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TIED-DOWN WIENER PROCESS APPROXIMATIONS FOR ALIGNED RANK ORDER PROCESSES AND SOME APPLICATIONS[1]

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For independent random variables distributed symmetrically around an unknown location parameter, aligned rank order statistics are constructed by using an estimator of the location parameter based on suitable rank statistics. The sequence of these aligned rank order statistics is then incorporated in the construction of suitable stochastic processes which converge weakly to some Gaussian functions, and, in particular, to tied-down Wiener processes in the most typical cases. The results are extended for contiguous alternatives and then applied in two specific problems in nonparametric inference. First, the problem of testing for shift at an unknown time point is treated, and then, some sequential type asymptotic nonparametric tests for symmetry around an unknown origin are considered.

1. Introduction. Let $\{X_i, i \geqq 1\}$ be a sequence of independent and identically distributed random variables (i.i.d. rv) with a continuous distribution function (df) $F(x)$, $x \in (-\infty, \infty)$. For every n $(\geqq 1)$, consider the *signed rank statistic*

$$T_n = T_n(X_1, \cdots, X_n) = \sum_{i=1}^n c_i \operatorname{sgn} X_i a_n(R_{ni}^+) \tag{1.1}$$

where $\{c_i, i \geqq 1\}$ is a sequence of known (real) constants, R_{ni}^+ is the rank of $|X_i|$ among $|X_1|, \cdots, |X_n|$, $i = 1, \cdots, n$, $a_n(1), \cdots, a_n(n)$ are defined by

$$a_n(i) = E\phi^+(U_{ni}), \qquad i = 1, \cdots, n, \tag{1.2}$$

$U_{n1} < \cdots < U_{nn}$ are the ordered rv's of a sample of size n from the rectangular $[0, 1]$ df and the *score function* $\phi^+(u)$, $0 < u < 1$ is assumed to be square integrable inside $(0, 1)$. Conventionally, we define $T_0 = 0$, and let

$$A_n^2 = n^{-1} \sum_{i=1}^n a_n^2(i) \quad \text{and} \quad C_n^2 = \sum_{i=1}^n c_i^2, \quad n \geqq 1 \quad \text{and} \quad A_0^2 = C_0^2 = 0. \tag{1.3}$$

Note that by (1.2) and the square integrability of ϕ^+, for every $n \geqq 1$,

$$\begin{aligned} A_n^2 &= n^{-1} \sum_{i=1}^n E\{\phi^+(U_{ni})\}^2 - n^{-1} \sum_{i=1}^n [E\{\phi^+(U_{ni})\}^2 - (E\phi^+(U_{ni}))^2] \\ &\leqq n^{-1} \sum_{i=1}^n E\{\phi^+(U_{ni})\}^2 = \int_0^1 \{\phi^+(u)\}^2 \, du = A^2, \quad \text{say,} \end{aligned} \tag{1.4}$$

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while by Theorem b (on page 158) of Hájek and Šidák (1967),

$$A_n^2 \to A^2, \quad \text{as} \quad n \to \infty. \tag{1.5}$$

For every $n (\geqq 1)$, consider a stochastic process $W_n = \{W_n(t), t \in I\}$, $I = [0, 1]$, by introducing a sequence of integer-valued, nondecreasing and right continuous functions $\{k_n(t), t \in I\}$ where

$$k_n(t) = \max\{k : A_k^2 C_k^2 \leqq t A_n^2 C_n^2\}, \qquad t \in I, \tag{1.6}$$

and then letting

$$W_n(t) = (A_n C_n)^{-1} T_{k_n(t)}, \qquad t \in I. \tag{1.7}$$

When F is symmetric about 0, ϕ^+ is square integrable (as has been assumed) and

$$\max_{1 \leqq k \leqq n} c_k^2 / C_n^2 \to 0 \quad \text{as} \quad n \to \infty, \tag{1.8}$$

then, on defining $W = \{W(t), t \in I\}$ as a standard Wiener process on I, we have

$$W_n \to_{\mathscr{L}} W, \quad \text{in the } J_1\text{-topology on} \quad D[0, 1], \tag{1.9}$$

where $D[0, 1]$ is the space of real functions on I with no discontinuities of the second kind and is endowed with the Skorokhod J_1-topology (see Billingsley (1968)). The proof of our Theorem 3.1 (to follow) insures (1.9) under the minimal conditions stated above; under more stringent regularity conditions and under a slightly different construction (using the $C[0, 1]$ space instead of the $D[0, 1]$ space), the same result was proved in [15]. We prefer to use the current construction as it leads to relatively simpler expressions for some statistics based on such processes, as will be considered in Section 4.

We are concerned here with the model

$$F(x) = F_0(x - \theta), \quad \theta \quad \text{unknown and} \quad F_0(x) + F_0(-x) = 1 \tag{1.10}$$

for all $x \geqq 0$. Since θ is unknown, we use the *aligned rank order statistics*

$$\hat{T}_{k,n} = T_k(X_1 - \tilde{\theta}_n, \cdots, X_k - \tilde{\theta}_n), \qquad 1 \leqq k \leqq n; \hat{T}_{0,n} = 0, \tag{1.11}$$

where $\tilde{\theta}_n$ is the estimator of θ based on a suitable signed rank statistic $\tilde{T}_n^0$ (see Section 2) and the T_k are defined as in (1.1). On replacing T_k by $\hat{T}_{k,n}$, $0 \leqq k \leqq n$ in (1.7), we define the stochastic process $\hat{W}_n = \{\hat{W}_n(t), t \in I\}$, for $n \geqq 1$; note that the observations $X_1 - \tilde{\theta}_n, \cdots, X_n - \tilde{\theta}_n$ are no longer independent so that the standard result on W_n does not readily apply to that on $\hat{W}_n$. We are primarily concerned in showing that under suitable regularity conditions, as $n \to \infty$, $\hat{W}_n$ converges weakly to an appropriate Gaussian function on $D[0, 1]$. In particular, when the c_i, $i \geqq 1$ are all equal, $\hat{W}_n$ weakly converges to a tied-down Wiener process $W_0 = \{W_0(t) = W(t) - tW(1), t \in I\}$. The preliminary notions and the basic regularity conditions are introduced in Section 2. The main weak convergence results are presented in Section 3. In this context, asymptotic linearity of signed rank statistics (in the shift parameter) is extended to appropriate stochastic processes constructed from these statistics. The last section deals with two important problems of nonparametric statistical inference where such aligned rank order processes are useful. Specifically, the problem of testing for a shift of location at an unknown point of time and the problem of testing symmetry about an unknown origin are treated and in both cases asymptotic nonparametric tests are developed.

2. Preliminary notions. We make the following assumptions: (i) the df F admits of an absolutely continuous probability density function (pdf) f with a finite Fisher information

$$\mathscr{I}(f) = \int_{-\infty}^{\infty} \{f'(x)/f(x)\}^2 \, dF(x) \qquad \text{where} \quad f'(x) = (d/dx)f(x) , \tag{2.1}$$

(ii) $\phi^+(u) = \phi(\frac{1}{2}(1+u))$, $0 < u < 1$ where the score function $\phi(u)$ is square integrable inside I and $\phi(u) + \phi(1-u) =$ constant, for all $0 < u < 1$. Without any loss of generality, we may let $\phi(\frac{1}{2}) = 0$, so that

$$A^2 = \int_0^1 \{\phi^+(u)\}^2 \, du = \int_0^1 \phi^2(u) \, du < \infty . \tag{2.2}$$

Concerning the c_i, we assume that (a) (1.8) holds and (b) there exist two real numbers c_0 and C_0 (> 0) such that

$$\lim_{n\to\infty} n^{-1} \sum_{i=1}^n c_i = c_0 \qquad \text{and} \qquad \lim_{n\to\infty} n^{-1} C_n^{\,2} = C_0^{\,2} \quad \text{both exist;} \tag{2.3}$$

we let

$$c^* = c_0/C_0^{\,2} \qquad \text{and} \qquad c^{**} = (c_0/C_0)^2 . \tag{2.4}$$

