*The Annals of Statistics*
1978, Vol. 6, No. 4, 813–819

# ASYMPTOTIC EFFICIENCIES OF SEQUENTIAL TESTS II

BY ROBERT H. BERK[1]

*Rutgers University*

An asymptotic expression is given for the log error probability of a sequential test based on a random walk. This may be used to compute limiting relative efficiencies of such tests. The results are illustrated for the one-sided normal testing problem with an asymptotic Bayes test due to Schwarz. Some numerical comparisons are given for five sequential tests of a normal mean.

**1. Introduction and main results.** In a previous paper, Berk (1976), it was argued that relative efficiencies for sequential tests are ratios of certain efficacies, the latter being a limiting ratio of a log error to an expècted sample size. (The discussion there is in the context of testing a normal mean, but applies generally.) The limit is taken as the stopping time of the sequential test becomes infinite in a suitable way. We extend the range of applicability of this idea by showing how to evaluate this limit for a large class of sequential tests based on cumulative sums with i.i.d. summands.

Let $Y, Y_1, \cdots$ be i.i.d. random variables and let $S_n = Y_1 + \cdots + Y_n$. We consider sequential tests with stopping time of the form

$$N = \min\{n : S_n \notin (-a\overline{g_a}(n/a), ag_a(n/a))\}. \tag{1.1}$$

Here $g_a$ and $\overline{g_a}$ are the boundary curves of the continuation region; $a$ is a parameter governing the size of the region. In taking limits, we let $a \to \infty$. Many sequential tests for one-parameter models have stopping times of the form (1.1). These include SPRTs, LMP sequential tests (Berk (1975)) and asymptotically Bayes sequential tests for one-parameter families (Schwarz (1962), Kiefer and Sacks (1963)). We consider the one-sided (hypothesis) case, for which the appropriate terminal decision, on stopping, is to reject the null hypothesis iff $S_N \geqq ag_a(N/a)$. *We write this as* $(S_N \geqq g)$ *for short.* (Implicit in our notation is that $-\overline{g_a} \leqq g_a$.) Interest then centers on the error probability, which is $P(S_N \geqq g)$ or $P(S_N \leqq -\overline{g})$ depending on which hypothesis obtains. We work explicitly with the former probability; it is seen, on replacing $Y$ by $-Y$ and interchanging boundaries, that the results apply to the latter as well. Our considerations apply to an upper boundary that has an asymptotic shape: As $a \to \infty$, $g_a(x) \to g(x)$ for $x > 0$. (In fact, we suppose that $g_a$ decreases to $g$.)

We suppose that under the distribution of interest, $EY < 0$. We also suppose that $Ee^{t|Y|} < \infty$ for some $t > 0$. Then $b(t) = \log Ee^{tY}$ is finite in some neighborhood of zero. It is well known that $b$ is convex and analytic on the

Received April 1976; revised July 1977.

[1] Prepared with the support of NSF Grants MPS72-05082-A02 and MCS76-82618.

*AMS* 1970 *subject classifications.* Primary 62L10, 62F05; Secondary 62F20, 60G40, 60F99.

*Key words and phrases.* Sequential test, asymptotic efficiency, efficacy.



Institute of Mathematical Statistics is collaborating with JSTOR to digitize, preserve, and extend access to *The Annals of Statistics.* www.jstor.org

interior of the interval $(b < \infty)$. Let $\beta(r) = \sup\{tr - b(t) : t \in R\}$; $\beta$ is the convex function dual to $b$ (Rockafellar (1970)). Since $EY = b'(0) < 0$, for $r > 0$, $\beta(r) = \sup\{tr - b(t) : t > 0\}$. Also, let $R(x) = g(x)/x$. We say $g \in \mathscr{G}$ if the following holds: for some $0 < \nu \leqq \infty$, $0 \leqq R$ is strictly decreasing on $(0, \nu) = (R > 0)$, $\lim_{x\to 0} R(x) = \infty$ and $\lim_{x\to\infty} R(x) = 0$.

