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STRONGLY OPTIMAL POLICIES IN SEQUENTIAL SEARCH WITH RANDOM OVERLOOK PROBABILITIES

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Assume a searcher is hunting for an object which has been hidden in one of N regions or cells, with initial prior probability p_i^1 that it is in cell i. Suppose that to each i there corresponds a sequence $\{\alpha_{ij}\}_{j\geqq 1}$ of random variables, where α_{ij} describes the chances that the searcher will fail to find the object on the jth search of i, given that the object is in i. The joint distribution of $\{\alpha_{ij}: 1 \leqq i \leqq N, j \geqq 1\}$ is known to the searcher. Under a certain monotonicity condition on the α_{ij}'s, it is shown that to maximize the probability of finding the object in at most n_0 stages of search, the one-stage look ahead rule is optimal. In an earlier paper concerning a related problem, Hall assumed $\{\alpha_{1j}\}_{j\geqq 1}, \cdots, \{\alpha_{Nj}\}_{j\geqq 1}$ were independent processes, whereas we allow them to be dependent. Our result is new for independent processes as well.

1. Introduction and summary. In Hall [6] the following search problem was considered: An object is hidden in one of N boxes, labeled $1, 2, \cdots, N$. The initial prior probability that it is hidden in box i is $p_i^1 \geqq 0$, where $\sum_{i=1}^N p_i^1 = 1$. There is a searcher who once a day selects a box to be searched for that day. The searcher knows that the object remains in the box in which it is hidden until it is found, and he also is informed of the initial location vector $\mathbf{p}^1 = (p_1^1, p_2^1, \cdots, p_N^1)$.

To each box i corresponds a sequence $\{\alpha_{ij}\}_{j=1}^\infty$ of random variables with values in the interval $[0, 1]$ and with known joint distribution. The random variable α_{ij} is the *random overlook probability* or *overlook random variable for the* jth *search of box* i, $1 \leqq i \leqq N$. This random variable describes the chances that the searcher will not find the object during the jth search of box i, given that the object is in box i.

One problem treated in Hall [6] is that in which each search of box i costs the searcher an amount $c_i > 0$. Assume that the searcher has already searched unsuccessfully for the object for $n - 1$ days and that the present location vector is $\mathbf{p}$. Suppose that the searcher has searched $m(l)$ times in box l, so that $0 \leqq m(l) \leqq n - 1$ for $1 \leqq l \leqq N$ and $\sum_{l=1}^N m(l) = n - 1$. The searcher knows the values of $\alpha_{l1}, \cdots, \alpha_{l,m(l)}$ for each box l, $1 \leqq l \leqq N$. If the searcher selects box i to be searched next, he pays cost $c_i > 0$ and learns a value t of $\alpha_{i,m(i)+1}$ during his search of box i.

If the searcher finds the object by this search of box i, search is terminated

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and no more cost is incurred. Otherwise the searcher uses the value $t = \alpha_{i,m(i)+1}$ observed to revise his opinion of the object's location (via Bayes' rule) and the search continues. The value t is the probability that the searcher will not find the object on this search of box i, given that it is in box i. The searcher's problem is to find an optimal policy to minimize the total expected search cost for finding the object.

The following definition was introduced in Hall [6]:

DEFINITION. The sequence $\{X_j\}_{j=1}^{\infty}$ of [0, 1]-valued random variables satisfies the strong monotonicity condition (S.M.C.) if $(\prod_{j=1}^{m-1} X_j)E(1 - X_m \mid X_1, \cdots, X_{m-1})$ is monotone nonincreasing in $m \geqq 1$, a.s.

It was proven in Hall [6] that if $\{\alpha_{1j}\}_{j=1}^{\infty}, \cdots, \{\alpha_{Nj}\}_{j=1}^{\infty}$ are N independent stochastic processes and if the sequence $\{\alpha_{ij}\}_{j=1}^{\infty}$ satisfies the S.M.C. for each i, $1 \leqq i \leqq N$, there is an optimal policy $f = \{f_n\}_{n=1}^{\infty}$ (minimizing the total expected search cost) such that f is an analog of optimal search procedures found by Blackwell [4], Black [3] and Kadane [8] (in these earlier papers $\{\alpha_{ij}\}_{j=1}^{\infty}$ was a sequence of constants, $1 \leqq i \leqq N$). It was also shown that if the S.M.C. was not satisfied, f may fail to be optimal. Roughly speaking, f is the policy which at each stage of search selects the box with highest present probability of success, per unit cost of search.

In this paper we treat the related problem in which the searcher's objective is to maximize the probability of finding the object in at most n_0 searches, for fixed $n_0 \geqq 1$. This type of search problem was first treated by Chew [5], under the assumption that $\alpha_{ij} \equiv \alpha_i$ is a constant, for each i. Chew defined a search policy π to be *strongly optimal* if for each $n \geqq 1$, $P_\pi[M > n \mid \mathbf{p}^1] = \inf_\sigma P_\sigma[M > n \mid \mathbf{p}^1]$, where $P_\sigma[M > n \mid \mathbf{p}^1]$ is the probability (under policy σ) that the random time M at which the object is found is greater than n. As Chew points out, if π is strongly optimal and if $c_i \equiv c$, $1 \leqq i \leqq N$, then π minimizes the total expected search cost for finding the object, since $E_\pi[\mathscr{C} \mid \mathbf{p}^1] = c \sum_{n=1}^{\infty} P_\pi[M \geqq n \mid \mathbf{p}^1]$, where $\mathscr{C}$ is the random cost of search.

We extend Chew's results to the problem in which the overlook probabilities are random. An example providing motivation for this is given by Example (a) of Hall [6]. In this problem one gold coin is hidden among brass coins in one of several boxes (box i having $n_i > 0$ coins). The searcher is allowed to select boxes sequentially to reach in and take out a batch of coins, in search of the gold coin. The distribution of each batch size (and hence the mean batch size) for each draw is determined by the index i and the number m_i of coins remaining (the searcher observes the batch size w_i after each draw). The value of the overlook rv is $1 - w_i/m_i$. We now wish to search optimally to find the gold coin in at most n_0 searches. In this problem and in all the problems treated in Hall [6], the N processes of overlook random variables are independent.

