The Annals of Statistics
1979, Vol. 7, No. 5, 955–959

A DIFFERENTIAL FOR L-STATISTICS[1]

BY DENNIS D. BOOS

North Carolina State University

The functional $T(F) = \int F^{-1}(t)J(t)\,dt$ associated with linear combinations of order statistics is shown to have a Frechet-type differential. As a corollary, the statistic $T(F_n)$ obtained by evaluating $T(\cdot)$ at the sample df F_n is seen to be asymptotically normal and to obey a law of the iterated logarithm.

1. Introduction. Let $X_{1n} \leqslant X_{2n} \leqslant \cdots \leqslant X_{nn}$ be an ordered sample from a distribution F and J a fixed score function on (0, 1). Statistics of the form

$$T_n = \sum_{i=1}^{n}\left(\int_{(i-1)/n}^{i/n} J(u)\,du\right)X_{in}$$

comprise an important subset of the general class $\sum c_{in}X_{in}$ of linear combinations of order statistics. In particular, T_n can be conveniently expressed as a functional of the empirical df F_n, $T_n = \int F_n^{-1}J(t)\,dt$.

In Sections 3 and 4 (Theorems 1 and 2) we show that the basic functional $T(F) = \int F^{-1}(t)J(t)\,dt$ possesses a differential with respect to (w.r.t.) the sup-norm $\|\cdot\|_\infty$ and w.r.t. the q-norm $\|\cdot\|_{q(F)}$. Both theorems require J to be bounded and continuous a.e. Lebesgue and a.e. F^{-1}. Theorem 2 requires, in addition, the tail condition $\int q(F(x))\,dx < \infty$. Corollaries to these theorems yield asymptotic normality and a law of the iterated logarithm (LIL) for $T(F_n)$. Section 2 defines the differential and motivates its statistical applications. Brief comparisons with related results are made in Section 5.

2. The differential. Let T be a real-valued functional defined on a *convex* set $\mathfrak{F}$ of df's. Denote by $\{\mathfrak{D}(\mathfrak{F}), \|\cdot\|\}$ the normed linear space generated by *differences* $H - G$ of members of $\mathfrak{F}$, i.e., $\mathfrak{D}(\mathfrak{F}) = \{\Delta : \Delta = a(H - G),\ H, G \in \mathfrak{F},\ a \in R\}$.

DEFINITION. The functional T has a *differential* at the point $F \in \mathfrak{F}$ w.r.t. the norm $\|\cdot\|$ and the set $\mathcal{G}_F \subset \mathfrak{F}$ if there exists a quantity $T(F; \Delta)$ defined on $\Delta \in \mathfrak{D}(\mathfrak{F})$, which is linear in the argument Δ and satisfies

$$\lim_{\|G-F\|\to 0;\ G\in\mathcal{G}_F} \frac{T(G) - T(F) - T(F; G - F)}{\|G - F\|} = 0. \tag{2.1}$$

$T(F; \Delta)$ is called the "differential." For $G = F_n = n^{-1}\sum\delta_{X_i}$, i.e., the sample df written in terms of point masses δ_x, the linearity property allows $T(F; F_n - F)$ to

---

Received November 1977; revised June 1978.

[1]Research supported by the Army, Navy, and Air Force under Office of Naval Research Contract No. N00014-76-C-0608.

AMS 1970 *subject classifications.* Primary 62E20; secondary 62G30.

Key words and phrases. Differential, linear combinations of order statistics, asymptotic normality, law of the iterated logarithm.

be expressed as an average of rv's,

$$(2.2)\qquad T(F; F_n - F) = T\Big(F; \frac{1}{n}\Sigma_{i=1}^n(\delta_{X_i} - F)\Big) = \frac{1}{n}\Sigma_{i=1}^n T(F; \delta_{X_i} - F).$$

If $n^{\frac{1}{2}}\|F_n - F\| = O_p(1)$ and $P(F_n \in \mathcal{G}_F) \to 1$, then (2.1) gives the approximation

$$(2.3)\qquad n^{\frac{1}{2}}\Big[T(F_n) - T(F) - \frac{1}{n}\Sigma_{i=1}^n T(F; \delta_{X_i} - F)\Big] \to_p 0, \qquad n \to \infty.$$

Slutsky's theorem and appropriate central limit theory then yield

$$(2.4)\qquad n^{\frac{1}{2}}[T(F_n) - T(F)] \to_d N(0, \sigma^2), \qquad n \to \infty,$$

where $\sigma^2 = \text{Var } T(F; \delta_X - F)$ and $E\ T(F; \delta_X - F)$ is assumed to be 0. Similar techniques and stronger assumptions on $\|F_n - F\|$ lead from (2.1) to an LIL

$$(2.5)\qquad \limsup_{n\to\infty} \frac{n^{\frac{1}{2}}[T(F_n) - T(F)]}{(2\sigma^2 \log\log n)^{\frac{1}{2}}} = 1 \text{ wp } 1.$$

Details can be found in Boos and Serfling (1980).