Under certain conditions on the distribution of $Y$ and the boundary curves, we show that as $a \to \infty$,

$$\log P(S_N \geqq g) \sim \max_n \log P(S_n \geqq ag_a(n/a)) . \tag{1.2}$$

That is, to first order, the log probability of the sequentially determined event $(S_N \geqq g)$ is determined by the point of the upper boundary "closest" to the random walk $\{S_n\}$; closest in that the probability of crossing there is maximum (but regardless of the past history of the walk). The following theorem paraphrases (1.2).

THEOREM. *Let $R_a(x) = g_a(x)/x \in \mathscr{G}$ and suppose too that as $a \uparrow \infty$, $R_a(x)$ decreases to $R(x) = g(x)/x \in \mathscr{G}$. Suppose also that for each fixed $n = 1, 2, \cdots$, $\lim_a a\underline{g_a}(n/a) = \infty$. Letting $b(t) = \log Ee^{tY}$, suppose that $(b < \infty)$ is an open interval containing zero and that $EY < 0$. Then as $a \to \infty$,*

$$\log P(S_N \geqq g) \sim -a\kappa , \tag{1.3}$$

*where*

$$\kappa = \inf_{x>0} x\beta(R(x)) = \inf_{x>0} \inf_t x[tR(x) - b(t)] . \tag{1.4}$$

PROOF. We observe first that

$$\begin{aligned} P(S_n \geqq ag_a(n/a)) &= P(S_n \geqq nR_a(n/a)) \\ &\leqq P(S_n \geqq nR(n/a)) \leqq \inf_{t>0} E\exp\{tS_n - ntR(n/a)\} \\ &= \inf_t \exp\{nb(t) - ntR(n/a)\} = \exp\{-n\beta(R(n/a))\} . \end{aligned}$$

Hence

$$\begin{aligned} \max_n P(S_n \geqq ag_a(n/a)) &\leqq \max_n \exp\{-n\beta(R(n/a))\} \\ &\leqq \sup_{x>0} \exp\{-ax\beta(R(x))\} = e^{-a\kappa} . \end{aligned}$$

Repeating the above argument with $R$ replaced by zero gives

$$P(S_n \geqq ag_a(n/a)) \leqq P(S_n \geqq 0) \leqq \inf_{t>0} \exp\{nb(t)\} = e^{-n\rho} ,$$

where $-\rho = \inf_t b(t) < 0$. (That $\inf_{t>0} b(t) = \inf_t b(t) < 0$ follows from the fact that $b'(0) < 0$, togeter with the fact that $b(0) = 0$.) Letting $k = \kappa/\rho$,

$$\begin{aligned} P(S_N \geqq g) &\leqq \textstyle\sum_{n=1}^{\infty} P(S_n \geqq ag_a(n/a)) \\ &\leqq ake^{-a\kappa} + \textstyle\sum_{n>ak} e^{-n\rho} = \exp\{-a\kappa + o(a)\} . \end{aligned} \tag{1.5}$$

For $t \in (b < \infty)$, let $P_t$ denote the distribution of the i.i.d. sequence $Y, Y_1, \cdots$ when they marginally have pdf $\exp\{ty - b(t)\}$ with respect to $P$. Let $\mu_t = b'(t) = E_t Y$. That $(b < \infty)$ is open insures that $\mu_t$ ranges (continuously and strictly monotonically) in $(-\infty, \infty)$ for $t \in (b < \infty)$. As is well known, the likelihood ratio for the sequentially obtained data is

$$dP_t^N/dP^N = \exp\{tS_N - Nb(t)\} .$$

The conditions on $g_a$ entail $P_t(N < \infty) = 1$ when $\mu_t > 0$ (for $a$ sufficiently large) and then

$$P(S_N \geqq g) = \int_{(S_N \geqq g)} \exp\{-tS_N + Nb(t)\}\, dP_t , \quad \mu_t > 0 . \tag{1.6}$$