In this paper we shall allow dependence of the overlook rv's among the boxes. Our results are that if $\{\alpha_{ij}: 1 \leqq i \leqq N, j \geqq 1\}$ satisfy a condition called the generalized strong monotonicity condition (or G.S.M.C.) then there is a strongly optimal policy g. The G.S.M.C. (defined in Section 2) is a generalization of the S.M.C., so that if $\{\alpha_{ij}\}_{j=1}^{\infty}$, $1 \leqq i \leqq N$ are independent processes and satisfy the S.M.C., the policy f of Hall is strongly optimal. We also give an example for $N = 2$ wherein $\{\alpha_{1j}\}_{j=1}^{\infty}$, $\{\alpha_{2j}\}_{j=1}^{\infty}$ are dependent processes such that the G.S.M.C. is satisfied. We compare this example to a problem given by Bellman [2] (page 90) and compare the optimal policy of our example to the solution of Bellman's problem. This gives an interesting comparison between the independence and dependence problems. Lastly, we give an example of an N-box search problem wherein $\{\alpha_{ij}\,;\, 1 \leqq i \leqq N, j \geqq 1\}$ are dependent and the G.S.M.C. is satisfied, using mixtures of Dirichlet processes (Antoniak [1]).

We remark that the results of this paper cannot be obtained from those of Hall [6], and that the results of this paper apply to the search problem with cost only when the cost of search is the same for all boxes. As in Chew, we assume $c_i \equiv 1$, $1 \leqq i \leqq N$.

2. The dynamic programming model. In Hall [6] the search problem with cost was formulated as a discrete time negative dynamic programming model. The problem treated in this paper is a positive dynamic programming problem with finite time horizon n_0. To describe the model we must give the action space A, state space S, transition probability q and reward function r.

The action space is $A = \{1, 2, \cdots, N\}$ where to take act i at stage m means to search the ith box then. Now let $\Delta^N = \{\mathbf{p} = (p_1, \cdots, p_N, 0) : p_i \geqq 0,\, 1 \leqq i \leqq N,$ and $\sum_{i=1}^{N} p_i = 1\}$ and $s^* = (0, 0, \cdots, 0, 1)$. Thus Δ^N is the set of initial possible states for the search problem. Moreover, s^* is the state into which the system is transformed once the object is found, meaning that when it is found it is situated at box $N + 1$, where no search can take place. For $i \in A$ and $t \in [0, 1]$, define the transition operator $T(i, t) : \Delta^N \cup \{s^*\} \to \Delta^N \cup \{s^*\}$ as follows: If $\mathbf{p} \in \Delta^N$, $T(i, t)\mathbf{p} = \mathbf{p}'$ where $p_i' = tp_i/(1 - (1 - t)p_i)$ and $p_l' = p_l/(1 - (1 - t)p_i)$ for $l \neq i$. This definition is valid except when $\mathbf{p} = \mathbf{e}^i$ and $t = 0$ ($\mathbf{e}^i$ is the vertex of Δ^N with 1 in the ith coordinate and 0 elsewhere). In this case we define $T(i, 0)\mathbf{e}^i = s^*$, and for completeness define $T(i, t)s^* = s^*$ for all i, t. If $\mathbf{p}$ is the prior location vector at some stage, action i is taken, the value t is observed and the search of box i is unsuccessful, then $T(i, t)\mathbf{p}$ is the posterior location vector for the object, computed by Bayes' rule.

Now define $S_1 = \Delta^N$ and for $n \geqq 2$, let $S_n = \{s_n = (\mathbf{p}^1, i_1, \mathbf{p}^2, i_2, \cdots, i_{n-1}, \mathbf{p}^n) :$ $\forall\, m$, $1 \leqq m \leqq n - 1$, $\mathbf{p}^{m+1} \in \Delta^N$ and $\exists\, u_m \in [0, 1] \ni \mathbf{p}^{m+1} = T(i_m, u_m)\mathbf{p}^m\}$. Then the state space $S = \bigcup_{n=1}^{\infty} S_n \cup \{s^*\}$, the disjoint union. To define the transition probability $q : S \times A \to S$, let μ_{i1} denote the (marginal) distribution of α_{i1} and for each $m \geqq 1$ let $\mu_{i,m+1}$ be a transition probability of $[0, 1]^m$ into $[0, 1]$ which is a conditional distribution of $\alpha_{i,m+1}$ given $\alpha_{i1}, \cdots, \alpha_{im}$, $1 \leqq i \leqq N$. Let $s \in S$ and $i \in A$. If $s = s_n \in S_n$ for some n, $t_{i1}, \cdots, t_{i,m(i)}$ denote the observed values of $\alpha_{i1}, \cdots, \alpha_{i,m(i)}$ occurring in s_n, where $m(i) = m(i; s_n)$ is the number of times

i is searched in s_n. For convenience of notation write $\mu_{i,m(i)+1}(\cdot \mid s_n) \equiv \mu_{i,m(i)+1}(\cdot \mid t_{i1}, \cdots, t_{i,m(i)})$. Then define $q(s^* \mid s_n, i) = p_i^n[1 - E(\alpha_{i,m(i)+1} \mid s_n)]$ where $E(\alpha_{i,m(i)+1} \mid s_n) = \int_0^1 t\mu_{i,m(i)+1}(dt \mid s_n)$. For B a Borel subset of Δ^N, define $q(\{(s_n, i)\} \times B \mid s_n, i) = \int_C [1 - (1 - t)p_i^n]\mu_{i,m(i)+1}(dt \mid s_n)$ where $C = \{t \in [0, 1]: T(i, t)\mathbf{p}^n \in B\}$. Hence $q(S_{n+1} \mid s_n, i) = 1 - q(s^* \mid s_n, i)$. Lastly, define $q(s^* \mid s^*, i) = 1$, for all i. Thus for $s_n \in S_n$, $q(s^* \mid s_n, i)$ is the probability that the object is found during the $(m(i) + 1)$st search of box i and $q(\{(s_n, i)\} \times B \mid s_n, i)$ is the probability that the object is not found and the posterior location vector lies in B.

For the reward function $r: S \times A \times S \to R^1$, if $s, s' \in S$, define $r(s, i, s^*) = 1$ for $s \in \bigcup_{n=1}^{n_0} S_n$ and $r(s, i, s') = 0$ otherwise. Thus the searcher receives a reward of one unit when and only when he finds the object by time n_0.

Now the sequential search problems for which solutions have been obtainable seem to share the following characteristic: At any stage of search, if a box i has the maximum current probability of a successful search and some box $k \neq i$ is searched, and if the search of box k is unsuccessful, then box i will have the maximum posterior probability of a successful search. As Furman Smith [12] puts it, the box which at some stage is "most inviting" will remain "most inviting" if some other box is searched unsuccessfully. Search problems which do not have this general property seem to be rather difficult to solve.