Also, for some nondecreasing score function $\tilde{\phi}^+(u)$, $0 < u < 1$, satisfying assumption (ii) (made on ϕ^+), we define as in (1.1),

$$\tilde{T}_n^{\,0} = \tilde{T}_n^{\,0}(X_1, \cdots, X_n) = \sum_{i=1}^n \operatorname{sgn} X_i \tilde{a}_n(R_{ni}^+) \quad (\text{note here } c_i = 1, i \geqq 1) \tag{2.5}$$

where $\tilde{a}_n(i)$ is defined as in (1.2) with ϕ^+ being replaced by $\tilde{\phi}^+$. Let then

$$\tilde{T}_n^{\,0}(a) = \tilde{T}_n^{\,0}(X_1 - a, \cdots, X_n - a) \qquad \text{for} \quad -\infty < a < \infty ; \tag{2.6}$$

by assumption, $\tilde{T}_n^{\,0}(a)$ is nonincreasing in a, and we define

$$\begin{aligned} &\tilde{\theta}_n^{\,(1)} = \sup\{a : \tilde{T}_n^{\,0}(a) > 0\} , \qquad \tilde{\theta}_n^{\,(2)} = \inf\{a : \tilde{T}_n^{\,0}(a) < 0\} ; \\ &\tilde{\theta}_n = \tfrac{1}{2}(\tilde{\theta}_n^{\,(1)} + \tilde{\theta}_n^{\,(2)}) . \end{aligned} \tag{2.7}$$

Then, $\tilde{\theta}_n$ is a robust, translation-invariant and consistent estimator of θ (viz., [7]) and in (1.11), we use $\tilde{\theta}_n$ for our alignment process.

3. Weak convergence of $\hat{W}_n$. First, we consider the following result which will be required in the sequel. Let us introduce

$$\begin{aligned} &H_0\colon \ \ (1.10) \text{ holds with } \ \theta = 0 \qquad \text{and} \\ &K_n^{(b)}\colon \ \ (1.10) \text{ holds with } \ \theta = C_n^{-1} b , \end{aligned} \tag{3.1}$$

where b is real and finite. Also, let

$$\nu(\phi, \psi) = \int_0^1 \phi(u)\psi(u) \, du \qquad \text{where} \quad \psi(u) = -f'(F^{-1}(u))/f(F^{-1}(u)), \ 0 < u < 1 . \tag{3.2}$$

THEOREM 3.1. *Suppose that* (A) *the* c_i *satisfy the conditions* (a) *and* (b) *of Section* 2, (B) *the* df F *satisfies the condition* (i) *and* (C) ϕ^+ *is nondegenerate and satisfies condition* (ii) *of Section* 2. *Then, on defining* W_n *by* (1.6)—(1.7) *and* W *as before by* (1.9), *we have under* $\{K_n^{(b)}\}$ *in* (3.1),

$$W_n \to_{\mathscr{L}} \{W(t) + c^* A^{-1} bt\nu(\phi, \psi), t \in I\}, \quad \text{as} \quad n \to \infty . \tag{3.3}$$

Under H_0 *in* (3.1), (3.3) *holds* (*with* $b = 0$) *for any continuous* F_0 *when* (C) *and* (1.8) *hold.*

PROOF. First, consider the null hypothesis case when $b = 0$. Define

$$S_k = \textstyle\sum_{i=1}^k c_i \operatorname{sgn} X_i \phi(F_0(|X_i|)), \qquad k \geqq 1;\ S_0 = 0 . \tag{3.4}$$

Then, note that under H_0, S_n has mean 0 and variance $A^2 C_n^2$, $n \geqq 1$. Also, let $\mathscr{B}_n$ be the sigma-field generated by $(R_{n1}^+, \cdots, R_{nn}^+)$ and $(\operatorname{sgn} X_1, \cdots, \operatorname{sgn} X_n)$. Then, a little examination reveals that $T_n = E(S_n \mid \mathscr{B}_n, H_0)$ and for every $k : 1 \leqq k \leqq n$, $A_n^{-2} C_n^{-2} E\{(T_k - S_k)^2 \mid H_0\} = A_n^{-2} C_n^{-2} C_k^2 (A^2 - A_k^2)$. Note that by (1.3) and (1.8), C_n^2 is $\nearrow$ in n and it goes to ∞ as $n \to \infty$, and we can always choose a sequence $\{m_n\}$ of positive integers such that as $n \to \infty$, $m_n \to \infty$ but $C_{m_n}^2 / C_n^2 \to 0$. Then, by (1.4) and (1.5),

$$\begin{aligned} &\max_{1 \leqq k \leqq n} C_n^{-2} A_n^{-2} (A^2 - A_k^2) C_k^2 \\ &\quad = \max \left\{ \max_{k \leqq m_n} \frac{C_k^2 (A^2 - A_k^2)}{C_n^2 A_n^2},\ \max_{m_n < k \leqq n} \frac{C_k^2 (A^2 - A_k^2)}{C_n^2 A_n^2} \right\} \\ &\quad \leqq \max \{ (A^2 / A_n^2) C_n^{-2} C_{m_n}^2,\ A_n^{-2} \max_{m_n < k \leqq n} (A^2 - A_k^2) \} \to 0 \end{aligned} \tag{3.5}$$

as $n \to \infty$. Thus,

$$\max_{1 \leqq k \leqq n} E\{(T_k - S_k)^2 \mid H_0\} / A_n^2 C_n^2 \to 0 \quad \text{as} \quad n \to \infty . \tag{3.6}$$

Also, note that by (1.5) and (1.8),

$$\begin{aligned} &\max_{1 \leqq k \leqq n} E\{(S_k - S_{k-1})^2 \mid H_0\} / A_n^2 C_n^2 \\ &\quad = \max_{1 \leqq k \leqq n} (c_k^2 / C_n^2)(A^2 / A_n^2) \to 0 \quad \text{as} \quad n \to \infty . \end{aligned} \tag{3.7}$$

Hence, if in (1.6)—(1.7), we replace the T_k by S_k $(0 \leqq k \leqq n)$ and denote the corresponding process by W_n^*, then the finite dimensional distributions (f.d.d.) of W_n and W_n^* are asymptotically the same. On the other hand, under (3.7), the special central limit theorem (on page 153) of Hájek and Šidák (1967) applies to finitely many S_k, and hence, the f.d.d.'s of W_n^* can easily be shown to be asymptotically the same as those of W. Thus, to prove (3.3) in this case, we need to show only that W_n is *tight*. For this note that by (3.6) and (3.7),

$$\max_{1 \leqq k \leqq n} E\{(T_k - T_{k-1})^2 \mid H_0\} / A_n^2 C_n^2 \to 0 \quad \text{as} \quad n \to \infty . \tag{3.8}$$

Let us define

$$\omega_\delta(W_n) = \sup \{ |W_n(t) - W_n(s)| : 0 \leqq s < t \leqq s + \delta \leqq 1 \}, \quad \text{for } 0 < \delta < 1 . \tag{3.9}$$

Then, by using inequality (14.9) (on page 110) and Theorem 15.2 (on page 125) of Billingsley (1968) and noting that here $W_n(0) = 0$, with probability 1, it suffices to show that for every $\varepsilon > 0$ and $\eta > 0$, there exist a $\delta: 0 < \delta < 1$ and an integer, n_0, such that

$$P\{\omega_\delta(W_n) > \varepsilon\} \leqq \eta, \quad \forall\, n \geqq n_0. \tag{3.10}$$

Further, as in Section 3 of Sen and Ghosh (1973), under H_0, whenever ϕ^+ is integrable inside I, $\{T_n, \mathscr{B}_n; n \geqq 1\}$ is a martingale (insuring the applicability of Lemma 4 of Brown (1971)). By virtue of (3.8), the convergence of the f.d.d.'s of $\{W_n\}$ to those of W and the martingale property of $\{T_n\}$, we may virtually repeat the proof of Theorem 3 (viz., (25)—(26)) of Brown (1971); for intended brevity, the details are omitted. This completes the proof of (3.3) for the null hypothesis case; note that here (C) and (1.8) suffice.

