The Annals of Statistics
1976, Vol. 4, No. 6, 1038–1050

CONSISTENCY IN CONCAVE REGRESSION

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For each t in some subinterval T of the real line let F_t be a distribution function with mean $m(t)$. Suppose $m(t)$ is concave. Let $t_1, t_2, \cdots$ be a sequence of points in T and let $Y_1, Y_2, \cdots$ be an independent sequence of random variables such that the distribution function of Y_k is F_{t_k}. We consider estimators $m_n(t) = m_n(t; Y_1, \cdots, Y_n)$ which are concave in t and which minimize $\sum_{i=1}^n [m_n(t_i; Y_1, \cdots, Y_n) - Y_i]^2$ over the class of concave functions. We investigate their consistency and the convergence of $\{m_n'(t)\}$ to $m'(t)$.

1. Introduction and summary. Concave and convex functions occur with some regularity. (Note that the negative of a convex function is concave.) In economics, utility functions are usually assumed to be concave; marginal utility is often assumed to be convex; and functions representing productivity, supply, and demand curves are often assumed to be either concave or convex. (See Hildreth [3] for more discussion of this point and a fairly detailed example.) Various functions which occur in statistics are sometimes assumed to be either concave or convex. This paper concerns the estimation of such functions.

Let T be a subinterval of the real line having positive length. For each t in T let F_t be a distribution function with mean $m(t)$.

(A1) Assume $m(t)$ is continuous and concave on T.

(Note that concavity implies continuity except possibly at the endpoints of T.) For each subset A of T define

$$N_n(A) = \sum_{k=1}^n I_A(t_k) = \sharp\{k : 1 \leqq k \leqq n \text{ and } t_k \text{ is in } A\} \tag{1.1}$$

where we use the notation $\sharp(S)$ to denote the number of elements in the set S.

(A2) Assume that $t_1, t_2, \cdots$ is a sequence of (not necessarily distinct) points from T. Assume that for each subinterval I of T having positive length

$$\liminf N_n(I)/n > 0\,. \tag{1.2}$$

(A3) Assume that $Y_1, Y_2, \cdots$ is an independent sequence of random variables such that Y_k has distribution function F_{t_k}.

We think of $Y_1, Y_2, \cdots$ as a sequence of random approximations to $m(t_1)$, $m(t_2), \cdots$ respectively. For each positive integer n we will define precisely (in the next section) an estimator $m_n(t) = m_n(t; Y_1, \cdots, Y_n)$ of $m(t)$ which minimizes a sum of squares.

Received August 1975; revised April 1976.
AMS 1970 *subject classifications.* Primary 62G05; Secondary 90C20.
Key words and phrases. Concave, convex, nonparametric regression, concave regression, convex regression, consistency, regression.

For $y \geqq 0$ define

$$G(y) = \sup_{t \in T} \{F_t[m(t) - y] + 1 - F_t[(m(t) + y)-]\} \tag{1.3a}$$

so that G provides a uniform bound on the tails of the distributions of the error random variables $Y_k - m(t_k)$. In particular,

$$P\{|Y_k - m(t_k)| \geqq y\} \leqq G(y) \qquad \text{for all} \quad k . \tag{1.3b}$$

(A4) Assume that

$$\lim_{y\to\infty} G(y) = 0 \tag{1.4}$$

and that

$$\int_0^\infty y^2 |dG(y)| = Q^2 < \infty . \tag{1.5}$$

The main purpose of this paper is to prove the following:

THEOREM. *Suppose* $T = [0, 1]$ *and that* $0 < \alpha < \beta < 1$. *Then*

$$P\{\limsup \max_{t \in T} [m_n(t) - m(t)] \leqq 0; \quad \liminf \min_{\alpha \leqq t \leqq \beta} [m_n(t) - m(t)] \geqq 0\} = 1 . \tag{1.6}$$

This theorem says that, with probability one, $m_n(t)$ converges to $m(t)$ uniformly on $[\alpha, \beta]$, and in addition $m_n(t)$ will not get too large at the ends of the interval T.