Under the assumptions on $g_a$ and $\overline{g_a}$, for $\mu_t > 0$, $P_t(\lim_a 1_{(S_N \geqq g)} = 1) = 1$, so also $\lim_a P_t(S_N \geqq g) = 1$. Moreover, the inequalities

$$S_{N-1}/N < R_a(N/a - 1/a) , \qquad (S_N/N)1_{(S_N \geqq g)} \geqq R(N/a)1_{(S_N \geqq g)} \tag{1.7}$$

entail

$$P_t(\lim_a N/a = R^{-1}(\mu_t)) = 1 \qquad \text{if} \quad \mu_t > 0 \tag{1.8}$$

as follows: we note that $P_t(\lim_a N = \infty) = 1$, so that $P_t(\lim_a S_N/N = \mu_t) = 1$. Thus (1.7) entails

$$\limsup_a R(N/a) \leqq \mu_t \leqq \liminf_a R_a(N/a - 1/a) \quad [P_t] , \tag{1.9}$$

which in turn implies (1.8). (Since $R$ is strictly decreasing, there is no ambiguity in defining $R^{-1}$. Note that $R(N/a)$ need not tend to $\mu_t$ for $P_t$ unless $R$ is continuous at $R^{-1}(\mu_t)$.)

To finish the proof, we need the following lemma which is proved in Berk (1976); cf. Lemma 7.3.

LEMMA. *Let $V_a$ be an indexed set of random variables so that for some constant $v$, $V_a/a \to_P v$ as $a \to \infty$. Let $B_a$ be any indexed set of events for which* $\liminf_a PB_a > 0$. *Then*

$$\int_{B_a} \exp\{V_a\}\, dP \geqq \exp\{av + o(a)\} . \tag{1.10}$$

We apply the lemma to $V_a = tS_N - Nb(t)$, the exponent in (1.6). It follows from (1.8) that w.p. 1 for $P_t$, $V_a/a \to R^{-1}(\mu_t)[t\mu_t - b(t)]$. Recall that $\beta(r) = \sup_s \{sr - b(s)\}$. If $r = \mu_t$, the derivative of the concave function $f(s) = sr - b(s)$ vanishes at $s = t$, so that $\beta(\mu_t) = t\mu_t - b(t)$. Thus $V_a/a \to \beta(\mu_t)$ $[P_t]$. Since $P_t(S_N \geqq g) \to 1$ for $\mu_t > 0$, (1.6) and (1.10) entail

$$\liminf_a a^{-1} \log P(S_N \geqq g) \geqq -R^{-1}(\mu_t)\beta(\mu_t) , \quad \mu_t > 0 ,$$

hence that

$$\begin{aligned} \liminf_a a^{-1} \log P(S_N \geqq g) &\geqq \sup\{-R^{-1}(\mu_t)\beta(\mu_t) : \mu_t > 0\} \\ &= -\inf_{x>0} x\beta(R(x)) = -\kappa . \end{aligned} \tag{1.11}$$

Together, (1.5) and (1.11) establish the theorem. □

When $EY > 0$, the corresponding result is

$$\log P(S_N \leqq -\bar{g}) \sim -a\bar{\kappa} , \tag{1.12}$$

where

$$\bar{\kappa} = \inf_{x>0} x\beta(\bar{R}(x)) = \inf_{x>0} \sup_t x[t\bar{R}(x) - \bar{b}(t)]\,. \tag{1.13}$$

Here $\bar{b}(t) = b(-t)$ and $\bar{R}$ is the $R$-function for the lower boundary.