It is this consideration which prompts us to make the following definition: Let $n_i \geqq 1$ be an integer, $1 \leqq i \leqq N$, and define $\mathscr{F}(n_1 - 1, n_2 - 1, \cdots, n_N - 1)$ as the Borel σ-algebra generated by $\{\alpha_{i,m_i}: 1 \leqq i \leqq N, 1 \leqq m_i \leqq n_i - 1\}$. Now define the random variables

$$F_i = E(1 - \alpha_{i,n_i} \mid \mathscr{F}(n_1 - 1, n_2 - 1, \cdots, n_N - 1)), \qquad 1 \leqq i \leqq N,$$

and

$$F_i^k = E(1 - \alpha_{i,n_i+\delta_{ik}} \mid \mathscr{F}(n_1 - 1, n_2 - 1, \cdots, n_k, \cdots, n_N - 1)), \qquad 1 \leqq i, k \leqq N,$$

where δ_{ik} is Kronecker's delta symbol. Note that if each box i has been searched $n_i - 1$ times and $\mathbf{p}$ is the present location vector, then $p_i F_i$ is the current probability of a successful search in box i.

For any event B, let $P_B(A) = P(A \mid B)$ if $P(B) > 0$, where P is the probability measure on the probability space of $\{\alpha_{ij}: 1 \leqq i \leqq N, j \geqq 1\}$.

Definition 1. The sequence $\{\alpha_{ij}: 1 \leqq i \leqq N, j \geqq 1\}$ of $[0, 1]$-valued random variables satisfies the generalized strong monotonicity condition (or G.S.M.C.) if for all integers $n_i \geqq 1$, $1 \leqq i \leqq N$, the following conditions are met:

(1.1) $\quad (\prod_{j=1}^{n_i-1} \alpha_{ij})F_i > 0$ implies $(\prod_{j=1}^{n_i-1} \alpha_{ij})F_i^k > 0$, for $1 \leqq i \neq k \leqq N$.

(1.2) $\quad$ Let $B_i = \{(\prod_{j=1}^{n_i-1} \alpha_{ij})F_i > 0\}$. If $P(B_i) > 0$ then

$$\frac{(\prod_{j=1}^{n_l-1} \alpha_{lj})F_l}{(\prod_{j=1}^{n_i-1} \alpha_{ij})F_i} \geqq \frac{(\prod_{j=1}^{n_l+\delta_{kl}-1} \alpha_{lj})F_l^k}{(\prod_{j=1}^{n_i-1} \alpha_{ij})F_i^k}, \quad \text{a.s.—}P_{B_i},$$

for all i, l, k, $(l \neq i \neq k)$ with strict equality if $k \neq l$.

If the sequence $\{\alpha_{ij}: 1 \leqq i \leqq N, j \geqq 1\}$ satisfies the G.S.M.C., then a box i which at some stage is "most inviting" remains "most inviting" if a box $k \neq i$ is searched at that stage. The reasoning is as follows: Suppose that at stage $m \geqq 1$ box l has been searched $n_l - 1$ times, so that $0 \leqq n_l - 1 \leqq m$ and $\sum_{l=1}^{N}(n_l - 1) = m$. Values of the random variables $\alpha_{l1}, \cdots, \alpha_{l,n_l-1}$ were observed for box l. By abuse of notation, we may write $\mathbf{p} = (p_1, \cdots, p_N)$ where the lth coordinate of this location vector is

$$p_l = p_l^1(\textstyle\prod_{j=1}^{n_l-1} \alpha_{lj})/(\sum_{r=1}^{N} p_r^1(\prod_{j=1}^{n_r-1} \alpha_{rj})) .$$

A box i is "most inviting" if

$$p_i F_i \geqq p_l F_l \tag{1.3}$$

for all $l \neq i$, where we assume $p_l F_l > 0$, for some l. But (1.3) is equivalent to

$$p_i^1(\textstyle\prod_{j=1}^{n_i-1} \alpha_{ij})F_i \geqq p_l^1(\prod_{j=1}^{n_l-1} \alpha_{lj})F_l , \qquad \text{for all} \quad l \neq i . \tag{1.4}$$

In order that box i remain "most inviting" after a search of box $k \neq i$, it is required that for $1 \leqq l \leqq N$ we have

$$(T(k, \alpha_{k,n_k})\mathbf{p})_i F_i^k \geqq (T(k, \alpha_{k,n_k})\mathbf{p})_l F_l^k . \tag{1.5}$$

By the definition of the transformation operator $T(\cdot, \cdot)$ and that of F_i, F_i^k, we see that (1.5) is equivalent to

$$p_i^1(\textstyle\prod_{j=1}^{n_i-1} \alpha_{ij})F_i^k \geqq p_l^1(\prod_{j=1}^{n_l+\delta_{kl}-1} \alpha_{lj})F_l^k . \tag{1.6}$$

But if the G.S.M.C. is satisfied then (1.4) implies (1.6) because of (1.1) and (1.2). Thus a "most inviting" box remains "most inviting" if some other box is unsuccessfully searched first.

REMARK. If we define $\alpha'_{i,j} = \alpha_{i,n_i-1+j}$ for $1 \leqq i \leqq N$ and $j \geqq 1$, and let $\{\alpha'_{ij}: 1 \leqq i \leqq N, j \geqq 1\}$ have the conditional distribution given $\{\alpha_{i,m_i}: 1 \leqq i \leqq N, 1 \leqq m_i \leqq n_i - 1\}$ then $\{\alpha'_{ij}: 1 \leqq i \leqq N, j \geqq 1\}$ satisfies the G.S.M.C. on the event $\{\prod_{j=1}^{n_i-1} \alpha_{ij} > 0, 1 \leqq i \leqq N.\}$

We are now prepared to prove that under the assumption of the G.S.M.C., there is a strongly optimal search policy which, loosely speaking, always searches the "most inviting" box at each stage. In fact, the search rule we are interested in is the following:

