \operatorname{Var} \bar{R}_i = {} & \tfrac{1}{12}k^2 n + \left(2r - p - \tfrac{1}{4}\right)kn + \left(4p - 2p^2 + q - 6r + \tfrac{1}{6}\right)n \\ & + \tfrac{1}{12}k - p + p^2 - q + 2r - \tfrac{1}{6} \end{aligned} \tag{2.3}$$

$$\operatorname{Cov}\left(\bar{R}_i, \bar{R}_j\right) = -\tfrac{1}{12}kn + \left(3p - p^2 - 4r + \tfrac{1}{12}\right)n - \tfrac{1}{12}. \tag{2.4}$$

So $n^{-\frac{1}{2}}(\bar{R}_1, \cdots, \bar{R}_{k-1})$ has an asymptotically normal distribution with covariance matrix

$$\begin{bmatrix} a_1 & a_2 & \cdot & \cdot & \cdot & \cdot & \cdot & a_2 \\ & a_1 & & & & & & \\ & & \cdot & & & & & \\ & & & \cdot & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & a_2 \\ & & & & & & & a_1 \end{bmatrix},$$

where $a_1 = k^2/12 + (2r - p - \frac{1}{4})k + 4p - 2p^2 + q - 6r + \frac{1}{6}$ and $a_2 = -k/12 + 3p - p^2 - 4r + \frac{1}{12}$.

If we define (see also Miller (1966), page 46) $\bar{R} = (k-1)^{-1}\Sigma_{i=1}^{k-1}\bar{R}_i$ and $\gamma = 1 \pm \{(a_1 - a_2)/(a_1 + (k-2)a_2)\}^{\frac{1}{2}}$, then $n^{-\frac{1}{2}}(\bar{R}_1 - \gamma\bar{R}, \cdots, \bar{R}_{k-1} - \gamma\bar{R})$ has an asymptotically normal distribution with covariance matrix $(a_1 - a_2)I_{k-1}$ (where I_{k-1} denotes the identity matrix of size $k - 1$). If we set $b = a_1 - a_2$, then we have found that the range of $(nb)^{-\frac{1}{2}}\bar{R}_1, \cdots, (nb)^{-\frac{1}{2}}\bar{R}_{k-1}$ has asymptotically the same distribution as the range of $k - 1$ independent standard normal random variables. Henceforth this last range will be denoted by Q_{k-1}. Since b depends on F and G, we shall write $b(F, G)$ and we may conclude

$$\alpha(F, G) = P\left[Q_{k-1} > q_k^\alpha\{k^2/12b(F, G)\}^{\frac{1}{2}}\right], \tag{2.5}$$

where

$$b(F,G) = k^2/12 + \left(2r - p - \tfrac{1}{6}\right)(k - 1) + q - p^2 - \tfrac{1}{12}. \tag{2.6}$$

Remarks.

1. If X has distribution function F, then:

$$\begin{aligned} 2r - p &= 2\operatorname{Cov}(F(X), G(X)) \\ q - p^2 &= \operatorname{Var} G(X). \end{aligned} \tag{2.7}$$

2. If $F = G$, then $b(F, G) = k^2/12$, so under H_0 we (naturally) have $\alpha(F, G) \leqslant \alpha$.

3. Maximum of $\alpha(F, G)$. Now we shall compute the maximum value of $\alpha(F, G)$, and we want to know whether it is larger than α. Note that this may depend on k and α.

From (2.5) we see that $\alpha(F, G)$ is maximal when $b(F, G)$ is maximal. Writing

$$(3.1) \qquad 2r - p = \int (2F - 1) G dF,$$

we see that $2r - p$ is maximal if F and G satisfy the following two conditions:

$$(3.2) \qquad \begin{aligned} &\text{if } F(x) < \tfrac{1}{2}, && \text{then } G(x) = 0, \text{ and} \\ &\text{if } F(x) > \tfrac{1}{2}, && \text{then } G(x) = 1; \end{aligned}$$

that is, $F = \frac{1}{2}$ on the support of G. Now it happens that $q - p^2$ is maximized by the same pairs (F, G), so from (2.5) and (2.6) it follows that $\alpha(F, G)$ is maximal for the pairs (F, G) satisfying (3.2). As for these pairs $2r - p$ and $q - p^2$ are both equal to $1/4$, we conclude that the maximum value of $\alpha(F, G)$ is equal to

$$P\Big[Q_{k-1} > q_k^\alpha \{ k^2 / (k^2 + k + 1) \}^{\frac{1}{2}} \Big].$$

With the aid of a table of the cdf of the range of independent standard normal variables (e.g., Harter (1969)), we can find these values for several values of k and α. From Table 3.1 we see that in general max $\alpha(F, G)$ is larger than α.