In the next section we present the estimators $m_n(t)$. The theorem stated above is proved in Section 3. Section 4 contains some corollaries giving similar results about the convergence of $\{m_n'(t)\}$ to $m'(t)$. Section 5 contains some concluding remarks including some discussion of our assumptions and of variations on the main theorem.

2. The estimators. Fix n and let $r_1, \cdots, r_\nu$ be the distinct elements of $t_1, \cdots, t_n$ ordered so that $r_i < r_j$ if $i < j$. Let $y_1, \cdots, y_n$ be real numbers which we can think of as the values taken on by $Y_1, \cdots, Y_n$ at some particular point ω in our underlying probability space.

Let U be the set of all real valued functions g on $T = [0, 1]$ such that

$$\frac{g(r_{j+1}) - g(r_j)}{r_{j+1} - r_j} \leqq \frac{g(r_j) - g(r_{j-1})}{r_j - r_{j-1}} \qquad \text{for} \quad 2 \leqq j \leqq \nu - 1 ; \tag{2.1}$$

(2.2) g is continuous;

(2.3) if $\nu = 1$ then $g(t) = g(r_1)$ for all t in T;

(2.4) if $\nu = 2$ then g has constant slope; and

(2.5) if $\nu \geqq 3$ then g has constant slope on each of the intervals $(0, r_2), (r_2, r_3), \cdots, (r_{\nu-2}, r_{\nu-1}), (r_{\nu-1}, 1)$.

Note that U is a collection of concave, continuous and piecewise linear functions on T. Observe also that if h is a concave function on $\{r_1, \cdots, r_\nu\}$ (i.e., if h

satisfies (2.1)) then there is a *unique* function g in U which agrees with h on $\{r_1, \cdots, r_\nu\}$.

For $j = 1, \cdots, \nu$ define

$$A_j = \{k : 1 \leqq k \leqq n \text{ and } t_k = r_j\}, \tag{2.6}$$

$$s_j = \#(A_j), \quad \text{and} \tag{2.7}$$

$$\bar{y}_j = \sum_{k \in A_j} y_k/s_k. \tag{2.8}$$

Note that

$$\sum_{k=1}^n [g(t_k) - y_k]^2 = \sum_{j=1}^\nu s_j[g(r_j) - \bar{y}_j]^2 + \sum_{j=1}^\nu \sum_{k \in A_j} (\bar{y}_j - y_k)^2 \tag{2.9}$$

and that the last term in (2.9) does not depend on g.

We now think of $(g(r_1), \cdots, g(r_\nu))$ as a point in ν-dimensional Euclidean space. It is well known (from quadratic programming) and fairly easy to show that there is a *unique* point $(g^*(r_1), \cdots, g^*(r_\nu))$ which minimizes

$$\sum_{j=1}^\nu s_j[g(r_j) - \bar{y}_j]^2 \tag{2.10}$$

subject to (2.1). We can think of g^* as a concave function of $\{r_1, \cdots, r_\nu\}$. We also use g^* to represent the unique function in U whose values on $r_1, \cdots, r_\nu$ are $g^*(r_1), \cdots, g^*(r_\nu)$. It follows from (2.9) that g^* is the unique function in U which minimizes

$$\sum_{k=1}^n [g(t_k) - y_k]^2. \tag{2.11}$$

We let

$$m_n(t; y_1, \cdots, y_n) = g^*(t). \tag{2.12}$$

Our estimators will be the functions $m_n(t; Y_1, \cdots, Y_n)$.

The computation of $m_n(t; y_1, \cdots, y_n)$ is a problem in quadratic programming. Hildreth [3] proposed an iterative procedure for obtaining m_n. The authors know of no closed form solution for m_n. Its minimizing property will be used in our proofs.

3. Proof of the main theorem.

We proceed via several lemmas.

