The Annals of Statistics
1977, Vol. 5, No. 6, 1099–1106

ON THE ASYMPTOTIC EQUIVALENCE OF TWO RANKING METHODS FOR *K*-SAMPLE LINEAR RANK STATISTICS

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Two methods of ranking K samples for rank tests comparing K populations are considered. The first method ranks the K samples jointly; the second ranks the K samples pairwise. These procedures were first suggested by Dunn (1964), and Steel (1960), respectively. It is shown that both ranking procedures are asymptotically equivalent for rank-sum tests satisfying certain nonrestrictive conditions. The problem is formulated in terms of multiple comparisons, but is applicable to other nonparametric procedures based on K-sample rank statistics.

1. Introduction. Rank tests for comparing K populations have been generalized from the corresponding two-sample linear rank statistics for both location and scale alternatives. The tests usually involve ranking the K samples jointly and examining some linear function of the ranks of the jth sample ($j = 1, \cdots, K$). These K-sample tests are described most generally in Hájek and Šidák (1967, Section III.4.1).

The question of ranking procedures for these K-sample tests will be formulated in the framework of the multiple comparisons problem. In many situations, it is of less importance to know whether the K samples are identically distributed than to determine which of the K samples differ on an individual basis. In these cases a multiple comparisons procedure is preferable to a single test. Two methods of ranking for multiple comparisons will be considered in this note. For the first method the K samples are combined, and populations π_i and $\pi_{i'}$ are compared by their respective rank scores in the joint ranking scheme. For the second method, samples are ranked in pairs: populations π_i and $\pi_{i'}$ are compared by their relevant two-sample statistic in the joint ranking of samples i and i'.

The first method of ranking was suggested by Dunn (1964) and the second by Steel (1960), for effecting multiple comparisons among K populations using the Wilcoxon test. In this context, Sherman (1965) showed that the two methods have the same asymptotic Pitman efficiency. A similar statement can be found in Mehra and Puri (1967). It is proved in this note that the two methods of ranking are asymptotically equivalent for rank-sum statistics satisfying certain nonrestrictive conditions. Implications of this result extend beyond the multiple

Received February 1, 1977; revised March 16, 1977.

[1] Research supported in part by the U.S. Energy Research and Development Administration, Contract No. E(11-1)-2751, to the Department of Statistics, University of Chicago.

AMS 1970 *subject classifications*. Primary 62G20; Secondary 62G10, 62E20.

Key words and phrases. Nonparametric statistics, linear rank tests, multiple comparisons, location, scale, asymptotic Pitman efficiency.

comparisons problem, to other nonparametric procedures formally reliant upon one or the other of the two ranking techniques.

2. Methods of multiple comparisons. For each i, $1 \leqq i \leqq K$, let X_{ij}, $1 \leqq j \leqq N_i$, be a random sample from a population π_i with an absolutely continuous cumulative distribution function F_i and associated density f_i. The null hypothesis is

$$H_0: f_1(\cdot) = \cdots = f_K(\cdot) = f_0(\cdot),$$

where f_0 is unknown. The alternative hypotheses to be considered are location alternatives:

$$H_{1N}: f_j(x) = f_0(x - \Delta_j) \qquad j = 1, \cdots, K,$$

and scale alternatives:

$$H_{2N}: f_j(x) = \exp(-\Delta_j) f_0(x \exp(-\Delta_j)) \qquad j = 1, \cdots, K,$$

with Δ_j arbitrary. If $\Delta_i \neq \Delta_{i'}$, populations π_i and $\pi_{i'}$ differ by a location shift under H_{1N} and a scale shift under H_{2N}.

From Hájek and Šidák (1967, Theorem II.4.4), the locally most powerful rank test of H_0 against two samples differing in location is the test with critical region $\sum_{j \in \text{sample } 2} a_N(R_j, f) \geqq k$, where R_j is the rank of the jth observation in the combined sample, and

$$a_N(i, f) = E\phi(U_N^{(i)}, f) \tag{2.1}$$

are the scores corresponding to the underlying density f. The scores are defined by the score generating function

$$\phi(u, f) = -\frac{f'(F^{-1}(u))}{f(F^{-1}(u))}, \qquad 0 < u < 1.$$

If $f(\cdot)$ is the logistic density, (2.1) defines the Wilcoxon test, whereas if $f(\cdot)$ is the normal density, (2.1) defines the normal scores test.