TABLE 3.1
Maximum values of $\alpha(F, G)$ for α = .01, .025, .05 and .10

	$k = 3$	4	5	6	7	8	9	10	12	15	20
$\alpha = .01$	.0153	.0181	.0182	.0178	.0172	.0167	.0162	.0158	.0151	.0143	.0134
.025	.0303	.0361	.0386	.0385	.0379	.0372	.0365	.0358	.0347	.0334	.0318
.05	.0512	.0643	.0682	.0690	.0688	.0682	.0674	.0667	.0652	.0633	.0612
.10	.0877	.1123	.1208	.1240	.1250	.1250	.1245	.1238	.1224	.1202	.1172

REMARK. If we keep in mind that $b(F, G) = \frac{1}{2}\lim_{n\to\infty} \operatorname{Var} n^{-\frac{1}{2}}(\bar{R}_i - \bar{R}_j)(1 \leq i, j \leq k - 1)$, then it is also clear intuitively, that $b(F, G)$ is maximal if F and G satisfy (3.2), since in that case the kth sample is expected to receive the midranks.

4. Shift altenatives. From this moment we shall consider only pairs (F, G) for which there exists an $a \in \mathbb{R}$ such that

$$(4.1) \qquad G(x) = F(x - a) \quad \text{for all } x \in \mathbb{R},$$

and again we ask ourselves whether $\alpha(F, G)$ may be larger than α. As now $\alpha(F, G)$ and $b(F,G)$ in fact depend on F and a, we shall modify our notation: $\alpha(F, a) = \alpha(F, G)$, $b(F, a) = b(F, G)$, where G is given by (4.1).

If X has distribution function F, then we define

$$(4.2) \qquad c(F,a) = \operatorname{Cov}(F(X), F(X - a)) = \int \big(F(x) - \tfrac{1}{2}\big) F(x - a) dF(x),$$

$$(4.3) \qquad v(F,a) = \operatorname{Var} F(X - a).$$

Now we can rewrite (2.5) and (2.6):

$$(4.4) \qquad \alpha(F,a) = P\Big\{ Q_{k-1} > q_k^\alpha (k^2/12 b(F,a))^{\frac{1}{2}} \Big\},$$

where

$$(4.5) \qquad b(F,a) = \tfrac{1}{12}k^2 + \big(2c(F,a) - \tfrac{1}{6}\big)(k - 1) + v(F,a) - \tfrac{1}{12}.$$

Furthermore we define

$$\alpha(F) = \sup_{a \in \mathbb{R}} \alpha(F,a) \tag{4.6}$$

and $b(F)$, $c(F)$ and $v(F)$ analogously (see also (1.4)).

First we try to maximize $c(F,a)$ over F and a. Suppose $a > 0$. Then $F(x - a) \leqslant F(x)$ for all $x \in \mathbb{R}$, and consequently,

$$\begin{aligned} c(F,a) &\leqslant \int_{\{x \mid F(x) > \frac{1}{2}\}} \left(F(x) - \tfrac{1}{2}\right) F(x-a)\, dF(x) \\ &\leqslant \int_{\{F > \frac{1}{2}\}} \left(F^2 - \tfrac{1}{2}F\right) dF = \tfrac{5}{48}. \end{aligned} \tag{4.7}$$

If $a < 0$, then also $c(F,a) < \frac{5}{48}$ for all F. On the other hand, $\frac{5}{48}$ tuns out to be the lowest upperbound, since for F_m defined below in (4.8), we have $c(F_m, \frac{1}{2}) = \frac{5}{48} - \mathcal{O}(m^{-1})$.