DEFINITION 2. The generalized Blackwell-Black-Kadane-Chew analog (or G.B.B.K.C. policy) is the following search rule $g = \{g_n\}_1^\infty$: At initial state $\mathbf{p}^1$, g_1 chooses any action $i \in A$ such that $p_i^1 E(1 - \alpha_{i,1}) = \max_{1 \leqq l \leqq N} p_l^1 E(1 - \alpha_{l1})$. Suppose that $n \geqq 2$, and let state $s_n = (\mathbf{p}^1, i_1, \mathbf{p}^2, i_2, \cdots, i_{n-1}, \mathbf{p}^n) \in S_n$. Assume that box l has been searched $m_l = m(l; s_n)$ times in state s_n, for $1 \leqq l \leqq N$. Then g_n selects any box i achieving the equality

$$\begin{aligned} p_i^1(\textstyle\prod_{j=1}^{m_i} \alpha_{ij})E(1 - \alpha_{i,m_i+1} | \mathscr{F}(m_1, m_2, \cdots, m_N)) \\ = \max_{1 \leqq l \leqq N} p_l^1(\textstyle\prod_{j=1}^{m_l} \alpha_{lj})E(1 - \alpha_{l,m_l+1} | \mathscr{F}(m_1, m_2, \cdots, m_N)) \end{aligned}$$

where (by abuse of notation) $\alpha_{l1}, \cdots, \alpha_{l,m_l}$ are the (observed) overlook random variables for box l in the state s_n.

We shall now prove that if $\{\alpha_{ij}: 1 \leqq i \leqq N, j \geqq 1\}$ satisfies the G.S.M.C., the G.B.B.K.C. policy g is strongly optimal. The first result in this direction (known to this author) was obtained by Y. C. Kan [10] under the assumption that for each i, $\{\alpha_{ij}\}_{j=1}^{\infty}$ are independent and identically distributed. Our result is thus an extension of Kan's result to the situation in which the G.S.M.C. is satisfied. The proof, which is by induction, is a generalization of Kan's proof.

THEOREM 1. *Suppose that the sequence* $\{\alpha_{ij}: 1 \leqq i \leqq N, j \geqq 1\}$ *of overlook random variables satisfies the* G.S.M.C. *Then the* G.B.B.K.C. *policy* g *is strongly optimal*, i.e., *for each fixed* $n \geqq 1$,

$$P_g[M \leqq n \mid \mathbf{p}^1] = \sup_\sigma P_\sigma[M \leqq n \mid \mathbf{p}^1]$$

where M *is the random time at which the object is found.*

PROOF. By induction on $n = n_0$.

For $n = 1$, the conclusion is clearly true, from the definition of g_1.

Assume that for some $n_0 \geqq 2$ the conclusion is true for all $n < n_0$. Now let $n = n_0$.

From Hinderer ([7], page 115, Theorem 17.10) we know that there is an optimal policy $\sigma^* = \{\sigma_j^*\}_{j\geqq 1}$ which maximizes

$$P_\sigma[M \leqq n \mid \mathbf{p}^1] = \textstyle\sum_{j=1}^{n} P_\sigma[M = j \mid \mathbf{p}^1] ,$$

where $\sigma_j^*: S_j \rightarrow A$, $j \geqq 1$. Let $g_1(\mathbf{p}^1) = i_0$, so that

$$p_{i_0}^1 E(1 - \alpha_{i_0,1}) = \max_{1\leqq l \leqq N} p_l^1 E(1 - \alpha_{l1}) ,$$

and suppose $\sigma_1^*(\mathbf{p}^1) = k_0 \neq i_0$. We will show that g does at least as well as σ^*.

Suppose box k_0 is searched at state $\mathbf{p}^1$. Then if this search is unsuccessful, the new state is $\mathbf{p}^2 = T(k_0, \alpha_{k_0,1})\mathbf{p}^1$ where $p_l^2 = p_l^1/(1 - (1 - \alpha_{k_0,1})p_{k_0}^1)$ for $l \neq k_0$ and $p_{k_0}^2 = p_{k_0}^1 \alpha_{k_0,1}/(1 - (1 - \alpha_{k_0,1})p_{k_0}^1)$. The new state $\mathbf{p}^2$ is the initial state for the remaining $n - 1$ stage search problem. Because of the G.S.M.C.,

$$\begin{aligned} &p_{i_0}^2 E(1 - \alpha_{i_0,1} \mid \mathscr{F}\{\alpha_{k_0,1}\}) \\ &\qquad = \max\left(\max_{l\neq k_0} p_l^2 E(1 - \alpha_{l1} \mid \mathscr{F}\{\alpha_{k_0,1}\}), p_{k_0}^2 E(1 - \alpha_{k_0,2} \mid \mathscr{F}\{\alpha_{k_0,1}\})\right) \end{aligned}$$

where $\mathscr{F}\{\alpha_{k_0,1}\}$ is the Borel σ-algebra generated by $\alpha_{k_0,1}$.

Let $\alpha'_{lj} = \alpha_{lj}$ for $l \neq k_0$, $j \geqq 1$ and let $\alpha'_{k_0,j} = \alpha_{k_0,j+1}$ for $j \geqq 1$ and let $\{\alpha'_{ij}: 1 \leqq i \leqq N, j \geqq 1\}$ have the conditional posterior distribution given $\alpha_{k_0,1}$. From the earlier remark, $\{\alpha'_{ij}: 1 \leqq i \leqq N, j \geqq 1\}$ satisfies the G.S.M.C., and this is the sequence of overlook random variables for the remaining $n - 1$ stage search problem.

Thus by the induction hypothesis, g is optimal in the $n - 1$ stage problem, regardless of the value of $\alpha_{k_0,1}$ observed. Also notice

$$p_{i_0}^2 E(1 - \alpha'_{i_0,1}) = \max_{1\leqq l \leqq N} p_l^2 E(1 - \alpha'_{l1}) .$$

We may thus take $i_0 = g_1(\mathbf{p}^2)$.

Let $\pi = (k_0, i_0, g)$ be the policy which first searches box k_0, then i_0, then follows policy g (i.e., $\pi = \{\pi_j\}_{j \geq 1}$, where $\pi_1 \equiv k_0$, $\pi_2 \equiv i_0$, and $\pi_j = g_{j-2}$ for $j \geqq 3$). Since by induction hypothesis g is also optimal for the $n - 2$ stage search problem, π is optimal at state $\mathbf{p}^1$, i.e.,

$$P_{\sigma^*}[M \leqq n \mid \mathbf{p}^1] = P_\pi[M \leqq n \mid \mathbf{p}^1] .$$

Let $\delta = (i_0, k_0, g)$ be the policy which first searches box i_0, then k_0, then utilizes policy g.