One advantage of the approach just outlined is the separation of the analytic differentiation in (2.1) from the stochastic results (2.3)–(2.5). This allows arbitrary sequences $\{X_i\}$ to be handled through the asymptotic properties of $\|F_n - F\|$ and $\Sigma T(F; \delta_{X_i} - F)$. Thus, although the corollaries of Sections 3 and 4 refer to only the independent case, specific "dependent" corollaries are easily formulated and proved.

3. Robust L-functionals. Suppose that the score function J is *trimmed* near 0 and 1,

$$(3.1)\qquad J(u) = 0 \qquad u \in [0, t_1) \cup (t_2, 1]$$

for $0 < t_1 < t_2 < 1$. Let F denote a fixed underlying df and $\mathcal{F} = \mathcal{G}_F = \{\text{all df's}\}$.

THEOREM 1. *If* (3.1) *holds and*

(3.2) *J is bounded and continuous* a.e. *Lebesgue and* a.e. F^{-1},

then the differential of $T(F) = \int F^{-1}(t)J(t)\,dt$ at F w.r.t. $\|\cdot\|_\infty$ *is given by*

$$(3.3)\qquad T(F; \Delta) = -\int_{-\infty}^{\infty}\Delta(x)J(F(x))\,dx.$$

PROOF. Since (3.3) is linear in Δ, we need only show

$$(3.4)\qquad T(G) - T(F) - \int[F(x) - G(x)]J(F(x))\,dx = o(\|G - F\|_\infty), \qquad \|G - F\|_\infty \to 0.$$

Using integration by parts, the left hand side of (3.4) can be rewritten as

$$-\int[K(G(x)) - K(F(x)) - (G(x) - F(x))J(F(x))]\,dx = -\int V_{G,F}(x)\,dx,$$

where $K(y) = \int_0^y J(u)\,du$. Let (a, b) be such that $G(a) < t_1$, $F(a) < t_1$, $G(b) > t_2$,

$F(b) > t_2$. Then $J(F(x))$ and $K(G(x)) - K(F(x))$ are 0 outside (a, b). Let $B = \{x : F(x)$ is a discontinuity point of $J\}$ and define

$$W_{G,F}(x) = \frac{V_{G,F}(x)}{G(x) - F(x)} \quad \text{if} \quad G(x) \neq F(x) \tag{3.5}$$
$$= 0 \qquad \text{if} \quad G(x) = F(x).$$

Since B is a Lebesgue-null set (using the fact that J is continuous a.e. F^{-1}), it follows that

$$|\int V_{G,F}(x)| = |\int_{(a,b)-B}[G(x) - F(x)]W_{G,F}(x)\,dx| \tag{3.6}$$
$$\leqslant \|G - F\|_\infty \int_{(a,b)-B}|W_{G,F}(x)|dx.$$

The derivative $K'(y) = J(y)$ exists at all continuity points of J and thus

$$\lim_{\|G-F\|_\infty \to 0}|W_{G,F}(x)| = 0 \qquad \forall x \in (a, b) - B.$$

The bound $\|W_{G,F}\|_\infty \leqslant 2\|J\|_\infty$ allows interchange of limit and integration in (3.6) through use of the theorem on bounded convergence for a finite interval. □

Under the conditions of Theorem 1, the rv's $T(F; \delta_{X_i} - F)$ have mean 0 and variance

$$\sigma^2 = \int\int[F(\min(s, t)) - F(s)F(t)]J(F(s))J(F(t))\,ds\,dt. \tag{3.7}$$

The following corollary follows directly from the classical central limit theorem and LIL and known results regarding $\|F_n - F\|_\infty$ (e.g., Chung (1949)).

COROLLARY. *Suppose that J and F satisfy* (3.1) *and* (3.2) *and* $\sigma^2 > 0$. *Let* $\{X_i\}$ *be a sequence of independent* rv's *having distribution F. Then* (2.4) *and* (2.5) *hold.*

EXAMPLE. The trimmed mean, $J(t) = I(\alpha_1 \leqslant t \leqslant 1 - \alpha_2)/(1 - \alpha_1 - \alpha_2)$, obviously satisfies (3.1). If F has unique quantiles $F^{-1}(\alpha_1)$ and $F^{-1}(1 - \alpha_2)$, then (3.2) is satisfied.

4. General L-functionals. In this section the trimming restrictions on J are removed, and the q-norms $\|\cdot\|_{q(F)} = \|(\cdot)/q(F)\|_\infty$ are used to deal with the weight placed on the extremes of F. A motivating class of q functions is

$$q(t) = [t(1 - t)]^{\frac{1}{2}-\delta}, \qquad 0 < \delta < \tfrac{1}{2}. \tag{4.1}$$

Let $\mathcal{F} = \{F : |\int F^{-1}(t)J(t)\,dt| < \infty\}$ and $\mathcal{G}_F = \{G : G \in \mathcal{F}$ and $S_G \subset S_F\}$, where S_F is the support of F. Let q be a bounded positive function on (0, 1).