LEMMA 1. *Let $r \geqq 1$ and let $X_1, X_2, \cdots$ be an independent sequence of random variables such that $EX_i = 0$ and $E|X_i|^{2r} < \infty$ for all i, and such that $\sum_{i=1}^\infty E|X_i|^{2r}/i^{r+1} < \infty$. Corresponding to each positive integer $n \geqq 2$ let $i_{1,n}, i_{2,n}, \cdots, i_{n,n}$ be a permutation of the integers $1, \cdots, n$ obtained by assigning a place to the integer n between two successive integers, or at the beginning, or at the end, of the permutation corresponding to the integer $n - 1$. Let B be a positive real number; let $a_{n,0} = 0$; and for $n = 1, 2, \cdots$ let $a_{n,1}, \cdots, a_{n,n}$ be random variables such that*

$$P\{\limsup \sum_{i=1}^n |a_{n,i} - a_{n,i-1}| \leqq B\} = 1.$$

Then

$$n^{-1} \sum_{k=1}^n a_{n,k} X_{i_{k,n}} \to 0 \quad \text{a.s.}$$

as $n \to \infty$.

PROOF.

$$\begin{aligned}|n^{-1}\textstyle\sum_{k=1}^{n} a_{n,k}X_{i_{k,n}}| &= |n^{-1}\textstyle\sum_{k=1}^{n}\sum_{j=1}^{k}(a_{n,j}-a_{n,j-1})X_{i_{k,n}}| \\ &= |\textstyle\sum_{j=1}^{n}(a_{n,j}-a_{n,j-1})n^{-1}\sum_{k=j}^{n}X_{i_{k,n}}| \\ &\leqq \textstyle\sum_{j=1}^{n}|a_{n,j}-a_{n,j-1}| \times |n^{-1}\sum_{k=1}^{n}X_{i_{k,n}} - n^{-1}\sum_{k=1}^{j-1}X_{i_{k,n}}| \\ &\leqq 2(\textstyle\sum_{j=1}^{n}|a_{n,j}-a_{n,j-1}|) \times (\max_{j=1,\cdots,n} n^{-1}|\sum_{k=1}^{j}X_{i_{k,n}}|)\,.\end{aligned}$$

The "lim sup" of the first expression in brackets is "$\leqq B$ a.s." by hypothesis. The "lim sup" of the second expression is zero a.s. by Brunk's theorem ([1], Theorem 6.1). Thus $\limsup |n^{-1}\sum_{k=1}^{n} a_{n,k}X_{i_{k,n}}| = 0$ a.s. proving this lemma.

For notational convenience we define $Z_k = Y_k - m(t_k)$ and $\sigma_k^2 = EZ_k^2$.

LEMMA 2. *If* $0 \leqq \alpha < \beta \leqq 1$ *then there exists a positive real number* M *such that* $P(\limsup\{|m_n(t) - m(t)| \geqq M \text{ for all } t \text{ in } [\alpha, \beta]\}) = 0$.

PROOF. From the minimizing property of m_n we see that

$$\limsup n^{-1}\textstyle\sum_{k=1}^{n}(Y_k - m_n(t_k))^2 \leqq \limsup n^{-1}\sum_{k=1}^{n} Z_k^2\,.$$

A version of the strong law of large numbers applied to the sequence $\{Z_k^2 - \sigma_k^2\}$ shows that $n^{-1}\sum_{k=1}^{n}(Z_k^2 - \sigma_k^2) \to 0$ a.s. so that $\limsup n^{-1}\sum_{k=1}^{n} Z_k^2 = \limsup n^{-1}\sum_{k=1}^{n}\sigma_k^2$ a.s. From the definitions of G and Q we have $n^{-1}\sum_{k=1}^{n}\sigma_k^2 \leqq Q^2$ for all n. Putting these together gives $\limsup n^{-1}\sum_{k=1}^{n}(Y_k - m_n(t_k))^2 \leqq Q^2$ a.s.

From Chebyshev's inequality $P\{|Z_k| \geqq a\} \leqq \sigma_k^2/a^2 \leqq Q^2/a^2$. Choose $a > 2^{\frac{1}{2}}Q$ so that $P\{|Z_k| < a\} \geqq \frac{1}{2}$ for all k. For reasons which will become apparent later choose M so that $M > a$ and, using (A2), so that $(M-a)^2(\liminf N_n[\alpha, \beta]/n) > 2Q^2$.