For any policy σ and any $\mathbf{p} \in \Delta^N$, $\varphi_\sigma{}^j(\mathbf{p}) = P_\sigma[M \leqq j \mid \mathbf{p}]$ denotes the probability of finding the object in j searches when $\mathbf{p}$ is the initial location vector and policy σ is used. Note that $\varphi_\sigma{}^j(s^*) = 0$. Since both g and δ search box i_0 at state $\mathbf{p}^1$, and since the conditional posterior distribution of the overlook random variables (given $\alpha_{i_0,1}$) satisfies the G.S.M.C., g is optimal for the remaining $n - 1$ stage search problem. Thus

$$\varphi_g{}^n(\mathbf{p}^1) \geqq \varphi_\delta{}^n(\mathbf{p}^1) . \tag{1.7}$$

Moreover, since π and σ^* both search box k_0 at state $\mathbf{p}^1$ and since (i_0, g) is optimal at the new state (again by induction hypothesis) we have

$$\varphi_\pi{}^n(\mathbf{p}^1) \geqq \varphi_{\sigma^*}^n(\mathbf{p}^1) . \tag{1.8}$$

We will show that

$$\varphi_\delta{}^n(\mathbf{p}^1) \geqq \varphi_\pi{}^n(\mathbf{p}^1) . \tag{1.9}$$

By (1.7) and (1.8), this will imply that g is optimal for the n-stage search problem. Now

$$\begin{aligned} \varphi_\delta{}^n(\mathbf{p}^1) &= p^1_{i_0} E(1 - \alpha_{i_0,1}) + E[(1 - (1 - \alpha_{i_0,1})p^1_{i_0})\varphi^{n-1}_{(k_0,g)}(T(i_0, \alpha_{i_0,1})\mathbf{p}^1)] \\ &= p^1_{i_0} E(1 - \alpha_{i_0,1}) + p^1_{k_0} E(1 - \alpha_{k_0,1}) + E[(1 - (1 - \alpha_{i_0,1})p^1_{i_0} \\ &\quad - (1 - \alpha_{k_0,1})p^1_{k_0})\varphi_g{}^{n-2}(T(k_0, \alpha_{k_0,1}) \circ T(i_0, \alpha_{i_0,1})\mathbf{p}^1)] . \end{aligned} \tag{1.10}$$

Similarly

$$\begin{aligned} \varphi_\pi{}^n(\mathbf{p}^1) &= p^1_{k_0} E(1 - \alpha_{k_0,1}) + p^1_{i_0} E(1 - \alpha_{i_0,1}) + E[(1 - (1 - \alpha_{k_0,1})p^1_{k_0} \\ &\quad - (1 - \alpha_{i_0,1})p^1_{i_0})\varphi_g{}^{n-2}(T(i_0, \alpha_{i_0,1}) \circ T(k_0, \alpha_{k_0,1})\mathbf{p}^1)] . \end{aligned} \tag{1.11}$$

Since $T(i_0, \alpha_{i_0,1}) \circ T(k_0, \alpha_{k_0,1})\mathbf{p}^1 = \mathbf{p}'$ is the location vector where $p_l' = p_l/(1 - (1 - \alpha_{k_0,1})p^1_{k_0} - (1 - \alpha_{i_0,1})p^1_{i_0})$ for $l \neq k_0$, $l \neq i_0$ and $p'_{k_0} = \alpha_{k_0,1}p^1_{k_0}/(1 - (1 - \alpha_{k_0,1})p^1_{k_0} - (1 - \alpha_{i_0,1})p^1_{i_0})$ and $p'_{i_0} = \alpha_{i_0,1}p^1_{i_0}/(1 - (1 - \alpha_{k_0,1})p^1_{k_0} - (1 - \alpha_{i_0,1})p^1_{i_0})$, it is clear that $T(i_0, \alpha_{i_0,1}) \circ T(k_0, \alpha_{k_0,1})\mathbf{p}^1$ and $T(k_0, \alpha_{k_0,1}) \circ T(i_0, \alpha_{i_0,1})\mathbf{p}^1$ have the same distribution. Hence in fact $\varphi_\pi{}^n(\mathbf{p}^1) = \varphi_\delta{}^n(\mathbf{p}^1)$, and the theorem is proved. □

We remark that in the above proof, in equation (1.10), it will not happen that $T(i_0, \alpha_{i_0,1})\mathbf{p}^1 = s^*$, unless $\mathbf{p}^1 = \mathbf{e}^{i_0}$, in which case it is clear that always searching box i_0 is optimal. A similar remark applies to equation (1.11). Ross [11] contains a very nice proof that there is an optimal policy σ^* in finite horizon problems.

3. Illustrations and examples. In this section we examine the special case where $\{\alpha_{1j}\}_{j\geqq 1}, \cdots, \{\alpha_{Nj}\}_{j\geqq 1}$ are N independent stochastic processes and give an example for $N = 2$ where $\{\alpha_{1j}\}_{j\geqq 1}, \{\alpha_{2j}\}_{j=1}^{\infty}$ are dependent processes. We also give an example of N dependent processes for $N \geqq 2$ arbitrary.

(a) *An illustration wherein the N stochastic processes are independent.* Suppose that $\{\alpha_{ij}\}_{j\geqq 1}$, $1 \leqq i \leqq N$ are N independent processes. Then $F_i \equiv F_i^k$, for $1 \leqq i \neq k \leqq N$. Suppose that $\{\alpha_{ij}\}_{j\geqq 1}$ satisfies the S.M.C. for each i. Then condition (1.1) of the G.S.M.C. is certainly satisfied. And condition (1.2) is true, by independence of the N processes and by the S.M.C. Thus, by Theorem 1 of Section 2, the Blackwell–Black–Kadane rule of Hall [6] is strongly optimal, and all of the examples found there provide examples for Theorem 1.

(b) *An example for* $N = 2$ *where* $\{\alpha_{1j}\}_{j\geqq 1}, \{\alpha_{2j}\}_{j\geqq 1}$ *are dependent.* Let us denote $\alpha_{1j} = X_j, j \geqq 1$ and $\alpha_{2j} = Y_j, j \geqq 1$. Assume that $\{X_j\}_{j\geqq 1}$ are i.i.d. Bernoulli rv's given an unknown parameter λ, i.e., given λ, each X_j has probability function

$$f(x \mid \lambda) = \lambda^x(1-\lambda)^{1-x}, \qquad x = 0, 1 .$$

Similarly, let $\{Y_j\}_{j\geqq 1}$ be i.i.d. Bernoulli rv's given an unknown parameter ζ, so that Y_j has the probability function

$$g(y \mid \zeta) = \zeta^y(1-\zeta)^{1-y}, \qquad y = 0, 1 .$$

Let the vector (λ, ζ) have a Dirichlet prior distribution, i.e., (λ, ζ) has density

$$h(\lambda, \zeta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \lambda^{a-1}\zeta^{b-1} I_{\{\lambda+\zeta=1\}}$$

where $a > 0$, $b > 0$ are known constants. Thus λ has a beta distribution $\mathscr{B}e(a, b)$, and $\zeta = 1 - \lambda$.