Next, let the joint distribution of $(X_1, \cdots, X_n)$ under $K_n^{(b)}$ be denoted by $P_{n,b}$, so that $P_{n,0}$ relates to the null case. By reference to Hájek and Šidák (1967, Chapter VI) (see also van Eeden (1972, page 797)), under the hypothesis of the theorem, $\{P_{n,b}\}$ is contiguous to $\{P_{n,0}\}$. Hence, if we define $B_n^{(\delta)} = [\omega_\delta(W_n) > \varepsilon]$, we have from (3.10) and the contiguity of $P_{n,b}$ to $P_{n,0}$ that

$$\lim_{\delta\to 0}\{\limsup_{n\to\infty} P(B_n^{(\delta)} \mid K_n^{(b)})\} = 0. \tag{3.11}$$

Also, by definition, $W_n(0) = 0$, with probability 1, for every $n \geqq 1$. Hence, (3.11) along with Theorem 15.2 of Billingsley (1968, page 125) insures that W_n remains tight under $K_n^{(b)}$ as well. Thus, to prove (3.3), all we need to show is that the f.d.d.'s of $\{W_n\}$ converge to those of $\{W(t) + c^*A^{-1}bt\nu(\phi, \psi), t \in I\}$ when $\{K_n^{(b)}\}$ holds. By (3.6) (insuring the asymptotic equivalence in probability, under H_0) and the contiguity of $\{P_{n,b}\}$ to $\{P_{n,0}\}$, we claim that for every (fixed) m $(\geqq 1)$ and $t_1, \cdots, t_m$ (all $\in I$), as $n \to \infty$,

$$\max\{|W_n(t_j) - W_n^*(t_j)| : 1 \leqq j \leqq m\} \to_p 0, \quad \text{under} \quad \{K_n^{(b)}\}. \tag{3.12}$$

On the other hand, under $K_n^{(b)}$, S_k involves a triangular array of independent rv's, so that under the assumptions (A, B, C) of the theorem, the classical central limit theorem yields that $(W_n^*(t_1), \cdots, W_n^*(t_m))$ is asymptotically multinormal with mean vector

$$A^{-1}c^*b\nu(\phi, \psi)(t_1, \cdots, t_m) \tag{3.13}$$

and dispersion matrix

$$(t_i \wedge t_j)_{i,j=1,\cdots,m} \qquad \text{where} \quad a \wedge b = \min(a, b). \tag{3.14}$$

Therefore, the proof of the theorem is complete. □

REMARK. (3.3) is an extension of a theorem in Hájek and Šidák (1967, page 220) where it is shown that under $K_n^{(b)}$, $W_n(1) \to_{\mathscr{L}} W(1) + A^{-1}c^*b\nu(\phi, \psi)$. Towards the end of this section we shall comment on the case where the scores may not be defined by (1.2).

Because of the translation-invariance of $\tilde{\theta}_n$ (viz., [7]), in the sequel, without any loss of generality, we let $\theta = 0$ in the model (1.10). For every real b, let us define

$$\bar{T}_{k,n}(b) = T_k(X_1 - b/C_n, \cdots, X_k - b/C_n), \qquad 1 \leqq k \leqq n;\ \bar{T}_{0,n}(b) = 0, \tag{3.15}$$

where the T_k are defined by (1.1). On replacing the T_k by $\bar{T}_{k,n}$, $1 \leqq k \leqq n$ in (1.7), we denote the corresponding stochastic process by $\bar{W}_{n,b} = \{\bar{W}_{n,b}(t), t \in I\}$, so that $\bar{W}_{n,0} = W_n$. Then, we prove the following.

THEOREM 3.2. *Under the assumptions* (A, B, C) *of Theorem* 3.1, *when* H_0 *in* (3.1) *holds, for every real and finite* b,

$$\{\bar{W}_{n,b}(t) + A^{-1}c^*bt\nu(\phi, \psi), t \in I\} \to_{\mathscr{L}} W, \qquad \text{as} \quad n \to \infty. \tag{3.16}$$

PROOF. Let $\prod_{n,b}$ be the distribution of $\bar{W}_{n,b}$ (under H_0 in (3.1)), defined for Borel subsets $\mathscr{D}$ of $D[0, 1]$ (relative to the Skorokhod metric), so that

$$\textstyle\prod_{n,b}(D) = P\{\bar{W}_{n,b} \in D \mid H_0\}, \qquad \text{for every} \quad D \in \mathscr{D}. \tag{3.17}$$

Also, let $\prod_{n,b}^0$ be the distribution of W_n in $D[0, 1]$ when $K_n^{(b)}$ in (3.1) holds, i.e.,

$$\textstyle\prod_{n,b}^0(D) = P\{W_n = \bar{W}_{n,0} \in D \mid K_n^{(b)}\}, \qquad \text{for every} \quad D \in \mathscr{D}. \tag{3.18}$$

If $R_{ni}^+(b)$ be the rank of $|X_i - b/C_n|$ among $|X_1 - b/C_n|, \cdots, |X_n - b/C_n|$, $i = 1, \cdots, n$, $\mathbf{R}_n^+(b) = (R_{n1}^+(b), \cdots, R_{nn}^+(b))$ and $\mathbf{Q}_n(b) = (\operatorname{sgn}(X_1 - b/C_n), \cdots, \operatorname{sgn}(X_n - b/C_n))$, then obviously, the process $\bar{W}_{n,b}$ is a mapping of $(\mathbf{R}_n^+(b), \mathbf{Q}_n(b))$ into the space $D[0, 1]$, and similarly, $W_n = \bar{W}_{n,0}$ is a mapping of $(\mathbf{R}_n^+(0), \mathbf{Q}_n(0))$ into the $D[0, 1]$ space. Also,

$$[\mathbf{R}_n^+(b), \mathbf{Q}_n(b)), \text{ under } H_0] =_{\mathscr{L}} [(\mathbf{R}_n^+(0), \mathbf{Q}_n(0)), \text{ under } K_n^{(-b)}], \tag{3.19}$$

where $=_{\mathscr{L}}$ stands for the identity of distributions. Thus, for every D in $\mathscr{D}$, there exists a E_n in E^n, the n-dimensional Euclidean space, such that by (3.19),

$$\begin{aligned} \textstyle\prod_{n,b}(D) &= P\{\bar{W}_{n,b} \in D \mid H_0\} \\ &= P\{A_n^{-1}C_n^{-1}[\bar{T}_{1,n}(b), \cdots, \bar{T}_{n,n}(b)] \in E_n \mid H_0\} \\ &= P\{A_n^{-1}C_n^{-1}[\bar{T}_{1,n}(0), \cdots, \bar{T}_{n,n}(0)] \in E_n \mid K_n^{(-b)}\} \\ &= P\{\bar{W}_{n,0} \in D \mid K_n^{(-b)}\} = \textstyle\prod_{n,(-b)}^0(D). \end{aligned} \tag{3.20}$$

Consequently,

$$\{\bar{W}_{n,b}, \text{ under } H_0\} =_{\mathscr{L}} \{W_n,' \text{ under } K_n^{(-b)}\}, \qquad \text{for every} \quad n \geqq 1, \tag{3.21}$$

and hence, the rest of the proof follows from Theorem 3.1. □

For b belonging to a bounded interval, the asymptotic linearity (in probability) of $C_n^{-1}(\bar{T}_{n,n}(b) - \bar{T}_{n,n}(0))$ has been studied by van Eeden (1972) along the lines of Jurečková (1969). By using our Theorem 3.2, we are able to present the following result strengthening the linearity to that of the process $\bar{W}_{n,b} - \bar{W}_{n,0}$.

THEOREM 3.3. *Suppose that in addition to the conditions stated in Theorem* 3.2, $\phi^+(u)$ *is* $\nearrow$ *in* u: $0 < u < 1$ *and the* c_i *are all nonnegative. Then, under* H_0 *in* (3.1) *and for every (fixed) K: $0 < K < \infty$,*

$$\sup\{|\bar{W}_{n,b}(t) - W_n(t) + c^* A^{-1} b\nu(\phi, \psi)t| : |b| \leqq K,\ t \in I\} \to_p 0\,, \quad \text{as } n \to \infty\,. \tag{3.22}$$

PROOF. For a given K and $\varepsilon > 0$, one can always select a set of $m = m(K, \varepsilon)$ points, such that

$$-K = b_1 > \cdots < b_m = K\,; \qquad (b_{j+1} - b_j)c^* A^{-1}\nu(\phi, \psi) \leqq \varepsilon/2\,, \quad 1 \leqq j \leqq m\,. \tag{3.23}$$

Since, by assumption, the c_i are nonnegative and $\bar{T}_{k,n}(b)$ is nonincreasing in b for every $k\ (\leqq n)$, proceeding as in van Eeden (1972, page 799), we obtain by a few standard steps that

$$\begin{aligned} &\sup\{|\bar{W}_{n,b}(t) - W_n(t) + c^* A^{-1} bt\nu(\phi, \psi)| : t \in I,\ |b| \leqq K\} \\ &\quad \leqq \max_{1 \leqq j \leqq m}\{\sup_{t \in I} |\bar{W}_{n,b_j}(t) - W_n(t) + c^* A^{-1} b_j t\nu(\phi, \psi)|\} + \varepsilon/2\,. \end{aligned} \tag{3.24}$$