The same result holds for scale alternatives (Hájek and Šidák, 1967, Theorem II.4.5), with the scores and score generating function defined by

$$\begin{aligned} a_{1N}(u, f) &= E\phi_1(U_N^{(i)}, f) \\ \phi_1(u, f) &= -1 - F^{-1}(u)\frac{f'(F^{-1}(u))}{f(F^{-1}(u))}. \end{aligned} \tag{2.2}$$

For $f(v) = e^{-v}$, (2.2) gives the exponential ordered scores test of Savage. (In definitions (2.1) and (2.2) $U_N^{(i)}$ is the ith observation of an ordered sample of size N from the uniform distribution.)

K-sample versions of the above test are described in Hájek and Šidák (1967, Section II.4). The question of ranking procedures for these K-sample tests will be approached through the problem of multiple comparisons among the K underlying populations. One nonparametric method of multiple comparisons is based on the following joint ranking techenique. Denote by R_{ij} the rank of X_{ij}

in the joint ranking of the K samples; that is,

$$R_{ij} = \sum_{i'=1}^{K} \sum_{j'=1}^{N_{i'}} u(X_{ij} - X_{i'j'}) ,$$

where $u(x) = 1$ if $x \geqq 0$, 0 otherwise. Let

$$S_i = \sum_{j=1}^{N_i} a_N(R_{ij}) , \quad (N = \sum_{i=1}^{K} N_i) ,$$

where $a_N(i)$ are some given scores converging in quadratic mean to a square-integrable function $\phi(u)$:

$$\lim_{N\to\infty} \int_0^1 \{a_N(1 + [uN]) - \phi(u)\}^2 \, du = 0 ,$$

and assume ϕ has a bounded second derivative.

From Hájek and Šidák (1967, Theorem II.3.1.c), $E(S_i \mid H_0) = N_i \bar{a}_N$, and $\operatorname{Var}(S_i \mid H_0) = \sigma_N^2 N_i(1 - N_i/N)$, where

$$\bar{a}_N = N^{-1} \sum_{j=1}^{N} a_N(j) = \int_0^1 \phi(u) \, du \equiv \bar{\phi}$$

$$\sigma_N^2 = (N-1)^{-1} \sum_{j=1}^{N} (a_N(j) - \bar{a}_N)^2 \to \int_0^1 (\phi(u) - \bar{\phi})^2 \, du .$$

Following Dunn (1964), who proposed a similar technique for comparing location parameters in K populations using Wilcoxon rank sums, populations π_i and $\pi_{i'}$ may be compared with the statistic $[S_i - E(S_i \mid H_0)] - [S_{i'} - E(S_{i'} \mid H_0)]$. Suppose all $K(K-1)/2$ pairwise comparisons among the individual populations are desired. Define the $K(K-1)/2 \times 1$ vector

$$\mathbf{S} = (S_{12}, S_{13}, \cdots, S_{1K}, S_{23}, \cdots, S_{2K}, \cdots, S_{K-1,K})'$$

by

$$S_{ii'} = N_i^{-1}[S_i - E(S_i \mid H_0)] - N_{i'}^{-1}[S_{i'} - E(S_{i'} \mid H_0)] .$$

THEOREM 2.1. *Under H_0, for* $\min(N_1, \cdots, N_K) \to \infty$, $N_i/N \to \lambda_i^2$, $0 < \lambda_i < 1$, $(N^{\frac{1}{2}}/\sigma_N)\mathbf{S}$ *is asymptotically normally distributed with mean* $\mathbf{0}$ *and covariances given by*:

$$\begin{aligned}
(N/\sigma_N^2)E(S_{ii'}S_{jj'} \mid H_0) &= 1/\lambda_i^2 + 1/\lambda_{i'}^2 && i = j, \quad i' = j' ; \\
&= 1/\lambda_i^2 && i \neq j, \quad i' = j' , \\
&&& i = j, \quad i' \neq j' ; \\
&= -1/\lambda_{i'}^2 && i \neq j', \quad i' = j , \\
&&& i = j', \quad i' \neq j ; \\
&= 0 && \textit{otherwise}.
\end{aligned} \tag{2.3}$$

PROOF. From Hájek and Šidák (1967), Theorem V.1.6.a,

$$\xi_j = \left(\frac{S_j - N_j \bar{a}_N}{(N_j)^{\frac{1}{2}} \sigma_N}\right)$$

is asymptotically $N(0, 1 - \lambda_j^2)$. Then Theorem V.2.2 proves the asymptotic normality of the vector $(\xi_1, \cdots, \xi_K)'$.