**2. Discussion.** The preceding applies to exponential families as follows. Suppose that under $P$, $Y$ marginally has pdf $\exp\{\theta y - c(\theta)\}$ with respect to some underlying measure. Then $b(t) = c(t + \theta) - c(\theta)$ and if $\eta(r) = \sup_t [\mathrm{tr} - c(t)]$ denotes the function dual to $c$,

$$\begin{aligned} \kappa &= \inf_{x>0} \sup_t x[tR(x) - c(t + \theta) + c(\theta)] \\ &= \inf_{x>0} x[\eta(R(x)) + c(\theta) - \theta R(x)]\,, \quad EY < 0 \end{aligned} \tag{2.1}$$

and

$$\bar{\kappa} = \inf_{x>0} x[\bar{\eta}(\bar{R}(x)) + c(\theta) + \theta\bar{R}(x)]\,, \quad EY > 0\,. \tag{2.2}$$

When $Y \sim N(\theta, 1)$, $c(\theta) = \bar{c}(\theta) = \eta(\theta) = \frac{1}{2}\theta^2$ and these become

$$\begin{aligned} \kappa &= \inf_{x>0} \tfrac{1}{2}x(R(x) - \theta)^2\,, \quad \theta < 0 \\ \bar{\kappa} &= \inf_{x>0} \tfrac{1}{2}x(\bar{R}(x) + \theta)^2\,, \quad \theta > 0\,. \end{aligned} \tag{2.3}$$

In Berk (1976), $\kappa$ was evaluated by different methods for the continuous-time normal case. Equation (2.3) shows that those results apply to discrete time as well. As a complement to the results of Berk (1976), we consider the continuation region given by $g_a(x) = \overline{g_a}(x) = (2x)^{\frac{1}{2}} - \mu x$. As shown by Schwarz (1962), this is the asymptotic shape of the Bayes test of $H_1: \theta \leqq 0$ vs $H_2: \theta > 0$ in the normal case (variance = one) when the prior for $\theta$ dominates Lebesgue measure and when $(-\mu, \mu)$ is an indifference region. (Schwarz did not show that the corresponding procedure defined by

$$N = \min\{n: |S_n| \geqq (2an)^{\frac{1}{2}} - \mu n\} \tag{2.4}$$

is asymptotically optimal (Bayes), although this follows from the results of Kiefer and Sacks (1963).) In this case $R(x) = \bar{R}(x) = (2/x)^{\frac{1}{2}} - \mu$ and (2.3) becomes

$$\begin{aligned} \kappa &= 2^{\frac{1}{2}}\theta\mu\,, \quad 0 < \theta \leqq \mu \\ &= 2^{\frac{1}{2}}\,, \qquad \theta > \mu\,. \end{aligned} \tag{2.5}$$

Let $\varepsilon(\theta)$ denote the probability of error of a sequential test when $\theta$ obtains. It is suggested in Berk (1976) that the limits (as $a \to \infty$) of $[-\log \varepsilon(\theta)]/E_\theta N$ and $[-\log \varepsilon(\theta)]/E_0 N$ are efficacies for judging the relative efficiency of the test. (As in other cases, the ratio of corresponding efficacies for two tests gives their relative efficiency. Using $E_0 N$ as a divisor is akin to standardizing tests to have the same level.) Using (1.8) for $N$ defined by (2.4), we see that under $P_\theta$, $N/a \to 2/(\mu + |\theta|)^2$. Since $N/a$ is bounded, also

$$E_\theta N \sim 2a/(\mu + |\theta|)^2\,,$$

hence (2.5) gives

$$
(2.6)\qquad
\begin{aligned}
e(\theta) = \lim\nolimits_a \left[-\log \varepsilon(\theta)\right]/E_\theta N &= (\mu + \theta)^2\theta^2/2\mu^2\,, && |\theta| \leqq \mu \\
&= \tfrac{1}{2}(\theta + \mu)^2\,, && |\theta| > \mu \\
e_0(\theta) = \lim\nolimits_a \left[-\log \varepsilon(\theta)\right]/E_0 N &= \tfrac{1}{2}\theta^2\,, && |\theta| \leqq \mu \\
&= \tfrac{1}{2}\mu^2\,, && |\theta| > \mu\,.
\end{aligned}
$$