Also, for each $j\ (1 \leqq j \leqq m)$, if one considers the points $(0 \leqq) t_1 < \cdots < t_q (\leqq 1)$, $q (\geqq 1)$ fixed, then, along the lines of van Eeden (1972), it can be shown that the joint distribution of $(\{\bar{W}_{n,b_j}(t_s) - \bar{W}_{n,0}(t_s) + c^* A^{-1} b_j t_s \nu(\phi, \psi)\},\ s = 1, \cdots, q)$ (under H_0 in (3.1)) is asymptotically a degenerate multinormal with null mean vector and null dispersion matrix. On the other hand, defining $\omega_\delta(x)$ as in (3.9), and noting that $\omega_\delta(x - y) \leqq \omega_\delta(x) + \omega_\delta(y)$, we obtain by using Theorem 3.2 that for each $j\ (1 \leqq j \leqq m)$,

$$\begin{aligned} &P\{\omega_\delta([\bar{W}_{n,b_j}(t) - \bar{W}_{n,0}(t) + c^* A^{-1} b_j t\nu(\phi, \psi),\ t \in I]) > \tfrac{1}{2}\varepsilon \mid H_0\} \\ &\quad \leqq P\{\omega_\delta([\bar{W}_{n,b_j}(t) + c^* A^{-1} b_j t\nu(\phi, \psi),\ t \in I]) > \tfrac{1}{4}\varepsilon \mid H_0\} \\ &\qquad + P\{\omega_\delta([\bar{W}_{n,0}(t),\ t \in I]) > \tfrac{1}{4}\varepsilon \mid H_0\} \\ &\quad \leqq \eta'\,, \qquad \text{for every} \quad n \geqq n_0(\varepsilon, \eta')\,, \end{aligned} \tag{3.25}$$

when $\delta\ (> 0)$ is chosen sufficiently small. From the above, we conclude that for every (fixed) $j\ (1 \leqq j \leqq m)$, as $n \to \infty$,

$$\sup\{|\bar{W}_{n,b_j}(t) - \bar{W}_{n,0}(t) + c^* A^{-1} b_j t\nu(\phi, \psi)| : t \in I\} \to_p 0\,. \tag{3.26}$$

Since, for given K and ε, m is fixed, (3.22) follows from (3.24) and (3.26). □

REMARK. The condition that ϕ^+ is nondecreasing can be replaced by the condition that $\phi^+(u)$ is the difference of two nondecreasing functions $\phi_1^+(u)$ and $\phi_2^+(u)$, $0 < u < 1$. In that case, $\bar{W}_{n,b} - \bar{W}_{n,0}$ can also be expressed as a difference of two such processes where, in each case, the score function is nondecreasing; and hence the proof sketched before stands valid.

We are now in a position to formulate the main results of this section. Let us define ϕ and $\tilde{\phi}$ as in Section 2, A^2 as in (2.2) and $\tilde{A}^2$ by the same equation with ϕ replaced by $\tilde{\phi}$. Let then

$$L(\phi, \tilde{\phi}) = \left(\textstyle\int_0^1 \phi(u)\tilde{\phi}(u)\, du\right)/(A\tilde{A})\,, \tag{3.27}$$

$$\gamma = \gamma(\phi, \tilde{\phi}, \psi) = \tilde{A}A^{-1}[\nu(\phi, \psi)/\nu(\tilde{\phi}, \psi)]\,, \tag{3.28}$$

and, finally, let $Y = \{Y(t), t \in I\}$ be a Gaussian function on the $C[0, 1]$ space with

$$EY = 0 \quad \text{and} \quad EY(s)Y(t) = s \wedge t - c^{**}st\{2\gamma L(\phi, \tilde{\phi}) - \gamma^2\}, \qquad s, t \in I \tag{3.29}$$

where c^{**} is defined by (2.4), and is bounded from above by 1.

THEOREM 3.4. *Under* (1.10) *and assumptions* (i), (ii) *and* (a, b, c) *of Section* 2, *as* $n \to \infty$,

$$\hat{W}_n \to_{\mathscr{L}} Y, \quad \text{in the Skorokhod } J_1\text{-topology on } D[0, 1]. \tag{3.30}$$

PROOF. As before, we let $\theta = 0$ and consider the estimator $\tilde{\theta}_n$, defined by (2.7). Then, it is known (viz., [7]) that $n^{\frac{1}{2}}\tilde{\theta}_n$ is asymptotically normal with mean 0 and variance $\tilde{A}^2/\nu^2(\tilde{\phi}, \psi)$, and by (2.3), $n^{-\frac{1}{2}}C_n \to C_0$ as $n \to \infty$. Thus, $C_n|\tilde{\theta}_n| = O_p(1)$. Thus, writing $b = C_n\tilde{\theta}_n$ and noting that $\bar{W}_{n,b} = \hat{W}_n$, we obtain from Theorem 3.3 that

$$\sup\{|\hat{W}_n(t) - W_n(t) + A^{-1}(c_0/C_0)n^{\frac{1}{2}}\tilde{\theta}_n t\nu(\phi, \psi)| : t \in I\} \to_p 0, \quad \text{as } n \to \infty. \tag{3.31}$$

Parallel to (1.6), let us define

$$\tilde{k}_n(t) = \max\{k : k\tilde{A}_k^2 \leqq tn\tilde{A}_n^2\}, \qquad t \in I, \tag{3.32}$$

and let $\tilde{W}_n^0 = \{\tilde{W}_n^0(t) = n^{-\frac{1}{2}}\tilde{A}_n^{-1}\tilde{T}^0_{\tilde{k}_n(t)}, t \in I\}$, where the $\tilde{T}_k^0$ are defined by (2.5). Then, using (2.7) and (3.31) (for this particular case), we obtain that as $n \to \infty$,

$$n^{\frac{1}{2}}\tilde{\theta}_n\nu(\tilde{\phi}, \psi) = \tilde{A}\tilde{W}_n^0(1) + o_p(1). \tag{3.33}$$

Hence, from (3.28), (3.31) and (3.33), we obtain that as $n \to \infty$,

$$\sup\{|\hat{W}_n(t) - W_n(t) + \gamma t(c_0/C_0)\tilde{W}_n^0(1)| : t \in I\} \to_p 0. \tag{3.34}$$

Note that by (3.32), $n^{-1}\tilde{k}_n(t) \cdot \tilde{A}_n^{-2}\tilde{A}^2_{\tilde{k}_n(t)} \leqq t < n^{-1}(\tilde{k}_n(t) + 1)\tilde{A}_n^{-2}\tilde{A}^2_{1+\tilde{k}_n(t)}$, and hence, using (1.5) (for the score function $\tilde{\phi}$), we obtain that

$$\lim_{n\to\infty} n^{-1}\tilde{k}_n(t) = t \quad \text{for every} \quad t \in I. \tag{3.35}$$

Similarly, by (1.6), (1.5), (1.8) and (2.3),

$$\lim_{n\to\infty} C_n^{-2}C^2_{k_n(t)} = \lim_{n\to\infty} n^{-1}k_n(t) = t, \quad \text{for every} \quad t \in I. \tag{3.36}$$

Further, for any $1 \leqq k, q \leqq n$, letting $k^* = k \wedge q$, we have under H_0 in (3.1),

$$\begin{aligned} &E\{[A_n^{-1}C_n^{-1} \textstyle\sum_{i=1}^k c_i \operatorname{sgn} X_i \phi(F_0(|X_i|))][\tilde{A}^{-1}n^{-\frac{1}{2}} \sum_{i=1}^q \operatorname{sgn} X_i \tilde{\phi}(F_0(|X_i|))] \mid H_0\} \\ &\quad = A_n^{-1}C_n^{-1}\tilde{A}_n^{-1}n^{-\frac{1}{2}} \textstyle\sum_{i=1}^{k^*} c_i E\{\tilde{\phi}(F_0(|X_i|))\phi(F_0(|X_i|)) \mid H_0\} \\ &\quad = (A_n^{-1}A)(\tilde{A}_n^{-1}\tilde{A})n^{-\frac{1}{2}}C_n^{-1}k^*\bar{c}_{k^*}L(\phi, \tilde{\phi}) \\ &\quad \sim (c_0/C_0)(n^{-1}(k \wedge q))L(\phi, \tilde{\phi})(\bar{c}_{k^*}/c_0), \end{aligned} \tag{3.37}$$

so that, by (3.35), (3.36), (3.37) and the definitions of W_n and $\tilde{W}_n^0$, we have

$$\lim_{n\to\infty} E\{W_n(s)\tilde{W}_n^0(t) \mid H_0\} = (c_0/C_0)(s \wedge t)L(\phi, \tilde{\phi}). \tag{3.38}$$