Hence the joint distribution of $X_1, \cdots, X_m, Y_1, \cdots, Y_n, \lambda$ and ζ is given by

$$f(x_1, \cdots, x_m, y_1, \cdots, y_n, \lambda, \zeta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \lambda^{a+\sum_{j=1}^m x_j+n-\sum_{i=1}^n y_i-1}(1-\lambda)^{b+\sum_{i=1}^n y_i+m-\sum_{j=1}^m x_j-1} .$$

Thus the joint distribution of $X_1, \cdots, X_m, Y_1, \cdots, Y_n$ is given by

$$f(x_1, \cdots, x_m, y_1, \cdots, y_n) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a+n+\sum_{j=1}^m x_j - \sum_{i=1}^{n'} y_i)\Gamma(b+m-(\sum_{j=1}^m x_j - \sum_{i=1}^n y_i))}{\Gamma(a+b+m+n)} .$$

Moreover, it is easy to see that the posterior distribution of λ given $X_j = x_j$, $1 \leqq j \leqq m$, $Y_l = y_l$, $1 \leqq l \leqq n$ is

$$\mathscr{B}e(a+n+\textstyle\sum_{j=1}^m x_j - \sum_{l=1}^n y_l, b+m-(\sum_{j=1}^m x_j - \sum_{l=1}^n y_l)) .$$

We will show that $\{X_j, Y_l, j \geqq 1, l \geqq 1\}$ satisfy the G.S.M.C. We show condition (1.1) holds while we are demonstrating that condition (1.2) holds. By symmetry, it suffices to show condition (1.2) for $(i, k) = (1, 2)$. Thus we must only show that

$$\text{(2.1)}\quad \frac{(\prod_{l=1}^{n-1} Y_l)E(1 - Y_n \mid X_1, \cdots, X_{m-1}, Y_1, \cdots, Y_{n-1})}{(\prod_{j=1}^{m-1} X_j)E(1 - X_m \mid X_1, \cdots, X_{m-1}, Y_1, \cdots, Y_{n-1})} \geqq \frac{(\prod_{l=1}^{n-1} Y_l)Y_n E(1 - Y_{n+1} \mid X_1, \cdots, X_{m-1}, Y_1, \cdots, Y_{n-1}, Y_n)}{(\prod_{j=1}^{m-1} X_j)E(1 - X_m \mid X_1, \cdots, X_{m-1}, Y_1, \cdots, Y_{n-1}, Y_n)}, \quad \text{a.s.},$$

conditional on the event where the two denominators are positive. Now from the above, we have

$$\text{(2.2)}\quad \begin{aligned} E(1 - Y_n \mid X_1, \cdots, X_{m-1}, Y_1, \cdots, Y_{n-1}) &= E[E(1 - Y_n \mid \lambda) \mid X_1, \cdots, X_{m-1}, Y_1, \cdots, Y_{n-1})] \\ &= E[\lambda \mid X_1, \cdots, X_{m-1}, Y_1, \cdots, Y_{n-1}] \\ &= \frac{a + n - 1 + \sum_{j=1}^{m-1} X_j - \sum_{l=1}^{n-1} Y_l}{a + b + n + m - 2} \end{aligned}$$

and

$$\text{(2.3)}\quad E(1 - Y_{n+1} \mid X_1, \cdots, X_{m-1}, Y_1, \cdots, Y_{n-1}, Y_n) = \frac{a + n + \sum_{j=1}^{m-1} X_j - \sum_{l=1}^{n} Y_l}{a + b + n + m - 1}.$$

Similarly,

$$\text{(2.4)}\quad E(1 - X_m \mid X_1, \cdots, X_{m-1}, Y_1, \cdots, Y_{n-1}) = \frac{b + m - 1 - (\sum_{j=1}^{m-1} X_j - \sum_{l=1}^{n-1} Y_l)}{a + b + n + m - 2}$$

and

$$\text{(2.5)}\quad E(1 - X_m \mid X_1, \cdots, X_{m-1}, Y_1, \cdots, Y_{n-1}, Y_n) = \frac{b + m - 1 - (\sum_{j=1}^{m-1} X_j - \sum_{l=1}^{n} Y_l)}{a + b + n + m - 1}.$$

It follows from (2.2) and (2.4) that (1.1) of the G.S.M.C. holds.

Let $\gamma = a + n - 1 + \sum_{j=1}^{m-1} X_j - \sum_{l=1}^{n-1} Y_l$ and $\delta = b + m - 1 - (\sum_{j=1}^{m-1} X_j - \sum_{l=1}^{n-1} Y_l)$, so that expression (2.2) is $\gamma/(\gamma + \delta)$, (2.3) is $(\gamma + 1 - Y_n)/(\gamma + \delta + 1)$, (2.4) is $\delta/(\gamma + \delta)$ and (2.5) is $(\delta + Y_n)/(\gamma + \delta + 1)$. Thus it suffices to show that

$$\text{(2.6)}\quad \frac{\dfrac{\gamma}{\gamma + \delta}}{\dfrac{\delta}{\gamma + \delta}} \geqq \frac{Y_n\left(\dfrac{\gamma + 1 - Y_n}{\gamma + \delta + 1}\right)}{\left(\dfrac{\delta + Y_n}{\gamma + \delta + 1}\right)}, \quad \text{a.s.}, \qquad \text{or}$$

$$\frac{\gamma}{\delta} \geqq \frac{Y_n(\gamma + 1 - Y_n)}{\delta + Y_n}, \quad \text{a.s.}$$

If $Y_n = 0$, there is nothing to show. If $Y_n = 1$, then $\gamma/\delta \geqq \gamma/(\delta + 1)$. Thus the G.S.M.C. is satisfied.