Consider the sequence of local alternatives $\{H_{1N}\}$ and define $\bar{\Delta} = N^{-1} \sum_{i=1}^{K} N_i \Delta_i$, the mean location change, and

$$I(f_0) = \int_{-\infty}^{\infty} [f_0'(x)/f_0(x)]^2 f_0(x) \, dx ,$$

the Fisher information for f_0.

THEOREM 2.2. *If* $\min(N_1, \cdots, N_K) \to \infty$, $N_i/N \to \lambda_i^2$, $0 < \lambda_i < 1$, $\max_{1 \le i \le K} (\Delta_i - \bar{\Delta})^2 \to 0$, *and* $I(f_0) \sum_{i=1}^K N_i(\Delta_i - \bar{\Delta})^2 \to c$, $0 < c < \infty$, *then under* $\{H_{1N}\}$, $(N^{\frac{1}{2}}/\sigma_N)\mathbf{S}$ *is asymptotically normally distributed with covariances* (2.3) *but with limiting means*

$$(N^{\frac{1}{2}}/\sigma_N)E(S_{ii'} \mid H_{1N}) \doteq N^{\frac{1}{2}}(\Delta_i - \Delta_{i'}) \textstyle\int_0^1 \phi(u)\phi(u, f_0)\, du . \tag{2.4}$$

PROOF. From Hájek and Šidák (1967), Theorem VI.3.1, the ξ_j of Theorem 2.1 have expected value

$$\begin{aligned} E(\xi_j \mid H_{1N}) &= N_j^{-\frac{1}{2}} N_j(\Delta_j - \bar{\Delta}) \textstyle\int_0^1 \phi(u)\phi(u, f_0)\, du / \{\int_0^1 (\phi(u) - \bar{\phi})^2\, du\}^{\frac{1}{2}} \\ &= N_j^{\frac{1}{2}}(\Delta_j - \bar{\Delta}) \textstyle\int_0^1 \phi(u)\phi(u, f_0)\, du / \sigma_N . \end{aligned}$$

Hence $\sigma_N/(N_j)^{\frac{1}{2}}\xi_j$ has expected value $(\Delta_j - \bar{\Delta}) \int_0^1 \phi(u)\phi(u, f_0)\, du$, under $\{H_{1N}\}$. Noting that $S_{ii'} = (\sigma_N/(N_i)^{\frac{1}{2}})\xi_i - (\sigma_N/(N_{i'})^{\frac{1}{2}})\xi_{i'}$ gives the desired mean. The covariances and asymptotic normality also follow from Theorem VI.3.1 of Hájek and Šidák (1967).

REMARK 2.1. Theorem 2.2 is also true for scale alternatives $\{H_{2N}\}$, with $\phi(u, f_0)$ and $I(f_0)$ replaced by $\phi_1(u, f_0)$ and $I_1(f_0)$;

$$I_1(f_0) = \int_{-\infty}^{\infty} \left[-1 - x\frac{f_0'(x)}{f_0(x)} \right]^2 f_0(x)\, dx .$$

Theorem VII.1.4 of Hájek and Šidák (1967), together with an application of the argument used in Lehmann (1963), yields the fact that the asymptotic Pitman efficiency of the multiple comparisons procedure based on $(N^{\frac{1}{2}}/\sigma_N)\mathbf{S}$ compared with that based .on the maximin most powerful test is ρ^2, where

$$\rho = \textstyle\int_0^1 \phi(u)\phi(u, f_0)\, du / [\int_0^1 \phi^2(u, f_0)\, du]^{\frac{1}{2}} ,$$

for location alternatives, and

$$\rho = \textstyle\int_0^1 \phi(u)\phi_1(u, f_0)\, du / [\int_0^1 \phi_1^2(u, f_0)\, du]^{\frac{1}{2}} ,$$

for scale alternatives. The proof of this assertion is outlined in Appendix A.

A second nonparametric method of multiple comparisons is suggested by a technique due to Steel (1960). Let $\mathbf{S}^* = (S_{12}^*, S_{13}^*, \cdots, S_{1K}^*, \cdots, S_{K-1,K}^*)'$, where

$$S_{ii'}^* = \left(\frac{1}{N_i} + \frac{1}{N_{i'}}\right) [\textstyle\sum_{j=1}^{N_i} a_{N_i+N_{i'}}(R_{ij}^{i'}) - N_i \bar{a}_{N_i+N_{i'}}] .$$

In this formula, $R_{ij}^{i'}$ denotes the rank of X_{ij} in the joint ranking of sample i and i' only; hence $S_{ii'}^*$ corresponds to a two-sample rank statistic for testing equality of F_i and $F_{i'}$. Under this pairwise ranking scheme, populations π_i and $\pi_{i'}$ are compared by a pairwise ranking of their respective samples.