As shown in Berk (1976), the limits in (2.6) cannot exceed $2\theta^2$ and $\frac{1}{2}\theta^2$, respectively, for symmetric tests. Thus $N$ in (2.4) is optimal for the $E_0N$-criterion for $|\theta| \leqq \mu$ but is optimal for the $E_\theta N$-criterion only for $|\theta| = \mu$. This is despite the fact that $N$ defined by (2.4) has a very strong optimality property: among all tests $\tau$ of $H_1\colon \theta \leqq 0$ vs $H_2\colon \theta > 0$ whose error rates $\varepsilon(\theta \mid \tau)$ satisfy

$$
(2.7)\qquad \liminf\nolimits_a \log\left[\varepsilon(\mu \mid \tau) + \varepsilon(-\mu \mid \tau)\right]/\log \varepsilon(\mu) \geqq 1\,,
$$

$$
(2.8)\qquad \liminf\nolimits_a E_\theta \tau / E_\theta N \geqq 1 \qquad \text{for all} \quad \theta\,.
$$

Cf. Lorden (1972) and Berk (1977).

**3. Numerical results.** We present here some numerical calculations to compare the actual efficacy (ratio of log error to expected sample size) with the limiting value. Five symmetric sequential tests for the one-sided normal testing problem were selected: SPRT, TPRT (truncated SPRT), AND (Anderson's triangular boundaries), TAPO (truncated APO), and SCHWARZ. The first four tests are discussed in Berk (1976) and limiting efficacies are given there. The

TPRT

| $\theta$ | 0 | .1 | .2 | .3 | .4 | .5 | .6 | .7 | .8 | .9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\varepsilon(\theta)$ | .5 | .268 | .112 | 3.86 (−2) | 1.21 (−2) | 3.83 (−3) | 1.24 (−3) | 4.07 (−4) | 1.34 (−4) | 4.41 (−5) | 1.45 (−5) |
| $E_\theta N$ | 26.9 | 25.3 | 21.5 | 17.3 | 13.9 | 11.3 | 9.6 | 8.2 | 7.3 | 6.5 | 5.9 |
| $\hat{e}(\theta)$ | 2.58 (−2) | 5.19 (−2) | .102 | .188 | .318 | .490 | .700 | .947 | 1.23 | 1.55 | 1.90 |
| $e(\theta)$ | 0 | .005 | .04 | .135 | .32* | .5* | .72* | .98* | 1.28* | 1.62* | 2* |
| $\hat{e}_0(\theta)$ | 2.58 (−2) | 4.90 (−2) | 8.13 (−2) | .121 | .164 | .207 | .249 | .290 | .331 | .373 | .414 |
| $e_0(\theta)$ | 0* | .005* | .02* | .045* | .08* | .1 | .12 | .14 | .16 | .18 | .20 |

SPRT

| $\theta$ | 0 | .1 | .2 | .3 | .4 | .5 | .6 | .7 | .8 | .9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\varepsilon(\theta)$ | .5 | .246 | 9.68 (−2) | 3.39 (−2) | 1.14 (−2) | 3.76 (−3) | 1.24 (−3) | 4.07 (−4) | 1.34 (−4) | 4.41 (−5) | 1.45 (−5) |
| $E_\theta N$ | 31.3 | 28.3 | 22.7 | 17.6 | 13.9 | 11.3 | 9.6 | 8.2 | 7.3 | 6.5 | 5.9 |
| $\hat{e}(\theta)$ | 2.22 (−2) | 4.96 (−2) | .103 | .192 | .322 | .492 | .701 | .947 | 1.23 | 1.55 | 1.90 |
| $e(\theta)$ | 0* | .02* | .08* | .18* | .32* | .5* | .72* | .98* | 1.28* | 1.62* | 2* |
| $\hat{e}_0(\theta)$ | 2.22 (−2) | 4.55 (−2) | 7.57 (−2) | .108 | .145 | .178 | .217 | .253 | .289 | .325 | .361 |
| $e_0(\theta)$ | 0* | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