$$\begin{aligned} F_m(x) &= x + \tfrac{1}{2} && \text{if } -\tfrac{1}{2} \leqslant x \leqslant 0, \\ &= \frac{x}{m} + \tfrac{1}{2} && \text{if } 0 \leqslant x \leqslant \frac{m}{2}. \end{aligned} \tag{4.8}$$

Furthermore we have that $\lim_{m \to \infty} v(F_m, \frac{1}{2}) = \frac{29}{192}$, and hence by (4.5)

$$\sup_{F,a} b(F,a) \geqslant \tfrac{1}{12}\left(k^2 + \tfrac{1}{2}k + \tfrac{5}{16},\right) \tag{4.9}$$

which implies

$$\sup_F \alpha(F) \geqslant P\left[Q_{k-1} > q_k^\alpha \left\{ k^2 / \left(k^2 + \tfrac{1}{2}k + \tfrac{5}{16}\right)\right\}^{\frac{1}{2}} \right]. \tag{4.10}$$

TABLE 4.1

Lower bounds for $\sup_F \alpha(F)$.

	$k = 3$	4	5	6	7	8	9	10	12	15	20
$\alpha = .01$	.0079	.0101	.0109	.0113	.0114	.0114	.0114	.0114	.0114	.0112	.0111
.25	.0175	.0230	.0253	.0263	.0268	.0271	.0273	.0273	.0273	.0272	.0270
.05	.0325	.0431	.0478	.0501	.0514	.0521	.0526	.0529	.0531	.0532	.0530
.10	.0612	.0816	.0909	.0958	.0987	.1005	.1019	.1025	.1034	.1039	.1041

From Table 4.1 we see that $\sup_F \alpha(F)$ is larger than α for several values of α and k. However the exceedances, if any, are rather small, much smaller than in the general case treated in Section 3.

It should be noticed here that (see *Statistica Neerlandica* (1977), page 189-191, solution of problem nr. 45)

$$\sup_{F,a} v(F,a) = \left(3 - (5)^{\frac{1}{2}}\right)\tfrac{5}{24}, \tag{4.11}$$

which value is reached (for $m \to \infty$) by the same F_m of (4.8) but for $a \neq \frac{1}{2}$. However, the value in (4.11) only slightly exceeds $\frac{29}{192}$ and moreover $2(k-1) \cdot c(F,a)$ is the dominant term in (4.5), so (4.10) is almost an equality, especially for k not too small. Consequently the lower bounds in Table 4.1 are practically equal to $\sup_F \alpha(F)$.

The next question is, which conditions on F are sufficient to guarantee $\alpha(F) \leqslant \alpha$? The first result stated here is due to Professor R. Doornbos.

THEOREM 4.1. *If F is symmetrical and unimodal, then $c(F) \leqslant \frac{1}{12}$, and hence $\alpha(F) \leqslant \alpha$ for the usual values of α and k.*

SHORT PROOF. Combining $c(F) \leqslant \frac{1}{12}$ (proof omitted here) with (4.11), one will see that in (4.4) $b(F,a)$ is not large enough to make $q_k^\alpha(k^2/12b(F,a))^{\frac{1}{2}}$ smaller than q_{k-1}^α. ▯

We would like to relax the conditions on F in Theorem 4.1, especially since the symmetry is often not fulfilled in practice. However, unimodality alone is not sufficient to ensure $\alpha(F) \leqslant \alpha$, since F_m of (4.8) is also unimodal. Theorem 4.1, together with the extreme skewness of F_m, may suggest that $\alpha(F)$ is larger when F is more skewed. In the next sections we shall see that this guess puts us on the right track. Here skewness will not be the normed third moment, but it is defined with the c-comparison, introduced by Van Zwet (1964).

5. Skewness and its relation to $c(F)$ and $v(F)$. We shall confine ourselves to the class $\mathcal{F}$ of continuous distribution functions F, for which there exists a finite or infinite interval $I_F = (x_1,x_2)$ such that the following three conditions are satisfied:

$$F(x_2) - F(x_1) = 1, \tag{5.1}$$

$$F \text{ is differentiable on } I_F, \tag{5.2}$$

$$F' > 0 \text{ on } I_F. \tag{5.3}$$

On this class $\mathcal{F}$ a weak order relation is defined, which is called the *c-comparison.*

DEFINITION 5.1. If $F_1,F_2 \in \mathcal{F}$, then $F_1 <_c F_2$ iff $F_2^{-1}F_1$ convex on I_{F_1}.