THEOREM 2.3. *Under* H_0, *as* $\min(N_1, \cdots, N_K) \to \infty$, $N_i/N \to \lambda_i^2$, $0 < \lambda_i < 1$, *the random variates* $(N^{\frac{1}{2}}/\sigma_{N_i+N_{i'}})S_{ii'}^*$ *are marginally asymptotically normal with mean* 0 *and variance* $(1/\lambda_i^2 + 1/\lambda_{i'}^2)$.

PROOF. Follows trivially from Hájek and Šidák (1967) Theorem V.1.6.a.

It remains to show that the vector $\mathbf{S}^*$, suitably normalized, is asymptotically normally distributed. To this end, the projection method of Hájek (1968) will be used.

Define

$$Z_{ij} = N_i^{-1} \int [u(x - X_{ij}) - F_0(x)]\phi'(F_0(x))\, dF_0(x)\,. \tag{2.5}$$

From Theorem 4.2 of Hájek (1968), $-\sum_{j=1}^{N_i} Z_{ij} + \sum_{j=1}^{N_{i'}} Z_{i'j}$ is asymptotically equivalent to $S_{ii'}^*$. Asymptotically then, since the Z_{ij} are independent, and for fixed i identically distributed random variates with mean 0,

$$\begin{aligned}
&(N/\sigma_N^2) \operatorname{Cov}(S_{ii'}^*, S_{il'}^*)\\
&\quad= (N/\sigma_N^2) \operatorname{Cov}\left(-\textstyle\sum_{j=1}^{N_i} Z_{ij} + \sum_{j=1}^{N_{i'}} Z_{i'j}, -\sum_{j=1}^{N_i} Z_{ij} + \sum_{j=1}^{N_{l'}} Z_{l'j}\right)\\
&\quad= (N/\sigma_N^2) \textstyle\sum_{j=1}^{N_i} \operatorname{Var} Z_{ij}\\
&\quad= (N/\sigma_N^2) \frac{1}{N_i^2} \textstyle\sum_{j=1}^{N_i} E\{[\int [u(x - X_{ij}) - F_0(x)]\phi'(F_0(x))\, dF_0(x)]\\
&\qquad\quad \times [\int [u(y - X_{ij}) - F_0(y)]\phi'(F_0(y))\, dF_0(y)]\}\\
&\quad= (N/\sigma_N^2) \frac{1}{N_i} E \iint [u(x - X_{ij}) - F_0(x)][u(y - X_{ij}) - F_0(y)]\\
&\qquad\quad \times \phi'(F_0(x))\phi'(F_0(y))\, dF_0(x)\, dF_0(y)\\
&\quad= (N/\sigma_N^2) \frac{1}{N_i} \iint \{F_0[\min(x, y)] - F_0(x)F_0(y)\}\phi'(F_0(x))\phi'(F_0(y))\, dF_0(x)\, dF_0(y)\\
&\quad\to 1/\lambda_i^2\sigma_N^2 \textstyle\int_0^1 \int_0^1 [\min(x, y) - xy]\phi'(x)\phi'(y)\, dx\, dy\\
&\quad= 1/\lambda_i^2\sigma_N^2 \textstyle\int_0^1 (\phi(u) - \bar{\phi})^2\, du = 1/\lambda_i^2\,.
\end{aligned}$$

(The details of the last step are outlined in Appendix B.) Similarly, $(N/\sigma_N^2) \operatorname{Cov}(S_{ii'}^*, S_{i'l}^*) = -1/\lambda_{i'}^2$. Hence $(N/\sigma_N^2)^{\frac{1}{2}}\mathbf{S}^*$ has the same asymptotic covariance matrix as $(N/\sigma_N^2)^{\frac{1}{2}}\mathbf{S}$, is summarized in (2.3).