THEOREM 2. *Suppose that J and $F \in \mathcal{F}$ satisfy* (3.2) *and*

$$\int_{-\infty}^{\infty} q(F(x))\,dx < \infty. \tag{4.2}$$

Then the differential of $T(F) = \int F^{-1}(t)J(t)\,dt$ at F w.r.t. $\|\cdot\|_{q(F)}$ *and $\mathcal{G}_F$ is given by* (3.3).

PROOF. Define B and $W_{G,F}$ as in the proof of Theorem 1, and let the closure of (x_1, x_2) be the smallest interval (possibly infinite) containing S_F. Then for $G \in \mathcal{G}_F$ we have

$$(4.3)\qquad |\int V_{G,F}(x)\,dx| = |\int_{(x_1,x_2)-B}\left(\frac{G(x)-F(x)}{q(F(x))}\right)(W_{G,F}(x))q(F(x))\,dx|$$
$$\leqslant \|G-F\|_{q(F)}\int_{(x_1,x_2)-B}|W_{G,F}(x)|q(F(x))\,dx.$$

The interchange of limit and integration in (4.3) is then justified by dominated convergence via the bound $\|W_{G,F}\|_\infty \leqslant 2\|J\|_\infty$ and (4.2). □

Note that the extension of Theorem 2 to unbounded J's requires only a justification of the interchange of limit and integration in (4.3).

Let Q_1 be the set of q functions given by O'Reilly (1974), Theorem 2. Let Q_2 be the set of q functions given in James (1975) ($q = w^{-1}$ in James' notation). Note that each member of (4.1) belongs to both Q_1 and Q_2.

COROLLARY. *Suppose that J and F satisfy* (3.2) *and* (4.2) *and* $0 < \sigma^2 < \infty$. *Let* $\{X_i\}$ *be a sequence of independent* rv's *having distribution F. If $q \in Q_1$, then* (2.4) *holds. If $q \in Q_2$, then* (2.5) *holds.*

EXAMPLES. (i) The mean, $J(t) = 1$. (ii) Gini's mean difference, $J(t) = 4(t - \frac{1}{2})$. (iii) The asymptotically efficient L-estimator for location for the logistic family, $J(t) = 6t(1-t)$.

Extension of results to the more general functional $T(F) = \int h(F^{-1}(t))J(t)\,dt$ is given in Boos (1977).

5. Comparisons with other results.

Reeds (1976) restricts attention to df's on [0, 1] and shows that $\int h(F^{-1}(t))J(t)\,dt$ has a differential at F_0 = uniform w.r.t. the L_2 norm. His condition on J is good, $\int|J(t)|dt < \infty$; however, statistical application via the rv's $F^{-1}(U_i)$ apparently requires F to be continuous.

Among asymptotic normality theorems allowing discontinuous F and fixed centering constants, Stigler's (1974) results are closest to the corollaries to Theorems 1 and 2. For the untrimmed version he has slightly weaker moment conditions, $\int[F(x)(1-F(x))]^{\frac{1}{2}}\,dx < \infty$ as compared to the combination of (4.1) and (4.2), and slightly stronger conditions on J (see his Theorem 4).

The LIL corollaries follow at no extra cost but are not as general as the Strassen type LIL results of Wellner (1977).

Acknowledgment. I am indebted to Professor R. J. Serfling for suggesting the problem and for careful reading of the manuscript as well as providing many helpful comments. I would also like to thank Professor George Policello for detecting an error in an earlier proof of Theorem 2.

REFERENCES

[1] Boos, D. D. (1977). The differential approach in statistical theory and robust inference. Unpublished dissertation, Florida State Univ.

[2] Boos, D. D. and Serfling, R. J. (1980). A note on differentials and the CLT and LIL for statistical functions, with application to M-estimates. To appear in *Ann. Statist.*

[3] Chung, K. L. (1949). An estimate concerning the Kolmogoroff limit distribution. *Trans. Amer. Math. Soc.* **67** 36–50.

[4] James, B. R. (1975). A functional law of the iterated logarithm for weighted empirical distributions. *Ann. Probability* **3** 762–772.

[5] O'Reilly, N. E. (1974). On the weak convergence of empirical processes in sup-norm metrics. *Ann. Probability* **2** 642–651.

[6] Reeds, J. (1976). On the definition of von Mises functionals. Ph.D. dissertation, Harvard Univ.

[7] Stigler, S. M. (1974). Linear functions of order statistics with smooth weight functions. *Ann. Statist.* **2** 676–693.

[8] Wellner, J. A. (1977). A law of the iterated logarithm for functions of order statistics. *Ann. Statist.* **5** 481–494.

Department of Statistics
North Carolina State University
Raleigh, North Carolina 27607