$F_1 <_c F_2$ should be interpreted as F_2 is more skewed to the right than F_1.

PROPERTY (Lemma 4.1.3, Van Zwet (1964)). If f_1 and f_2 are the densities of F_1 and F_2 respectively, then

$$F_1 <_c F_2 \text{ iff } (F_2^{-1})'/(F_1^{-1})' = f_1(F_1^{-1})/f_2(F_2^{-1}) \text{ is nondecreasing on } (0,1). \tag{5.4}$$

For $F \in \mathcal{F}$ we define $\bar{F} \in \mathcal{F}$ by

$$\bar{F}(x) = 1 - F(-x) \text{ for all } x \in \mathbb{R}. \tag{5.5}$$

Then we can prove the following property:

LEMMA 5.1. *If $F_1,F_2 \in \mathcal{F}$, then $F_1 <_c F_2$* iff *$\bar{F}_2 <_c \bar{F}_1$.*

PROOF.

$$\Rightarrow: F_2^{-1}F_1(-x) \text{ convex in } x \text{ implies } F_1^{-1}F_2(-x) \text{ concave in } x.$$

$$\text{Hence } (\bar{F}_1)^{-1}\bar{F}_2(x) = -F_1^{-1}F_2(-x) \text{ is convex.}$$

$\Leftarrow$: Note that $\bar{\bar{F}} = F$. ▯

Using the c-comparison, we now define skewness on $\mathcal{F}$.

DEFINITION 5.2. F_2 is *more skewed* than F_1 iff $\bar{F}_2 <_c F_1 <_c F_2$ or $F_2 <_c F_1 <_c \bar{F}_2$.

Notice that, if we only have $F_1 <_c F_2$, F_1 still may be very skewed to the left.

Now we want to prove that $c(F)$ and $v(F)$ are increasing according as F is more skewed. But first we have to state two lemmas.

LEMMA 5.2.. *Let f and g be real functions on an interval $I \subset \mathbb{R}$ (g positive), such that f/g is nondecreasing on I. If furthermore $x_1, x_2, x_3, x_4 \in I$, such that $x_1 \leqslant x_3$ and $x_2 \leqslant x_4$, then*

$$\int_{x_1}^{x_2} f / \int_{x_1}^{x_2} g \leqslant \int_{x_3}^{x_4} f / \int_{x_3}^{x_4} g.$$

PROOF. Elementary calculus. □

LEMMA 5.3. *Let f and g be real functions on* $(0,1)$ *such that*:

(i) $\int_0^1 f = \int_0^1 g < \infty$,

(ii) *there exists* $x_0 \in (0,1)$ *such that* $f \leqslant g$ *on* $(0,x_0)$ *and* $f \geqslant g$ *on* $(x_0,1)$.

Then

$$\int_0^1 x f(x)dx \geqslant \int_0^1 x g(x)dx.$$

This lemma is a special case of a theorem due to J. F. Steffenson (see Mitrinović (1970), page 114, Theorem 13).

THEOREM 5.1. *If F_2 is more skewed than F_1 ($F_1, F_2 \in \mathcal{F}$), then*

$$(a)\ c(F_1) \leqslant c(F_2),$$
$$(b)\ v(F_1) \leqslant v(F_2).$$

PROOF. First we shall prove (a). After a change of variables (4.2) gives

$$c(F,a) = \int_0^1 \left(u - \tfrac{1}{2}\right) F\left(F^{-1}(u) - a\right) du. \tag{5.6}$$

Suppose

$$\bar{F}_2 <_c F_1 <_c F_2. \tag{5.7}$$

We shall start with showing

$$\begin{aligned} F_1 <_c F_2 \Rightarrow \sup_{a\in(0,\infty)} &\int_0^1 \left(u - \tfrac{1}{2}\right) F_1\left(F_1^{-1}(u) - a\right) du \\ \leqslant \sup_{a\in(0,\infty)} &\int_0^1 \left(u - \tfrac{1}{2}\right) F_2\left(F_2^{-1}(u) - a\right) du \end{aligned} \tag{5.8}$$

which has been proved if for any $a_1 > 0$ there exists $a_2 > 0$ such that the following two relationships are satisfied:

$$F_1\left(F_1^{-1}(u) - a_1\right) \geqslant F_2\left(F_2^{-1}(u) - a_2\right) \text{ for } u \in \left(0, \tfrac{1}{2}\right), \tag{5.9}$$

$$F_1\left(F_1^{-1}(u) - a_1\right) \leqslant F_2\left(F_2^{-1}(u) - a_2\right) \text{ for } u \in \left(\tfrac{1}{2}, 1\right). \tag{5.10}$$