THEOREM 2.4. *Under H_0, the vector $(N^{\frac{1}{2}}/\sigma_N)\mathbf{S}^*$ is asymptotically normally distributed.*

PROOF. It is sufficient to prove that $(N^{\frac{1}{2}}/\sigma_N) \sum_{i=1}^{K-1} \sum_{i'=i+1}^{K} \gamma_{ii'} S_{ii'}^*$ is asymptotically normal for any choice of $\{\gamma_{ii'}\}$. Without loss of generality, assume that $(N/\sigma_N^2) \operatorname{Var}[\sum\sum \gamma_{ii'} S_{ii'}^*] > 0$ (using the asymptotic covariance formulas (2.3)); if this is not the case, covariance to the degenerate normal is assured. Define z_{ij} as in (2.5), but replace $\phi(u)$ with its polynomial approximation $\psi(u)$, where $\psi(u)$ satisfies the regularity conditions of Lemma 5.1 of Hájek (1968). Asymptotically, $(N^{\frac{1}{2}}/\sigma_N) \sum_{i=1}^{K-1} \sum_{i'=i+1}^{K} \gamma_{ii'} S_{ii'}^*$ is equivalent to

$$\begin{aligned}
&(N^{\frac{1}{2}}/\sigma_N) \textstyle\sum_{i=1}^{K-1} \sum_{i'=i+1}^{K} \gamma_{ii'}(-\sum_{j=1}^{N_i} z_{ij} + \sum_{j=1}^{N_{i'}} z_{i'j})\\
&\quad= (N^{\frac{1}{2}}/\sigma_N)\{\textstyle\sum_{i=1}^{K-1} (-\sum_{i'=i+1}^{K} \gamma_{ii'}) \sum_{j=1}^{N_i} z_{ij} + \sum_{i'=2}^{K} (\sum_{i=1}^{i'-1} \gamma_{ii'}) \sum_{j=1}^{N_{i'}} z_{i'j}\}\\
&\quad= (N^{\frac{1}{2}}/\sigma_N)\{[-\textstyle\sum_{i'=3}^{K} \gamma_{ii'}] \sum_{j=1}^{N_1} z_{ij} + \sum_{i=2}^{K-1} [\sum_{i'=1}^{i-1} \gamma_{ii'} - \sum_{i'=i+1}^{K} \gamma_{ii'}] \sum_{j=1}^{N_i} z_{ij}\\
&\qquad\quad + [\textstyle\sum_{i'=1}^{K-1} \gamma_{Ki'}] \sum_{j=1}^{N_K} z_{Kj}\}\,.
\end{aligned}$$

This latter term may be shown to be asymptotically normally distributed by applying the Lindeberg central limit theorem, as in Theorem 2.1 of Hájek (1968).

This establishes that $(N^{\frac{1}{2}}/\sigma_N)\mathbf{S}^*$ has the same asymptotic distibution as $(N^{\frac{1}{2}}/\sigma_N)\mathbf{S}$ under the null hypothesis. Furthermore, under the sequences of local alternatives $\{H_{1N}\}$ and $\{H_{2N}\}$ considered previously, the $(N^{\frac{1}{2}}/\sigma_N)S^*_{ii'}$ are jointly asymptotically normal with the same covariance structure as under H_0, and with means as in (2.4). Thus, since $(N^{\frac{1}{2}}/\sigma_N)\mathbf{S}$ and $(N^{\frac{1}{2}}/\sigma_N)\mathbf{S}^*$ have the identical asymptotic distribution under location and scale alternatives, it follows that they are equivalent in terms of asymptotic Pitman efficiency.

3. Discussion. It has been shown that the joint and pairwise ranking schemes for multiple comparisons have the same asymptotic Pitman efficiency. Since the two methods of ranking described herein yield procedures that are asymptotically equivalent for the problem of comparing all pairs of populations, practical considerations might dictate choice of methods for the user. The joint ranking technique is more convenient computationally if all pairwise combinations are desired. However, the theory in the previous section could instead have been developed in terms of an arbitrary set of prespecified contrasts. The pairwise ranking technique could be computationally advantageous in such situations.

If some approximate scores $a_N(i)$ for a particular test have an associated $\phi(u)$ that does not have bounded second derivative, it may still be possible to show the asymptotic equivalence of the two ranking methods. Such is the case, for example, with the Savage test, in which $\phi(u) = -\log(1-u)$. By considering an appropriately linearized version of ϕ near $u = 1$, it can be shown that Theorem 2.4 continues to hold. Alternatively, if ϕ is absolutely continuous, then Theorem 2.4 can be proved by a method analogous to that of Theorem 2.3 of Hájek (1968).

For various other aspects of distribution-free methods of multiple comparisons the reader is directed to Sherman (1965), Miller (1966), and Lehmann (1975).