Finally, by using (3.35)—(3.38), we may consider a bivariate extension of the proof of the first part of Theorem 3.1, and show that under H_0 in (3.1), the bivariate process $\{(W_n(t), \tilde{W}_n^0(t))', t \in I\}$ converges weakly to a bivariate Gaussian function $\{(W(t), \tilde{W}^0(t))', t \in I\}$, where marginally each of W and $\tilde{W}^0$ is a standard Wiener process and $E\{W(t)\tilde{W}^0(s)\} = (c_0/C_0)(s \wedge t)L(\phi, \tilde{\phi})$, for every $s, t \in I$. Hence, $\{W_n(t) - \gamma(c_0/C_0)t\tilde{W}_n^0(1) : t \in I\} \rightarrow_{\mathscr{L}} Y$, and the proof of the theorem follows from (3.34).

COROLLARY 3.4.1. *Under the hypothesis of Theorem* 3.4, *if* $c^{**} = 1$ *and* $\phi = \tilde{\phi}$, *then* $\hat{W}_n$ *converges weakly to a standard tied-down Wiener process* W_0.

REMARKS. (i) Consider the case where the c_i are not necessarily nonnegative, and let

$$c_i^+ = \max\{c_i, 0\} \quad \text{and} \quad c_i^- = \max\{0, -c_i\}, \quad \text{for } i \geqq 1. \tag{3.39}$$

Note that in Theorems 3.1 and 3.2, we do not need the c_i to be all $\geqq 0$; however, we need the assumption in Theorems 3.3 and 3.4 where the monotonicity of $\bar{T}_{k,n}(b)$ in b rests on this condition. By using (3.39), it is possible to express $T_{k,n}(b)$ (and hence, $\bar{W}_{n,b}$) as a difference of two similar quantities where in each case the coefficients are all nonnegative. As such, if each of the two sequences $\{c_i^+, i \geqq 1\}$ and $\{c_i^-, i \geqq 1\}$ satisfies the conditions (a) and (b) of Section 2, then the conclusions of Theorems 3.3 and 3.4 remain valid.

(ii) So far, we have specialized ourselves to scores in (1.2). Often, scores are taken as $\phi(i/(n+1))$, $i = 1, \cdots, n$, or in some other forms. For such scores, the martingale property, mentioned after (3.10), may not hold; and hence, the tightness of W_n (under H_0 in (3.1)) may require a different proof. Towards this, we note that if the actual scores are denoted by $a_n^*(i)$, $i = 1, \cdots, n$ and the $a_n(i)$ are defined by (1.2), then letting $b_n(i) = a_n^*(i) - a_n(i)$, $i = 1, \cdots, n$, $B_n^2 = n^{-1}\sum_{i=1}^n b_n^2(i)$, we obtain on using (3.4) of Sen and Ghosh (1973) that for the special case of $c_i = 1$, $i \geqq 1$,

$$\begin{aligned} P_0\{\max\nolimits_{1\leq k\leq n} |\textstyle\sum_{i=1}^k \operatorname{sgn} X_i b_k(R_{ki}^+)| > \varepsilon n^{\frac{1}{2}}\} &\leqq 2\textstyle\sum_{k=1}^n [\inf_{t>0}\{\exp(-n^{\frac{1}{2}}\varepsilon t + \frac{1}{2}t^2kB_k^2)\}] \\ &= 2\textstyle\sum_{k=1}^n \{\exp(-\frac{1}{2}(n/k)\varepsilon^2/B_k^2)\}. \end{aligned} \tag{3.40}$$

Now, by Theorem b (page 158) and Lemmas a, b (pages 164–165) of Hájek and Šidák (1967), under assumption (ii) of Section 2, $B_n^2 \rightarrow 0$ for $a_n^*(i) = \phi(i/(n+1))$ or some related scores. We assume that for some $r > 1$,

$$B_n^2 \leqq K(\log n)^{-r}, \quad \text{for all } n \geqq n_0 \text{ where } K \text{ is a positive (finite) constant.} \tag{3.41}$$

Under (3.41), it is easy to show that the right-hand side of (3.40) converges to 0 as $n \rightarrow \infty$, so that the tightness of W_n, based on the scores in (1.2), implies the same for the scores $a_n^*(i)$. Now, it follows from Hoeffding (1973) that (3.41) holds whenever the square integrability assumption of ϕ in Section 2 is replaced by

$$(3.42)\qquad \int_0^1 \phi^2(u)\{\log(1+|\phi(u)|)\}^r\,du < \infty \qquad \text{for some} \quad r > 1\,,$$

and, in this sense, (3.41) is not very restrictive. When the c_i are not equal, the inequality (3.40) is difficult to extend in this full generality. However, on noting that by the Schwarz inequality

$$(3.43)\qquad \begin{aligned} &A_n^{-1}C_n^{-1}\{\max_{1\leq k\leq n}|\sum_{i=1}^k c_i \operatorname{sgn} X_i b_k(R_{ki}^+)|\} \\ &\qquad \leqq A_n^{-1}\{\max_{1\leq k\leq n}(C_k/C_n)k^{\frac{1}{2}}B_k\}\,, \end{aligned}$$

we claim that $\lim_{n\to\infty} nB_n^2 = 0$ implies that the tightness of W_n for scores in (1.2) implies the tightness for the scores $a_n^*(i)$. Such a condition on B_n, of course, demands much more than (3.42) or (2.2). However, under the usual Chernoff-Savage condition on the score function, it holds (cf. Section 10.5 of [12]).

(iii) As in van Eeden (1972), one may also be interested in replacing in (1.1) $\{c_i\}$ by a double sequence $\{c_{n,i}, i \leqq n; n \geqq 1\}$. Here also, if $c_{n,i} = c_i/C_n$ for all $i \leqq n(\geqq 1)$, then the results go through easily. On the other hand, for arbitrary $c_{n,i}$, in the absence of any martingale or related properties of the $T_{k,n}$, $k \leqq n$, it may be considerably difficult to establish the tightness property of W_n. This, in turn, introduces difficulties in the proof of the other theorems of this section.

(iv) In all the four theorems, the stochastic processes considered belong to the $D[0, 1]$ space. We could have constructed (by linear interpolation) parallel processes belonging to the $C[0, 1]$ space; in view of inequality (14.9) of Billingsley (1968, page 110), the same proofs will be applicable for such processes too. On the other hand, the current construction, besides being more general, leads to some simplifications of the expressions for certain functionals of these processes, as will be considered in Section 4. At this stage, we may refer to McLeish (1974) for certain invariance principles for dependent rv's which include the results of Brown (1971) as special cases.

4. Some asymptotically distribution-free tests based on the weak convergence of aligned rank order processes. We consider the following problems where the tests are based on appropriate stochastic processes constructed from suitable aligned rank order statistics. We shall incorporate the results of Section 3 to study the properties of these tests; some allied new results will also be deduced in this context.

4.1. *Some nonparametric tests for shift at an unknown time point.* Let $\{X_i, i \geqq 1\}$ be independent random variables with continuous df's $\{F_i(x), x \in E; i \geqq 1\}$, where

$$(4.1)\qquad \begin{aligned} &F_i(x) = F(x - \theta - \beta_i)\,, \\ &\qquad\qquad i = 1, \cdots, n\,; \quad F \;\text{ symmetric,}\quad \theta \;\text{ unknown} \end{aligned}$$

and $\beta_1, \cdots, \beta_n$ are also unknown. Our null hypothesis is $H_0\colon \beta_1 = \cdots = \beta_n = 0$ and we are interested in the alternative $K = \bigcup_{m=1}^{n-1} K_m$, where

$$(4.2)\qquad \begin{aligned} &K_m\colon \beta_1 = \cdots = \beta_m = 0\,, \\ &\beta_{m+1} = \cdots = \beta_n = \beta \neq 0 \qquad \text{for} \quad m = 1, \cdots, n-1\,. \end{aligned}$$

Thus, K relates to a shift at an unknown point of time. Bhattacharyya and Johnson (1968) have considered some simple nonparametric tests for this problem along with a review of available parametric tests. By reference to the criterion of best average power (where averaging is taken over arbitrary weighings), they obtained locally most powerful invariant rank tests; by nature, these tests are Bayesian in structure. Without any appeal to this criterion of best average power, we may consider some alternative tests as follows.