TAPO

| $\theta$ | 0 | .1 | .2 | .3 | .4 | .5 | .6 | .7 | .8 | .9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\varepsilon(\theta)$ | .5 | .256 | 9.91 (−2) | 3.22 (−2) | 1.12 (−2) | 4.80 (−3) | 2.35 (−3) | 1.20 (−3) | 6.20 (−4) | 3.24 (−4) | 1.71 (−4) |
| $E_\theta N$ | 36.8 | 34.9 | 29.7 | 23.2 | 17.4 | 13.1 | 10.1 | 8.2 | 6.8 | 5.8 | 5.1 |
| $\tilde{e}(\theta)$ | 1.88 (−2) | 3.91 (−2) | 7.79 (−2) | .148 | .258 | .408 | .597 | .822 | 1.08 | 1.38 | 1.71 |
| $e(\theta)$ | **0*** | **.005** | **.02** | **7.39 (−2)** | **.222** | **.373** | **.567** | **.804** | **1.08** | **1.40** | **1.77** |
| $\tilde{e}_0(\theta)$ | 1.88 (−2) | 3.70 (−2) | 6.28 (−2) | 9.32 (−2) | .122 | .145 | .164 | .183 | .201 | .218 | .235 |
| $e_0(\theta)$ | **0*** | **.005*** | **.02*** | **.045*** | **.08*** | **9.11 (−2)** | **.102** | **.113** | **.123** | **.134** | **.144** |

ANDERSON

| $\theta$ | 0 | .1 | .2 | .3 | .4 | .5 | .6 | .7 | .8 | .9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\varepsilon(\theta)$ | .5 | .267 | .107 | 3.21 (−2) | 7.23 (−2) | 1.27 (−3) | 1.82 (−4) | 2.46 (−5) | 3.05 (−6) | 3.71 (−7) | 4.49 (−8) |
| $E_\theta N$ | 28.3 | 27.1 | 24.2 | 20.8 | 17.8 | 15.4 | 13.5 | 12.0 | 10.9 | 9.9 | 9.1 |
| $\tilde{e}(\theta)$ | 2.45 (−2) | 4.90 (−2) | 9.20 (−2) | .165 | .277 | .434 | .638 | .881 | 1.17 | 1.49 | 1.86 |
| $e(\theta)$ | **0** | **7.5 (−2)** | **.04** | **.112** | **.24** | **.42** | **.64** | **.9** | **1.2** | **1.54** | **1.92** |
| $\tilde{e}_0(\theta)$ | 2.45 (−2) | 4.67 (−2) | 7.89 (−2) | .122 | .174 | .236 | .304 | .375 | .449 | .523 | .598 |
| $e_0(\theta)$ | **0*** | **.005*** | **.02*** | **.045*** | **.08*** | **.12** | **.16** | **.2** | **.24** | **.28** | **.32** |

SCHWARZ

| $\theta$ | 0 | .1 | .2 | .3 | .4 | .5 | .6 | .7 | .8 | .9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\varepsilon(\theta)$ | .5 | .288 | .135 | 5.47 (−2) | 2.11 (−2) | 8.75 (−3) | 4.16 (−3) | 2.22 (−3) | 1.27 (−3) | 7.63 (−4) | 4.70 (−4) |
| $E_\theta N$ | 21.5 | 20.5 | 18.1 | 15.1 | 12.4 | 10.2 | 8.5 | 7.2 | 6.2 | 5.4 | 4.8 |
| $\tilde{e}(\theta)$ | 3.22 (−2) | 6.07 (−2) | .111 | .193 | .312 | .466 | .645 | .848 | 1.08 | 1.33 | 1.61 |
| $e(\theta)$ | **0*** | **7.81 (−3)** | **.045** | **.138** | **.32*** | **.405** | **.5** | **.605** | **.72** | **.845** | **.98** |
| $\tilde{e}_0(\theta)$ | 3.22 (−2) | 5.79 (−2) | 9.29 (−2) | .135 | .179 | .220 | .255 | .284 | .310 | .334 | .356 |
| $e_0(\theta)$ | ***0** | **.005*** | **.02*** | **.045*** | **.08*** | **.08** | **.08** | **.08** | **.08** | **.08** | **.08** |