For this we take a_2 such that we have equalities for $u = \frac{1}{2}$. So

$$a_2 = F_2^{-1}\left(\tfrac{1}{2}\right) - F_2^{-1}\left(F_1\left(F_1^{-1}\left(\tfrac{1}{2}\right) - a_1\right)\right). \tag{5.11}$$

To prove (5.10) we use Lemma 5.2 with $f = (F_2^{-1})'$, $g = (F_1^{-1})'$, $x_1 = F_1(F_1^{-1}(\frac{1}{2}) - a_1)$, $x_2 = \frac{1}{2}$, $x_3 = F_1(F_1^{-1}(u) - a_1)$, $x_4 = u$. Then f/g is nondecreasing because of (5.4) and (5.7). To prove (5.9) we only need an interchangement of x_1 and x_2 and of

x_3 and x_4. Thus (5.8) has been proved. For negative a we have to make use of $\bar{F}_2 <_c f_1$. By Lemma 5.1 this is equivalent to $\bar{F}_1 <_c F_2$, so (5.8) gives

$$\sup_{a\in(0,\infty)}\int_0^1\left(u-\tfrac{1}{2}\right)\bar{F}_1\left(\bar{F}_1^{-1}(u)-a\right)du \leqslant \sup_{a\in(0,\infty)}\int_0^1\left(u-\tfrac{1}{2}\right)F_2\left(F_2^{-1}(u)-a\right)du. \tag{5.12}$$

Using $\bar{F}_1(\bar{F}_1^{-1}(u)-a) = 1 - F_1(F_1^{-1}(1-u)+a)$, we have

$$\int_0^1\left(u-\tfrac{1}{2}\right)\bar{F}_1\left(\bar{F}_1^{-1}(u)-a\right)du = \int_0^1\left(u-\tfrac{1}{2}\right)F_1\left(F_1^{-1}(u)+a\right)du.$$

Hence (5.12) gives

$$\bar{F}_2 <_c F_1 \Rightarrow \sup_{a\in(-\infty,0)}\int_0^1\left(u-\tfrac{1}{2}\right)F_1\left(F_1^{-1}(u)-a\right)du \leqslant \sup_{a\in(0,\infty)}\int_0^1\left(u-\tfrac{1}{2}\right)F_2\left(F_2^{-1}(u)-a\right)du. \tag{5.13}$$

Combining (5.8) and (5.13), we see that (5.7) implies $c(F_1) \leqslant c(F_2)$. This is also implied by $F_2 <_c F_1 <_c \bar{F}_2$, as $c(\bar{F}_2) = c(F_2)$. So the proof of (a) has been completed.

To prove (b), we take random variables X_1 and X_2 with distribution functions F_1 and F_2. As $F_1(X_1-a)$ has distribution function H_1, defined by $H_1(\mathrm{u}) = F_1(F_1^{-1}(u)+a)$, we have

$$\mathcal{E}F_1(X_1-a) = 1 - \int_0^1 H_1(u)du = 1 - \int_0^1 F_1\left(F_1^{-1}(u)+a\right)du, \tag{5.14}$$

$$\mathcal{E}\left\{(F_1(X_1-a))^2\right\} = 1 - 2\int_0^1 uH_1(u)du = 1 - 2\int_0^1 uF_1\left(F_1^{-1}(u)+a\right)du, \tag{5.15}$$

and similarly for $F_2(X_2-a)$. First we prove that $F_1 <_c F_2$ implies that for any $a_1 > 0$ there exists $a_2 \geqslant 0$ such that

$$\operatorname{Var}F_1(X_1-a_1) \leqslant \operatorname{Var}F_2(X_2-a_2). \tag{5.16}$$