Under H_0, the common location θ may be estimated by $\tilde{\theta}_n$ in (2.7), assuming that F is symmetric about 0. Also, we construct the $\hat{T}^0_{k,n}$ as in (1.1) with $c_i = 1$, $i \geqq 1$. If H_0 is not true, $\tilde{\theta}_n$ will estimate some intermediate point between θ and $\theta + \beta$, so that the $\hat{T}^0_{k,n}$ will be consistently deflected for $k \leqq m$; while for $k > m$, this deflection will be gradually drifted. As such, suitable functionals of $\hat{W}_n$ may be used to test for H_0 against K. Since $A_n^2 \to A^2$ (viz., (1.4)), in (1.5), one could have defined $k_n(t) = \max\{k : C_k^2 \leqq tC_n^2\}$, $t \in I$ and as in this case, $c_i = 1, \forall i \geqq 1$, we can take $k_n(t) = \max\{k : k/n \leqq t\}$, $t \in I$. With this modification in the definition of W_n and $\hat{W}_n$, we propose the following test statistics:

$$D_n^+ = \sup_{t\in I} \hat{W}_n(t) = n^{-\frac{1}{2}}A_n^{-1}\{\max_{1\leqq k\leqq n} \hat{T}^0_{k,n}\}\,, \tag{4.3}$$

$$D_n = \sup_{t\in I} |\hat{W}_n(t)| = n^{-\frac{1}{2}}A_n^{-1}\{\max_{1\leqq k\leqq n} |\hat{T}^0_{k,n}|\}\,, \tag{4.4}$$

$$D_n^* = \textstyle\int_0^1 \hat{W}_n^2(t)\,dt = n^{-2}A_n^{-1}\{\sum_{k=1}^{n-1} (\hat{T}^0_{k,n})^2\}\,. \tag{4.5}$$

D_n^+ is suitable for one-sided alternative ($\beta < 0$), while D_n and D_n^* are for two-sided ones. Now, in (2.5), we use the same score function ϕ, so that $\tilde{a}_n(i) = a_n(i)$, $i = 1, \cdots, n$. Also, here all the c_i are equal to 1, so that we are in a position to use Corollary 3.4.1 when the null hypothesis holds. Thus, under H_0, D_n^+ and D_n have respectively the same asymptotic distributions as that of the one-sided and two-sided Kolmogorov–Smirnov statistics for the classical goodness of fit problem (for the latter, see pages 199–200 of Hájek and Šidák (1967)). Also, under H_0, D_n^* has the same asymptotic distribution as that of the classical Cramér–von Mises goodness of fit statistic [cf. page 192 of Hájek and Šidák (1967)]. Thus, large sample tests based on D_n^+, D_n and D_n^* can be carried out by appeal to the critical values of the classical goodness of fit tests for the one sample problem.

Next, we consider the asymptotic nonnull distributions of D_n^+, D_n and D_n^* for contiguous alternatives $\{K_n^*\}$, where

$$\begin{aligned} &K_n^* : \text{(4.2) holds for some} \quad m = m_n \quad \text{and} \quad \beta = n^{-\frac{1}{2}}\delta\,, \quad \delta \neq 0\,, \\ &\text{with} \quad n^{-1}m_n \to \gamma : 0 < \gamma < 1 \quad \text{and} \quad (\gamma, \delta) \quad \text{is fixed.} \end{aligned} \tag{4.6}$$

We note that under K_n^*, $X_1, \cdots, X_{m_n}$ are i.i.d. rv's with df $F(x - \theta)$ and $X_{m_n+1}, \cdots, X_n$ are i.i.d. rv's with df $F(x - \theta - n^{-\frac{1}{2}}\delta)$, while the assumptions (i) and (ii) of Section 2 insure contiguity under (4.6). As such, proceeding as in the proof of Theorem 3.1 and noting that here $c_i = 1$, $i \geqq 1$, we obtain that, under assumptions (2.1)—(2.3) of Section 2, (4.1) and (4.6) and for $\theta = 0$, as

$n \to \infty$,

$$W_n \to_{\mathscr{L}} \{W(t),\ 0 \leqq t \leqq \gamma;\ W(t) + (t-\gamma)\delta\nu(\phi, \psi)/A,\ \gamma \leqq t \leqq 1\}, \tag{4.7}$$

where $W = \{W(t), t \in I\}$ is a standard Wiener process. Secondly, by using Theorems 4.4.3 and 6.2.3 of Puri and Sen (1971), it follows by some standard steps that under (4.6) with $\theta = 0$ and with $\phi = \tilde{\phi}$ in (2.5),

$$n^{\frac{1}{2}}\tilde{\theta}_n \sim \mathscr{N}((1-\gamma)\delta,\ A^2/\nu^2(\phi, \psi)), \tag{4.8}$$

and hence, $n^{\frac{1}{2}}|\tilde{\theta}_n| = O_p(1)$. Thirdly, (3.22) is also valid in this case with the further simplification that $c^* = 1$, and as a result,

$$W_n(1) = n^{\frac{1}{2}}\tilde{\theta}_n \nu(\phi, \psi)/A + o_p(1) \qquad \text{as} \quad n \to \infty; \tag{4.9}$$

$$\sup_{t \in I} |\hat{W}_n(t) - W_n(t) + tW_n(1)| \to_p 0 \qquad \text{as} \quad n \to \infty. \tag{4.10}$$

From (4.7) through (4.10), we obtain that under the assumptions (1.7) and (2.1)—(2.3), (4.1) and (4.6), as $n \to \infty$,

$$\begin{aligned} \hat{W}_n &\to_{\mathscr{L}} W_0(t) - t(1-\gamma)\delta\nu(\phi, \psi)/A, \quad 0 \leqq t \leqq \gamma; \\ &\to_{\mathscr{L}} W_0(t) - \gamma(1-t)\delta\nu(\phi, \psi)/A, \quad \gamma \leqq t \leqq 1, \end{aligned} \tag{4.11}$$

where $W_0 = \{W_0(t) = \{W(t) - tW(1), t \in I\}\}$ is a standard tied-down Wiener process.

Now, (4.11) is the key result for the study of the asymptotic power functions of the tests based on D_n^+, D_n and D_n^*. For example, if D_α^+ be the upper $100\alpha\%$ point of the limiting null distribution of D_N^+ (actually, equal to $(-\frac{1}{2}\log \alpha)^{\frac{1}{2}}$), then

$$\lim_{n\to\infty} P\{D_n^+ \geqq D_\alpha^+ \mid K_n^*\} = P\{W_0(t) - \lambda(t) \geqq D_\alpha^+, \ \text{for some} \ t \in I\}, \tag{4.12}$$

where $\lambda(t) = t(1-\gamma)\delta\nu(\phi, \psi)/A$ for $0 \leqq t \leqq \gamma$ and $= \gamma(1-t)\delta\nu(\phi, \psi)/A$ for $\gamma \leqq t \leqq 1$. Further, on introducing the process $\xi(t) = (t+1)W_0(t/(t+1))$, $0 \leqq t < \infty$, the right-hand side of (4.12) can be expressed as

$$P\{\xi(t) \geqq (t+1)D_\alpha^+ + \lambda^*(t) \ \text{for some} \ t \in [0, \infty)\}, \tag{4.13}$$

where $\xi = \{\xi(t), t \in I\}$ is a standard Wiener process on $[0, \infty)$ and

$$\begin{aligned} \lambda^*(t) &= t(1-\gamma)\delta\nu(\phi, \psi)/A \quad \text{for} \quad t \leqq \gamma/(1-\gamma) \qquad \text{and} \\ &= \gamma\delta\nu(\phi, \psi)/A \qquad \text{for} \quad t > \gamma/(1-\gamma). \end{aligned} \tag{4.14}$$

Note that $(t+1)D_\alpha^+ + \lambda^*(t)$, $t \in [0, \infty)$ represents two straight lines with a common intercept at $t = \gamma/(1-\gamma)$. For simple linear boundaries, Anderson (1960, Section 4) has obtained explicit expressions for the boundary-crossing probabilities for standard Wiener processes, and, in his Section 6, he has indicated that the results can be extended to segmented straight lines, as is the case in our problem. However, in general, this is quite complicated to write down explicitly. The case of D_n follows similarly where instead of the hitting probability for one boundary, we will have to deal with the case of two parallel or trapezoidal boundaries. The case of D_n^* is more complicated. However, for general contiguous alternatives, asymptotic power of the Cramér-von Mises test has recently been studied by Neuhaus (1976) and his results can perhaps be adapted in our problem too.