Schwarz test is discussed above. For the Schwarz test, $\mu$ was chosen to be 0.4 and $a = 4.0$. Thus

$$N_{\text{SCH}} = \min\{n : |S_n| \geqq (8n)^{\frac{1}{2}} - 0.4n\}\,.$$

It is easily seen that $N \leqq m = 50$. The boundaries for the other tests were chosen so as to circumscribe the Schwarz continuation region. This is because, asymptotically, any such region has the same log error rate at $\pm\mu$ as the Schwarz procedure and an $E_0N$ (= maximum expected sample size) which is, to first order, as small as possible; cf. Berk (1977). Thus

$$\begin{aligned} N_{\text{SPRT}} &= \min\{n : |S_n| \geqq 5\} \\ N_{\text{TPRT}} &= N_{\text{SPRT}} \wedge 50 \\ N_{\text{AND}} &= \min\{n : |S_n| \geqq 10 - 0.2n\} \leqq 50 \\ N_{\text{TAPO}} &= \min\{n : |S_n| \geqq (2.532n + 6.331)^{\frac{1}{2}}\} \wedge 50 \,. \end{aligned}$$

The null hypothesis is rejected if $S_N \geqq 0$. For these tests the probability of error $\varepsilon(\theta)$ and the expected sample size were computed for $\theta = 0(0.1)1.0$. This was done using successive convolutions, a procedure alluded to by Aroian (1968) as the "direct method"; see Aroian and Robinson (1969). From these values we obtain the actual efficacies (ratios of log error to expected sample size), $\tilde{e}(\theta)$ and $\tilde{e}_0(\theta)$, which may be compared with the limiting values given in the tables below. Values of the limiting efficacies that achieve the appropriate bound $2\theta^2$ or $\frac{1}{2}\theta^2$ are marked with an asterisk.

The tables indicate a good qualitative agreement between the actual and limiting efficacies and reasonably good quantitative agreement in most cases. The values for $\tilde{e}(\theta)$ agree better with their limiting values than those for $\tilde{e}_0(\theta)$. In part, this can be attributed to the fact that $E_0 N$ appears to tend to its limiting value rather slowly. In all of the above tests $E_0 N \sim m$, the common truncation value, but the actual $E_0 N$ is substantially less in all cases. Nevertheless, the agreement is good enough to warrant guarded optimism that the limiting efficacies do represent the actual characteristics of the test.

**Acknowledgment.** The author wishes to thank Letita Chow Lai and Gary Myers for assistance with the calculations. He is also indebted to Richard Monahan for providing the program used.

## REFERENCES

AROIAN, L. (1968). Sequential analysis, direct method. *Technometrics* **10** 125–132.

AROIAN, L. and ROBINSON, D. E. (1969). Direct methods for exact truncated sequential tests of the mean of a normal distribution. *Technometrics* **11** 661–675.

BERK, R. H. (1975). Locally most powerful sequential tests. *Ann. Statist.* **3** 373–381.

BERK, R. H. (1976). Asymptotic efficiencies of sequential tests. *Ann. Statist.* **4** 891–911.

BERK, R. H. (1977). On asymptotically optimal sequential tests. Unpublished manuscript.

KIEFER, J. and SACKS, J. (1963). Asymptotically optimum inference and design. *Ann. Math. Statist.* **34** 705–750.

LORDEN, G. (1972). Likelihood ratio tests for sequential $k$-decision problems. *Ann. Math. Statist.* **43** 1412–1427.

ROCKAFELLAR, R. T. (1970). *Convex Analysis*. Princeton Univ. Press.

SCHWARZ, G. (1962). Asymptotic shapes of Bayes sequential testing regions. *Ann. Math. Statist.* **33** 224–236.

DEPARTMENT OF STATISTICS
RUTGERS UNIVERSITY
NEW BRUNSWICK, NEW JERSEY 08903