For that purpose we take a_2 such that $\mathcal{E}F_1(X_1-a_1) = \mathcal{E}F_2(X_2-a_2)$, that is,

$$\int_0^1 F_1\left(F_1^{-1}(u)+a_1\right)du = \int_0^1 F_2\left(F_2^{-1}(u)+a_2\right)du \tag{5.17}$$

(a_2 exists, since F_1 and F_2 are continuous). Then (5.16) is satisfied if

$$\int_0^1 uF_1\left(F_1^{-1}(u)+a_1\right)du \geqslant \int_0^1 uF_2\left(F_2^{-1}(u)+a_2\right)du. \tag{5.18}$$

This follows from Lemma 5.3 if we substitute

$$f(u) = F_1\left(F_1^{-1}(u)+a_1\right) \text{ and } g(u) = F_2\left(F_2^{-1}(u)+a_2\right).$$

Condition (i) is satisfied by (5.17) and condition (ii) is satisfied because:

(1). According to (5.17) there exists $u_0 \in (0, 1)$ such that $F_1(F_1^{-1}(u_0)+a_1) = F_2(F_2^{-1}(u_0)+a_2)$, as F_1 and F_2 and their inverses are continuous.

(2). As $F_1 <_c F_2$, we can use Lemma 5.2 in the same way as in the proof of part (a) with $\frac{1}{2}$ replaced by u_0. This gives $F_1(F_1^{-1}(u)+a_1) \leqslant F_2(F_2^{-1}(u)+a_2)$ for $u \in (0, u_0)$ and the reverse inequality for $u \in (u_0, 1)$.

Hence we now have

$$F_1 <_c F_2 \Rightarrow \sup_{a\in(0,\infty)}\operatorname{Var}F_1(X_1-a) \leqslant \sup_{a\in(0,\infty)}\operatorname{Var}F_2(X_2-a). \tag{5.19}$$

For negative a again we use $\bar{F}_2 <_c F_1$ (or $\bar{F}_1 <_c F_2$). As $-X_1$ has distribution function $\bar{F}_1$ and furthermore

$$\mathrm{Var}\bar{F}_1(-X_1 - a) = \mathrm{Var}F_1(X_1 + a),$$

we find

$$\bar{F}_2 <_c F_1 \Rightarrow \sup_{a\in(-\infty,0)}\mathrm{Var}F_1(X_1 - a) \leq \sup_{a\in(0,\infty)}\mathrm{Var}F_2(X_2 - a).$$

Together with (5.19) this completes the proof of Theorem 5.1. □

6. Sufficient conditions on F such that $\alpha(F) \leq \alpha$. Now an application of Theorem 5.1 to our multiple comparisons problem is given. Let F_e be the negative exponential distribution (which is rather skewed), so $F_e(x) = 1 - e^{-x}$ ($x > 0$). Since $c(F_e) = \frac{3}{32}$ and $v(F_e) = \frac{1}{9}$, we have by (4.5)

$$b(F_e, a) \leq k^2/12 + \left(2c(F_e) - \tfrac{1}{6}\right)(k-1) + v(F_e) - \tfrac{1}{12} = \left(k^2 + k/4 + \tfrac{1}{12}\right)/12$$

and substituted in (4.4), this gives the upperbounds for $\alpha(F_e)$ in Table 6.1 (see below). In that table we see that $\alpha(F_e)$ is smaller than α for the usual values of α and k. As $F_e \in \mathfrak{F}$, we now have, by Theorem 5.1, that $(k^2 + k/4 + \frac{1}{12})/12$ is also an upperbound for $b(F, a)$, for all $F \in \mathfrak{F}$ which are less skewed than the exponential distribution. Translation of "F less skewed than F_e" gives

THEOREM 6.1. *If* $\log F$ *and* $\log(1 - F)$ *are both concave, then* $\alpha(F) \leq \alpha$ *(for* $\alpha = .01, .025, .05$ *and* $.10$*) and upperbounds are given in Table* 6.1.