4.2. *Test for symmetry: location unknown.* For the model (1.9) the problem is to test for the symmetry of F_0 about 0 where θ is regarded as a nuisance parameter. Gupta (1967) considered an asymptotically nonparametric test based on a special case of $\hat{W}_n(1)$ where he took $c_i = 1$, $i = 1, \cdots, n$, $\phi(u) = u$ and $\tilde{\phi}(u) = \operatorname{sgn}(u - \frac{1}{2})$, $u \in I$. We are basically interested in a general class of asymptotic sequential tests for this problem.

We define here $\{T_k\}$ by (1.1) with $c_i = 1$ for all i and let

$$\ddot{T}_k = T_k(X_1 - \tilde{\theta}_k, \cdots, X_k - \tilde{\theta}_k)\,, \qquad k \geqq 1\,, \tag{4.15}$$

where the $\tilde{\theta}_k$ are defined by (2.7) and are based on the score function $\tilde{\phi}$. Let $\tilde{T}_{k,\alpha}$ be the smallest positive number such that for some given α $(0 < \alpha < 1)$, $P\{|\tilde{T}_k| \leqq \tilde{T}_{k,\alpha} \mid H_0 \text{ in } (3.1)\} \geqq 1 - \alpha$, and let

$$\tilde{\theta}_{k,L} = \inf\{a : \tilde{T}_k(a) \leqq \tilde{T}_{k,\alpha}\}\,, \qquad \tilde{\theta}_{k,U} = \sup\{a : \tilde{T}_k(a) \geqq -\tilde{T}_{k,\alpha}\}\,; \tag{4.16}$$

$$\hat{\nu}_k(\tilde{\phi}, \psi) = 2\tilde{T}_{k,\alpha}/k(\tilde{\theta}_{k,U} - \tilde{\theta}_{k,L})\,, \tag{4.17}$$

where $\tilde{A}^{-1}k^{-\frac{1}{2}}\tilde{T}_{k,\alpha} \to \tau_{\alpha/2}$, the $50\alpha\%$ point of the standard normal df. Then, $\hat{\nu}_k(\tilde{\phi}, \psi)$ is a consistent estimator of $\nu(\tilde{\phi}, \psi)$ (see Chapter 6 of [12] and [17]). Similarly, replacing the function $\tilde{\phi}$ by ϕ, we define the estimator $\hat{\nu}_k(\phi, \psi)$, and we let

$$v_k = A^{-1}\tilde{A}[\hat{\nu}_k(\phi, \psi)/\hat{\nu}_k(\tilde{\phi}, \psi)]\,, \qquad k \geqq 1\,. \tag{4.18}$$

The tests to be considered now are suitable when all the n observations are not available at the same time. For example, in clinical trials, patients undergoing a specified treatment enter into the clinic at different points of time. Since the observations are available in a sequential plan, it may be profitable to stop the experiment at an early stage (i.e., prior to attaining the target sample size n) when the accumulated statistical evidence at that stage leads to a clear decision in favor of the alternative hypothesis. Thus, as one progresses with experimentation, at each stage a test statistic is constructed and used to test the null hypothesis; by nature, these statistics based on the partial sequence of sample sizes are correlated; and hence, this repeated test of significance calls for extra care in handling the allied distribution theory. On the other hand, if the experiment is curtailed at an intermediate stage, savings of time and cost of experimentation may be quite important. With that idea, starting with an initial sample of size n_0 (at least moderately large), we continue drawing observations one by one so long as $Z_k = \ddot{T}_k(1 - 2v_k L(\phi, \tilde{\phi}) + v_k^2)^{-\frac{1}{2}}$ (or $|Z_k|$), $k \geqq n_0$ lies below a constant $\zeta^+_{n,\alpha}$ (or $\zeta_{n,\alpha}$) where $L(\phi, \tilde{\phi})$ is defined by (3.27). If, for the first time, for $k = N(\leqq n)$, Z_N is $\geqq \zeta^+_{n,\alpha}$ (or $|Z_N|$ is $\geqq \zeta_{n,\alpha}$), we stop sampling along with the rejection of the null hypothesis; if no such $N(\leqq n)$ exists, we stop at the nth stage and accept the null hypothesis. Thus, basically, our test statistics are

$$L_n^+ = \max_{n_0 \le k \le n} Z_k \quad \text{and} \quad L_n = \max_{n_0 \le k \le n} |Z_k| . \tag{4.19}$$

We consider first the following theorem.

THEOREM 4.1. *Under* (1.2), (1.10) *and assumptions* (i) *and* (ii) *of Section* 2, *when* $n_0 = [n\varepsilon] + 1$, $\varepsilon > 0$, *as* $n \to \infty$,

$$\begin{aligned} n^{-\frac{1}{2}} A_n^{-1} L_n^+ &\to_{\mathscr{L}} \sup_{\varepsilon \le t < 1} W(t) \quad \text{and} \\ n^{-\frac{1}{2}} A_n^{-1} L_n &\to_{\mathscr{L}} \sup_{\varepsilon \le t \le 1} |W(t)| , \end{aligned} \tag{4.20}$$

where $W_\varepsilon = \{W(t), \varepsilon \le t \le 1\}$ *is a standard Wiener process on* $[\varepsilon, 1]$.

OUTLINE OF THE PROOF. Let us choose $K_\varepsilon = K/\varepsilon$, $0 < K < \infty$. Then, by (4.16),

$$\begin{aligned} P\{\tilde{\theta}_{k,L} < \theta - n^{\frac{1}{2}}K_\varepsilon \ &\text{for some } k : n_0 \le k \le n\} \\ = P\{\tilde{T}_k(X_1 + n^{-\frac{1}{2}}K_\varepsilon, \cdots, X_k + n^{-\frac{1}{2}}K_\varepsilon) &\le \tilde{T}_{k,\alpha} \\ &\text{for some } k : n_0 \le k \le n \mid H_0\} \\ = P\{\bar{W}_{n,(-K_\varepsilon)}(t) - \tilde{A}^{-1}K_\varepsilon t\nu(\tilde{\phi}, \psi) &< \tilde{A}^{-1} n^{-\frac{1}{2}} \tilde{T}_{k_n(t),\alpha} - \tilde{A}^{-1}K_\varepsilon t\nu(\tilde{\phi}, \psi), \\ &\text{for some } t : \varepsilon \le t \le 1 \mid H_0\} \end{aligned} \tag{4.21}$$

where $\bar{W}_{n,(-K_\varepsilon)}$ and $k_n(t)$ are defined as in Theorem 3.2 (but for the special case of $c_i = 1$, for all $i \ge 1$ and $\phi = \tilde{\phi}$). As has been noted after (4.17), $\tilde{A}^{-1} n^{-\frac{1}{2}} \tilde{T}_{k_n(t),\alpha} \to t\tau_{\alpha/2}$ $(\le \tau_{\alpha/2})$, while, for every $t \in [\varepsilon, 1]$, $\tilde{A}^{-1}K_\varepsilon t\nu(\tilde{\phi}, \psi) = \tilde{A}^{-1}K\nu(\tilde{\phi}, \psi)(t/\varepsilon) \ge \tilde{A}^{-1}K\nu(\tilde{\phi}, \psi)$. Hence, by using (3.16) and the fact that $\sup\{|W(t)| : t \in [\varepsilon, 1]\} = O_p(1)$, for every $\eta > 0$, we can choose K appropriately large, so that the right-hand side of (4.21) is $\le \eta/2$ when n is large. A similar case holds for $P\{\tilde{\theta}_{k,U} > \theta + n^{-\frac{1}{2}}K_\varepsilon$, for some $k : n_0 \le k \le n\}$. Hence, for every $\varepsilon > 0$ and $\eta > 0$, there exist a $K^*(< \infty)$ and an integer n^*, such that for $n_0 \ge n\varepsilon$ and $n \ge n^*$,

$$P\{\theta - n^{-\frac{1}{2}}K^* \le \tilde{\theta}_{k,L} \le \tilde{\theta}_{k,U} \le \theta + n^{-\frac{1}{2}}K^*, n_0 \le k \le n\} \ge 1 - \eta . \tag{4.22}$$