It should be noted that the theoretical results of Section 2 are not only applicable to multiple comparisons. For example, consider the problem of testing the equality of K continuous distributions against the alternative that they are stochastically ordered in a particular manner. A nonparametric test of this hypothesis, proposed independently by Terpstra (1952) and Jonckheere (1954), is based on the statistic $\sum_{i<i'} W_{ii'}$, where $W_{ii'}$ is the Mann–Whitney rank sum for comparing samples i and i'. Generalized versions of this statistic which include arbitrary rank scores have been investigated by Puri (1965) and Tryon and Hettmansperger (1973). Note that, in the notation of Section 2, these statistics are of the form $\mathbf{a}'\mathbf{S}^*$. Furthermore, it is clear that asymptotically equivalent versions of these test statistics may be formed by jointly ranking the K samples and taking the appropriate linear combinations, $\mathbf{a}'\mathbf{S}$ of the K marginal rank scores. The coefficients of $\mathbf{a}$ are in fact proportional to the values of the first (linear) orthogonal polynomials of degree K. That is, the Terpstra–Jonckheere

statistic tests for a linear trend in **S** by considering the linear contrast defined by the appropriate orthogonal polynomial. This suggests that higher order trends could be tested by contrasts defined by the higher order orthogonal polynomials of degree K. The asymptotic distributions follow directly.

APPENDIX A

The argument of Sherman (1965) and Lehmann (1963) yields the fact that the asymptotic efficiency of the multiple comparisons procedure equals that of the test of H_0 based on $(S_1, \cdots, S_K)'$, namely

$$Q = (N-1)[\sum_{j=1}^{N} (a_N(j) - \bar{a}_N)^2]^{-1} \sum_{i=1}^{K} \left(\frac{S_i - N_i \bar{a}_N}{N_i}\right)^2 .$$

It remains to show that the asymptotic efficiency of the Q-test equals ρ^2. In this regard, Theorem VII.1.4 of Hájek and Šidák (1967) can be applied directly for location alternatives

$$q = \prod_{i=1}^{K} \prod_{j=1}^{N_i} f_0(X_{ij} - \Delta_i) .$$

For scale alternatives

$$q = \sum_{i=1}^{K} \prod_{j=1}^{N_i} \exp(-\Delta_i) f_0(X_{ij} \exp(-\Delta_i))$$

the proof is easily modified by replacing $I(f_0)$ and $a_N(R_{Ni}, f_0)$ with $I_1(f_0)$ and $a_{1N}(R_{Ni}, f_0)$. In addition, $p_{\nu\bar{\Delta}}$ must be redefined as on page 245 of Hájek and Šidák (1967):

$$p_{\nu\bar{\Delta}} = \exp(-N_\nu \bar{\Delta}_\nu) \prod_{i=1}^{N_\nu} f_0(x_i \exp(-\bar{\Delta}_\nu)) .$$

Now Theorem VII.2.2 can be applied to give the desired result.

APPENDIX B

It is required to show that

$$\int_0^1 \int_0^1 [\min(x, y) - xy)] \phi'(x)\phi'(y)\, dy\, dx = \int_0^1 (\phi(u) - \bar{\phi})^2\, du .$$

First consider the integral over y:

$$\begin{aligned}
&\int_0^1 [\min(x, y) - xy]\phi'(x)\phi'(y)\, dy \\
&\quad = \int_0^x y(1-x)\phi'(y)\phi'(x)\, dy + \int_x^1 x(1-y)\phi'(y)\phi'(x)\, dy \\
&\quad = (1-x)\phi'(x)[x\phi(x) - \int_0^x \phi(y)\, dy] + x\phi'(x)[-(1-x)\phi(x) + \int_x^1 \phi(y)\, dy] \\
&\quad = (-1-x)\phi'(x) \int_0^x \phi(y)\, dy + x\phi'(x) \int_x^1 \phi(y)\, dy \\
&\quad = x\phi'(x) \int_0^1 \phi(y)\, dy - \phi'(x) \int_0^x \phi(y)\, dy .
\end{aligned}$$

Now integrating this last expression over x:

$$\begin{aligned}
\int_0^1 x\phi'(x)\bar{\phi}\, dx - \int_0^1 \phi'(x) \int_0^x (y)\, dy &= \bar{\phi}\phi(1) - \bar{\phi}^2 - [\bar{\phi}\phi(1) - \int_0^1 \phi^2(x)\, dx] \\
&= \int_0^1 (\phi^2(x) - \bar{\phi}^2)\, dx = \int_0^1 [\phi(x) - \bar{\phi}]^2\, dx .
\end{aligned}$$

Acknowledgment. The authors would like to thank the editor and the referee for their careful reading of the paper and many helpful suggestions.

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