Suppose now that $p_1^1 E(1 - X_1) \geqq p_2^1 E(1 - Y_1)$, so that box 1 is searched at stage 1. If the value of X_1 observed is 0, then either the object is found (because it actually was hidden in box 1) or the searcher is sure it is not in box 1, and proceeds to search box 2. If $X_1 = 1$, then the searcher searches box 1 if

$$p_1^1 \cdot 1 \cdot E(1 - X_2 | X_1 = 1) > p_2^1 E(1 - Y_1 | X_1 = 1)$$

and searches box 2 otherwise. Notice that

$$\begin{aligned} E(1 - X_m | X_1 = 1, \cdots, X_{m-1} = 1, Y_1 = y_1, \cdots, Y_{n-1} = y_{n-1}) \\ = \frac{b + m - 1 - (m - 1 - \sum_{l=1}^{n-1} y_l)}{a + b + n + m - 2} \\ = \frac{b - \sum_{l=1}^{n-1} y_l}{a + b + n + m - 2} \end{aligned}$$

is decreasing in m as m increases, and that

$$\begin{aligned} E(1 - Y_n | X_1 = 1, \cdots, X_{m-1} = 1, Y_1 = y_1, \cdots, Y_{n-1} = y_{n-1}) \\ = \frac{a + n - 1 + m - 1 - \sum_{l=1}^{n-1} y_l}{a + b + n + m - 2} \end{aligned}$$

is increasing in m.

Thus if the searcher, following the optimal rule g, starts by searching in box 1, and if he observes a sequence of 1's as values of the overlook random variables for box 1, it may be optimal to switch to box 2 to search before he ever observes the value 0 for an overlook random variable of box 1.

Contrast that situation to the following search problem, resembling one found in Bellman [2] (page 90) and solved by Kadane [9]. Suppose both λ and ζ are known constants, $\lambda + \zeta$ not necessarily equal to 1, and that $X_1, X_2, \cdots$ are i.i.d. Bernoulli random variables with parameter λ and $Y_1, Y_2, \cdots$ are i.i.d. Bernoulli random variables with parameter ζ. Suppose again that $p_1^1 E(1 - X_1) \geqq p_2^1 E(1 - Y_1)$. Then the searcher starts again by searching box 1. But in this problem, the optimal procedure tells the searcher to continue searching box 1 until the first time that the value 0 is observed for the overlook random variable (or until n_0 searches have been performed). Thus the searcher should not switch to box 2 until the value 0 is observed for box 1.

We now give an example of a nonparametric adaptive search problem such that $\{\alpha_{ij}\}_{j \geqq 1}$, $1 \leqq i \leqq N$ are dependent processes.

(c) *An example for $N \geqq 2$ using mixtures of Dirichlet processes.* Assume that there are N boxes labeled $1, 2, \cdots, N$. Let $\lambda_1, \lambda_2, \cdots, \lambda_N$ be N (different) finite nonnull positive σ-additive measures on [0, 1] such that $\nu_i = \int x\lambda_i(dx)$ and $w_i = \lambda_i([0, 1])$ are independent of i, $1 \leqq i \leqq N$. Let Π denote the set of $N!$ permutations of the symbols $1, 2, \cdots, N$, and let $\pi_1, \pi_2, \cdots, \pi_l, \cdots, \pi_{N!}$ be a list of the members of Π. Let π be one of these permutations, π unknown, and assume box i is assigned measure $\lambda_{\pi(i)}$, where $\pi(i)$ denotes the image of i under π. The initial prior probability that π is π_l is $h(\pi_l)$, $1 \leqq l \leqq N!$

As before, $\{\alpha_{ij}\}_{j \geqq 1}$ is the sequence of overlook random variables for box i.

Given π, the N stochastic processes $\{\alpha_{1j}\}_{j\geqq 1}, \cdots, \{\alpha_{Nj}\}_{j\geqq 1}$ are assumed to be mutually independent, and, given π, $\{\alpha_{ij}\}_{j\geqq 1}$ are i.i.d. according to a Dirichlet process prior $P_{\pi(i)} \in \mathscr{D}(\lambda_{\pi(i)})$. Since in general π is not known but the prior $\{h(l)\}_{1\leqq l \leqq N!}$ is known, $\{\alpha_{1j}\}_{j\geqq 1}, \cdots, \{\alpha_{Nj}\}_{j\geqq 1}$ are not mutually independent. The marginal distribution of $\{\alpha_{ij}\}_{j\geqq 1}$ is determined by a random probability measure P_i which is a mixture of Dirichlet processes, i.e., $P_i \in \sum_{\pi_l \in \Pi} h(\pi_l)\mathscr{D}(\lambda_{\pi_l(i)})$ (see Antoniak [1] for results concerning mixtures of Dirichlet processes).

We seek conditions under which (1.1) and (1.2) of the G.S.M.C. will be satisfied. Assume box i has been searched $n_i - 1$ times, and let $\boldsymbol{\alpha}_{i,n_i-1}$ denote the vector of observed values $(\alpha_{i1}, \cdots, \alpha_{i,n_i-1})$ for box i. Now the conditional distribution of $\{\alpha_{i,j_i} : j_i \geqq n_i, 1 \leqq i \leqq N\}$ given π and $\boldsymbol{\alpha}_{1,n_1-1}, \cdots, \boldsymbol{\alpha}_{N,n_N-1}$ is determined by the Dirichlet processes whose parameters are $\lambda_{\pi(1)} + \sum_1^{n_1-1}\delta_{\alpha_{1j}}, \cdots, \lambda_{\pi(N)} + \sum_1^{n_N-1}\delta_{\alpha_{Nj}}$ (since, given π and $\boldsymbol{\alpha}_{1,n_1-1}, \cdots, \boldsymbol{\alpha}_{N,n_N-1}$, $\{\alpha_{1j}\}_{j\geqq n_1}, \cdots, \{\alpha_{Nj}\}_{j \geqq n_N}$ are N mutually independent processes, and $\{\alpha_{ij}\}_{j\geqq n_i}$ is distributed according to a measure $Q_1 \in \mathscr{D}(\lambda_{\pi(i)} + \sum_1^{n_i-1}\delta_{\alpha_{ij}}))$. From Lemma 1, Part 4, in the paper of Antoniak, the posterior distribution of π given $\boldsymbol{\alpha}_{1,n_1-1}, \cdots, \boldsymbol{\alpha}_{N,n_N-1}$ is given by $h(\pi_l \mid \boldsymbol{\alpha}_{1,n_1-1}, \cdots, \boldsymbol{\alpha}_{N,n_N-1}) = t_l / \sum_m t_m$, where $\pi_l \in \Pi$ and