TABLE 6.1
Upper bounds for $\alpha(F)$ *when* $\log F$ *and* $\log(1 - F)$ *both concave*

	$k = 3$	4	5	6	7	8	9	10	12	15	20
$\alpha = .01$	.0053	.0073	.0083	.0088	.0092	.0094	.0095	.0097	.0098	.0099	.0100
.025	.0127	.0176	.0200	.0214	.0223	.0229	.0234	.0237	.0241	.0245	.0248
.05	.0249	.0345	.0393	.0422	.0440	.0453	.0462	.0468	.0478	.0486	.0493
.10	.0496	.0682	.0777	.0834	.0870	.0895	.0914	.0928	.0947	.0965	.0979

To show that this class of distribution functions is not too small, we remark that it contains all the strongly unimodal distributions.

COROLLARY. *If F is strongly unimodal, then* $\log F$ *and* $\log(1 - F)$ *are both concave, so Table* 6.1 *is also valid for strongly unimodal F.*

PROOF. Prékopa (1973) proved that strong unimodality (that is, $\log f$ concave) implies the log-concavity of F. F is strongly unimodal if and only if $\bar{F}$ is strongly unimodal; hence $\log(1 - F)$ is also concave. □

REMARKS.

(1). This corollary is the other version of Theorem 4.1 we were looking for at the end of Section 4. Symmetry is not required but unimodal is replaced by strongly unimodal. Nevertheless Theorem 6.1 is more general.

(2). Again the situation of Section 4 occurs: $c(F_e, a)$ and $v(F_e, a)$ are not maximal for the same value of a. However, since $v(F_e, a)$ is almost maximal when $c(F_e, a)$ is maximal ($\frac{7}{64}$ versus $\frac{1}{9}$), we see that the values in Table 6.1 are practically equal to $\alpha(F_e)$.

7. Friedman-type simultaneous rank tests. Now we shall treat a multiple comparison procedure, also proposed by Nemenyi, but for another model, namely, when blocks are present. Let X_{ij}, $i = 1, \cdots, k$; $j = 1, \cdots, n$ be independent random variables, with continuous distribution functions F_{ij}, where we assume that there exist numbers $\theta_1, \cdots, \theta_k$, $\beta_1, \cdots, \beta_n$ and a distribution function F such that $F_{ij}(x) = F(x - \theta_i - \beta_j)$. The β's are called block parameters and we want to know which θ's are different.

Let R_{ij} denote the rank of X_{ij} among the jth block $(X_{1j}, \cdots, X_{kj})$. Then we define $\bar{R}_i = n^{-1}\Sigma_{j=1}^{n} R_{ij}$. Again n is assumed to be large and under the null hypothesis $H_0 : \theta_1 = \cdots = \theta_k$ we have for $n \to \infty$

$$(7.1) \qquad P\Big[\max_{1 \leqslant i,j \leqslant k}|\bar{R}_i - \bar{R}_j| < q_k^{\alpha}\{k(k+1)/(12n)\}^{\frac{1}{2}}\Big] = 1 - \alpha.$$

We are interested again in the $(k-1)$-mean significance level. Suppose $\theta_1 = \cdots = \theta_{k-1}$ and $\theta_k = \theta_1 + a$ $(a \neq 0)$. What in that case is the value of $\alpha^*(F, a)$, defined by

$$(7.2) \quad \alpha^*(F, a) = \lim_{n\to\infty} P\Big[\max_{1 \leqslant i,j \leqslant k-1}|\bar{R}_i - \bar{R}_j| > q_k^{\alpha}\{k(k+1)/(12n)\}^{\frac{1}{2}}\Big],$$

and is it larger than α for some α, k, F and a?

To answer this question we shall compute the supremum of $\alpha^*(F, a)$ over F and a. The vectors $(R_{1j}, \cdots, R_{kj})$ for $j = 1, \cdots, n$ are i.i.d., so $(\bar{R}_1, \cdots, \bar{R}_k)$ has an asymptotically normal distribution for $n \to \infty$. After computation of the variances of $\bar{R}_1, \cdots, \bar{R}_{k-1}$ the same arguments used in Section 2 lead to

$$(7.3) \quad \alpha^*(F,a) = P\Big[Q_{k-1} > q_k^{\alpha}\big\{(k^2+k)/\big(k^2 + \big(2c(F,a) - \tfrac{1}{12}\big)k/12\big)\big\}^{\frac{1}{2}}\Big].$$

Since $\frac{5}{48}$ is the supremum of $c(F, a)$ over F and a (see Section 4), we have

$$(7.4) \qquad \sup_{F,a}\alpha^*(F, a) = P\left[Q_{k-1} > q_k^{\alpha}\left\{(k^2+k)/\left(k^2 + \frac{3}{2}k\right)\right\}^{\frac{1}{2}}\right]$$

whose values are given in Table 7.1.