A similar probability statement holds for the partial sequence $\{\hat{\theta}_{k,L}, \hat{\theta}_{k,U}; n_0 \le k \le n\}$. As such, by Theorem 3.3, (4.17), (4.18) and some routine computations, we obtain that

$$\max\{|v_k\gamma - 1| : n_0 \le k \le n\} \to_p 0 , \quad \text{as} \quad n \to \infty , \tag{4.23}$$

where γ is defined by (3.28). Let us now define $k_n(t) = \max\{k : k/n \le t\}$, $\varepsilon \le t \le 1$, and let

$$\ddot{W}_{n,\varepsilon} = \{\ddot{W}_n(t) = n^{-\frac{1}{2}} A_n^{-1} \ddot{T}_{k_n(t)} (1 - 2\gamma^{-1} L(\phi, \tilde{\phi}) + \gamma^{-2})^{-\frac{1}{2}}, \varepsilon \le t \le 1\} , \tag{4.24}$$

$$\check{W}_{n,\varepsilon} = \{\check{W}_n(t) = n^{-\frac{1}{2}} A_n^{-1} \ddot{T}_{k_n(t)} (1 - 2v_k L(\phi, \tilde{\phi}) + v_k^2)^{-\frac{1}{2}}, \varepsilon \le t \le 1\} . \tag{4.25}$$

Then, we have

$$\begin{aligned} \sup\{|\ddot{W}_n(t) - \check{W}_n(t)| &: \varepsilon \le t \le 1\} \\ \le [\sup\{|\ddot{W}_n(t)| &: \varepsilon < t \le 1\}][\max_{n_0 \le k \le n} |\{(1 - 2\gamma^{-1}L(\phi, \tilde{\phi}) \\ &+ \gamma^{-2})/(1 - 2v_k L(\phi, \tilde{\phi}) + v_k^2)\}^{\frac{1}{2}} - 1|] . \end{aligned} \tag{4.26}$$

Thus, if we show that as $n \to \infty$,

$$\ddot{W}_{n,\varepsilon} \to_{\mathscr{L}} W_\varepsilon\,, \quad \text{in the } J_1\text{-topology on} \quad D[\varepsilon, 1] \tag{4.27}$$

(insuring that $\sup\{|\ddot{W}_n(t)| : \varepsilon \leqq t \leqq 1\} = O_p(1)$), then, by (4.23) and (4.27), the right-hand side of (4.26) converges in probability to 0 as $n \to \infty$; and hence, $\ddot{W}_{n,\varepsilon}$ and $\check{W}_{n,\varepsilon}$ being asymptotically convergent equivalent, each satisfies (4.27).

To prove (4.27), we note that by virtue of (4.15), (4.22) and the fact that $\tilde{\theta}_{k,L} \leqq \tilde{\theta}_k \leqq \tilde{\theta}_{k,U}$, for every $k \geqq 1$, we are in a position to use Theorem 3.4 (under the assumptions made in the theorem and the simplification due to the fact that $c_i = 1$ for all $i \geqq 1$), and obtain by a few standard steps that as $n \to \infty$,

$$\begin{aligned}\sup\{|\ddot{W}_n(t) - (1 - 2\gamma^{-1}L(\phi, \tilde{\phi}) + \gamma^{-2})^{-\frac{1}{2}}[W_n(t) - \gamma^{-1}\tilde{W}_n(t)]| :\\ \varepsilon \leqq t \leqq 1\} \to_p 0\,,\end{aligned} \tag{4.28}$$

where W_n is defined by (1.6)—(1.7) and $\tilde{W}_n(t) = n^{-\frac{1}{2}}\tilde{A}^{-1}\tilde{T}_{k_n(t)}$, $\varepsilon \leqq t \leqq 1$, and in both cases, $k_n(t)$ is defined as it was before (4.24). Finally, along the lines of the proof of the later part of Theorem 3.4, it follows that under (1.10) with $\theta = 0$, $W_{n,\varepsilon} - \gamma^{-1}\tilde{W}_{n,\varepsilon}$ converges in law to $(1 - 2\gamma^{-1}L(\phi, \tilde{\phi}) + \gamma^{-2})^{\frac{1}{2}}W_\varepsilon$, as $n \to \infty$, and (4.27) then follows from (4.24), (4.28) and the above convergence. We complete the proof of the theorem by noting that, by definition, the two statistics in (4.20) are $\sup\{\check{W}_n(t) : \varepsilon \leqq t \leqq 1\}$ and $\sup\{|\check{W}_n(t)| : \varepsilon \leqq t \leqq 1\}$, so that (4.27) and the asymptotic equivalence of $\ddot{W}_{n,\varepsilon}$ and $\check{W}_{n,\varepsilon}$ imply (4.20). □

Note that for every $\lambda > 0$ and $0 < \varepsilon < 1$,

$$\begin{aligned}P\{\sup\{|W(t)| : 0 \leqq t \leqq 1\} > \lambda\} &\geqq P\{\sup\{|W(t)| : \varepsilon \leqq t \leqq 1\} > \lambda\}\\ &\geqq P\{\sup\{|W(t)| : 0 \leqq t \leqq 1\} > \lambda\}\\ &\quad - P\{\sup\{|W(t)| : 0 \leqq t \leqq \varepsilon\} > \lambda\}\,,\end{aligned} \tag{4.23}$$

and a similar case holds for $\sup\{W(t) : \varepsilon \leqq t \leqq 1\}$. It is well known (cf. Section 11 of [3]) that for every $\lambda > 0$,

$$\begin{aligned}P\{\sup\{|W(t)| : 0 \leqq t \leqq 1\} > \lambda\}\\ = 1 - \textstyle\sum_{k=-\infty}^{\infty} (-1)^k\{\Phi((2k+1)\lambda) - \Phi((2k-1)\lambda)\}\,,\end{aligned} \tag{4.24}$$

$$P\{\sup\{|W(t)| : 0 \leqq t \leqq \varepsilon\} > \lambda\} \leqq 4\{1 - \Phi(\lambda/\varepsilon^{\frac{1}{2}})\}\,, \tag{4.25}$$

where $\Phi(x)$ is the standard normal df. Since, for every (fixed) $\lambda > 0$, $\lambda/\varepsilon^{\frac{1}{2}} \to \infty$ as $\varepsilon \to 0$, (4.25) can be made arbitrarily small by choosing ε small. Hence, the central term in (4.23) converges to (4.24) as $\varepsilon \to 0$; a similar case holds for the one-sided case. Hence, from (4.20), we conclude that whenever n_0 goes to ∞ with n but $n_0/n \to 0$ as $n \to \infty$, the limiting null distributions of $n^{-\frac{1}{2}}A_n^{-1}L_n^+$ and $n^{-\frac{1}{2}}A_n^{-1}L_n$ converge to those of D_n^+ and D_n, respectively. As such, large sample tests can be based on these statistics with the critical values derived from the distributions of D_n^+ and D_n.

Note that here the alternative hypothesis relates to asymmetry of F_0, and it may be quite arbitrary in nature, excluding the case of symmetry of F_0 around some point different from 0. For any such F_0, we may show according to the criterion of Bahadur efficiency (cf. [12, page 122]) that the relative efficiency of the proposed sequential tests with respect to the fixed sample procedures based on Z_n or $|Z_n|$ is equal to one. On the other hand, the average sample sizes of the proposed sequential procedures are less than n, indicating a saving in cost of the experimentation through lesser amount of sampling. Finally, in the tests proposed, we need to assume that either $L^2(\phi, \tilde{\phi}) < 1$ or $\gamma \neq 1$, and this can always be made by proper choice of ϕ and $\tilde{\phi}$. Specifically, we choose $\tilde{\phi}$ in such a way that the estimators $\tilde{\theta}_k$ are efficient and ϕ such that the T_k in (1.1) are sensitive to some specific type of asymmetry, which we might have in mind.

REMARK. In the context of sequential estimation of θ, Sen and Ghosh (1971) studied the almost sure convergence of $\hat{\nu}_k(\phi, \psi)$ in (4.17) under more restrictive regularity conditions on the score function ϕ. It appears that in their case as well as in many others, all we need is the convergence of the type in (4.22) and for this, the basic assumptions of Section 2 suffice and are much less restrictive in nature.

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