$$t_m = h(\pi_m) \prod_{i=1}^N \frac{1}{M_{\pi_m(i)}^{(n_i-1)}} \prod_{j=1}^{r_i} \lambda'_{\pi_m(i)}(\alpha_{i(j)})(\mu_{\pi_m(i)}(\alpha_{i(j)}) + 1)^{(n(\alpha_{i(j)})-1)} \tag{2.7}$$

and where $\lambda'_{\pi_m(i)}$ denotes the Radon–Nikodym derivative of $\lambda_{\pi_m(i)}$ with respect to $\sum_1^N \lambda_i$; $\alpha_{i(j)}$ is the jth distinct value in $\boldsymbol{\alpha}_{i,n_i-1}$; $n(\alpha_{i(j)})$ is the number of times the value $\alpha_{i(j)}$ occurs in $\boldsymbol{\alpha}_{i,n_i-1}$; $M_{\pi_m(i)}^{(n_i-1)} = \lambda_{\pi_m(i)}([0, 1]) \cdot (\lambda_{\pi_m(i)}([0, 1]) + 1) \cdot \cdots \cdot (\lambda_{\pi_m(i)}([0, 1]) + n_i - 2)$, and $\mu_{\pi_m(i)}(\alpha_{i(j)}) = \lambda'_{\pi_m(i)}(\alpha_{i(j)})$ if $\alpha_{i(j)}$ is an atom of $\lambda_{\pi_m(i)}$, zero otherwise. We assume here that $\sum_1^N \lambda_i$ has mass 1 at each atom of λ_l, $1 \leqq l \leqq N$, in order to apply Antoniak's result.

Thus

$$\begin{aligned} E(1 - \alpha_{i,n_i} \mid \boldsymbol{\alpha}_{1,n_1-1}, \cdots, \boldsymbol{\alpha}_{N,n_N-1}) &= 1 - (\textstyle\sum t_m)^{-1} \sum_{\pi_l} t_l \dfrac{\nu_{\pi_l(i)} + \sum_1^{n_i-1}\alpha_{ij}}{w_{\pi_l(i)+n_i-1}} \\ &= (\textstyle\sum t_m)^{-1} \sum_{\pi_l} t_l \left(1 - \dfrac{\nu_{\pi_l(i)} + \sum_1^{n_i-1}\alpha_{ij}}{w_{\pi_l(i)+n_i-1}}\right). \end{aligned}$$

Now condition (1.2) of the G.S.M.C. essentially states that

$$\begin{aligned} &\frac{E(1 - \alpha_{i,n_i} \mid \boldsymbol{\alpha}_{1,n_1-1}, \cdots, \boldsymbol{\alpha}_{N,n_N-1})}{E(1 - \alpha_{u,n_u} \mid \boldsymbol{\alpha}_{1,n_1-1}, \cdots, \boldsymbol{\alpha}_{k,n_k-1}, \cdots, \boldsymbol{\alpha}_{N,n_N-1})} \\ &\qquad = \frac{E(1 - \alpha_{i,n_i} \mid \boldsymbol{\alpha}_{1,n_1-1}, \cdots, \boldsymbol{\alpha}_{k,n_k}, \cdots, \boldsymbol{\alpha}_{N,n_N-1})}{E(1 - \alpha_{u,n_u} \mid \boldsymbol{\alpha}_{1,n_1-1}, \cdots, \boldsymbol{\alpha}_{k,n_k}, \cdots, \boldsymbol{\alpha}_{N,n_N-1})} \end{aligned}$$

for $1 \leqq i, u, k \leqq N$, $i \neq k \neq u$, $i \neq u$. To ensure this, we use the assumption that $w_{\pi_l(i)} \equiv w$, $\nu_{\pi_l(i)} \equiv \nu$, independent of π_l. Thus

$$E(1 - \alpha_{i,n_i} \mid \boldsymbol{\alpha}_{1,n_1-1}, \cdots, \boldsymbol{\alpha}_{N,n_N-1}) = 1 - \frac{\nu + \sum_1^{n_i-1}\alpha_{ij}}{w + n_i - 1} = E(1 - \alpha_{i,n_i} \mid \boldsymbol{\alpha}_{i,n_i-1}).$$

Hence (1.1) is satisfied, and the remainder of condition (1.2) follows by assuming that (w, ν) satisfies the strong monotonicity condition of Hall [6, Example (c)]. Hence the Blackwell–Black–Kadane–Chew search rule is strongly optimal in this problem, and we have an example in which the N processes are dependent and the G.S.M.C. holds.

Finally, we point out that Section 5, Example (d) of Hall provides an example of an adaptive search problem with $N = 2$ such that $\{\alpha_{1j}\}_{j\geqq 1}$, $\{\alpha_{2j}\}_{j\geqq 1}$ are independent processes, neither the S.M.C. nor the G.S.M.C. is satisfied, and, for $n_0 = 4$ in Theorem 1 above, the one-stage look ahead rule (i.e., the G.B.B.K. policy) is not optimal.

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REFERENCES

[1] ANTONIAK, CHARLES E. (1974). Mixtures of Dirichlet processes with applications to Bayesian nonparametric problems. *Ann. Statist.* **2** 1152–1174.

[2] BELLMAN, RICHARD (1957). *Dynamic Programming*. Princeton Univ. Press.

[3] BLACK, WILLIAM L. (1965). Discrete sequential search. *Information and Control* **8** 159–162.

[4] BLACKWELL, DAVID (1962). Notes on Dynamic Programming. Unpublished notes, Univ. of California, Berkeley.

[5] CHEW, MILTON C. (1967). A sequential search procedure. *Ann. Math. Statist.* **38** 494–502.

[6] HALL, GAINEFORD J., JR. (1976). Sequential search with random overlook probabilities. *Ann. Statist.* **4** 807–816.

[7] HINDERER, K. (1970). *Foundations of Non-stationary Dynamic Programming with Discrete Time Parameter*. Springer-Verlag, New York.

[8] KADANE, JOSEPH B. (1968). Discrete search and the Neyman-Pearson lemma. *J. Math. Anal. Appl.* **22** 156–171.

[9] KADANE, JOSEPH B. (1968). Optimal whereabouts search. *Operations Res.* **19** 894–904.

[10] KAN, Y. C. (1972). Optimal search models. Ph. D. dissertation, Univ. of California, Berkeley.

[11] ROSS, SHELDON M. (1972). Dynamic programming and gambling models. ORC 72-24, Department of Industrial Engineering and Operations Research, Univ. of California, Berkeley.

[12] SMITH, FURMAN S. (1972). Binomial searching for a random number of multinomially hidden balls. Ph. D. dissertation, Florida State Univ.

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