TABLE 7.1

$\sup_{F,a}\alpha^*(F, a)$ *for several values of* α *and* k.

	$k = 3$	4	5	6	7	8	9	10	12	15	20
$\alpha = .01$	.0060	.0084	.0096	.0101	.0105	.0107	.0108	.0109	.0110	.0110	.0109
.025	.0141	.0198	.0227	.0242	.0251	.0257	.0260	.0263	.0265	.0267	.0267
.05	.0271	.0380	.0435	.0457	.0483	.0498	.0506	.0511	.0518	.0523	.0524
.10	.0530	.0738	.0843	.0904	.0942	.0967	.0985	.0997	.1013	.1025	.1032

We see that $\alpha^*(F, a)$ may be larger than α, but the exceedance is never large. Once having this result, again the following question arises: if we define $\alpha^*(F)$ by

$$\alpha^*(F) = \sup_{a \in \mathbf{R}} \alpha^*(F, a),$$

what conditions on F are sufficient to guarantee $\alpha^*(F) \leqslant \alpha$? From (7.3) and Theorem 5.1, one can conclude:

THEOREM 7.1. *If F_2 is more skewed than F_1, then $\alpha^*(F_2) \geqslant \alpha^*(F_1)(F_1, F_2 \in \mathcal{F})$.*

Note that such a conclusion is not right for $\alpha(F)$, since $\alpha(F, a)$ depends on both $c(F, a)$ and $v(F, a)$, which are not always maximized by the same value of a (although in practice they almost are!).

Again the comparison with the exponential distribution gives:

THEOREM 7.2. *If* $\log F$ *and* $\log (1 - F)$ *both concave, then $\alpha^*(F) \leqslant \alpha$ for the usual values of α and k.*

It turns out that $\alpha^*(F_e)$ is slightly smaller than the values given in Table 6.1.

8. Finite sample studies. In order to investigate how far the asymptotic results are valid for finite n, Monte Carlo studies have been made for $n = 5$ and $k = 3, \cdots, 10$ in the situation where block parameters are absent. Here I am much indebted to Kees van der Hoeven, who wrote the computer programs.

Firstly the exact critical values have been estimated (from 40,000 simulations under H_0 for each k) in order to make the simultaneous significance level equal to α. It turned out that for $n = 5$ the critical value used in (1.2) is an acceptable approximation. Its exact significance level was systematically somewhat smaller than α, so it seems to be safe to use the asymptotic approximation of (1.1), if exact critical values are not available. Another critical value, which is sometimes used, namely $\{h_{k-1}^{\alpha} k(kn + 1)/6\}^{\frac{1}{2}}$, where h_{k-1}^{α} is the upper α point of the distribution of the Kruskal-Wallis statistic, proved to be bad; the significance level is much smaller than the nominal one, especially for larger k.

Once having obtained the exact critical values (of course randomization was necessary), the $(k - 1)$-mean significance levels have been estimated for the pair F, G given in (3.2) and also for a shift with an amount $\frac{1}{2}$ of F_m defined by (4.8), where $m \to \infty$. For both alternatives also 40,000 simulations were made for each k. In both cases the $(k - 1)$-mean significance levels for $n = 5$ are systematically a little bit larger than the values given in the Tables 3.1 and 4.1, but the difference was so small that one may conclude that already for $n = 5$ these levels behave as if n were infinity.

9. Final remark. As the $(k - 1)$-mean significance levels of both multiple comparison methods do not exceed α very much, these results may not appear very alarming to a practical statistician, the more so as (for shift alternatives) $\alpha(F)$ and $a^*(F)$ are smaller than α for a large class of distribution functions (Theorems 6.1 and 7.2). However, the most serious disadvantage of the methods remains, namely, the dependence of the distribution of $\bar{R}_i - \bar{R}_j$ on the other F_i's respective θ_i's.

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