Now let

$$A = \{\limsup n^{-1}\textstyle\sum_{k=1}^{n}(Y_k - m_n(t_k))^2 \leqq Q^2\}\,,$$
$$B = \{N_n[\alpha, \beta]^{-1}\textstyle\sum_{k=1}^{n} I_{[\alpha,\beta]}(t_k)[I_{[0,a]}(|Z_k|) - P(|Z_k| \leqq a)] \to 0\}\,,$$

and

$$C = \limsup\{|m_n(t) - m(t)| \geqq M \text{ for all } t \text{ in } [\alpha, \beta]\}\,.$$

We have proved that $P(A) = 1$. By the strong law of large numbers $P(B) = 1$. Thus $P(ABC) = P(C)$. We will show that ABC is empty so that $P(C) = 0$. Suppose not and suppose ω is a fixed point in ABC. For the remainder of this argument all random variables will be evaluated at this fixed ω but we will not explicitly exhibit ω in our expressions. Since ω is in C there exists a subsequence $\{n_j\}$ such that "$|m_{n_j}(t) - m(t)| \geqq M$ for all t in $[\alpha, \beta]$" for each j. Note that

$$\begin{aligned}n^{-1}&\textstyle\sum_{k=1}^{n}(Y_k - m_n(t_k))^2 \\ &\geqq n^{-1}\textstyle\sum_{k=1}^{n} I_{[\alpha,\beta]}(t_k)(Y_k - m_n(t_k))^2 \\ &\geqq \left(\frac{N_n[\alpha, \beta]}{n}\right) N_n[\alpha, \beta]^{-1}\textstyle\sum_{k=1}^{n} I_{[\alpha,\beta]}(t_k)I_{[0,a]}(|Z_k|)(Y_k - m_n(t_k))^2\,.\end{aligned}$$

For notational convenience we use j in place of n_j. Using this notation we see that

$$\begin{aligned}&\liminf j^{-1} \sum_{k=1}^{j} (Y_k - m_j(t_k))^2 \\ &\quad \geqq (\liminf N_j[\alpha, \beta]/j)(\liminf N_j[\alpha, \beta]^{-1} \sum_{k=1}^{j} I_{[\alpha,\beta]}(t_k) I_{[0,a]}(|Z_k|)(M - a)^2) \\ &\quad = (M - a)^2 (\liminf N_j[\alpha, \beta]/j)(\liminf N_j[\alpha, \beta]^{-1} \sum_{k=1}^{j} I_{[\alpha,\beta]}(t_k) P(|Z_k| \leqq a)) \\ &\quad \geqq (M - a)^2 (\liminf N_j[\alpha, \beta]/j)(\tfrac{1}{2}) > Q^2 .\end{aligned}$$

This contradicts $\limsup n^{-1} \sum_{k=1}^{n} (Y_k - m_n(t_k))^2 \leqq Q^2$ for ω in A. Thus ABC is empty.

LEMMA 3. *There exists a positive real number K such that*

$$P(\limsup \{\sup_{t \in T} m_n(t) \geqq K\}) = 0 ,$$

PROOF. Because m is continuous on $T = [0, 1]$ there exists a positive real number M_0 such that $\max_{t \in T} |m(t)| \leqq M_0$. Now let $I_1 = [\frac{1}{16}, \frac{1}{8}]$, $I_2 = [\frac{3}{8}, \frac{1}{2}]$, $I_3 = [\frac{1}{2}, \frac{5}{8}]$, and $I_4 = [\frac{7}{8}, \frac{15}{16}]$. From Lemma 2 there exists a positive real number M_1 such that

$$P(\limsup \{|m_n(t) - m(t)| \geqq M_1 \text{ for all } t \text{ in } I_k\}) = 0$$

for $k = 1$ *and* for $k = 2$ *and* for $k = 3$ *and* for $k = 4$. Now let $M^* = M_0 + M_1$ and let $K = 6M^*$. Because of the concavity of m_n, if $m_n(s) \geqq K$ for any $s \leqq \frac{1}{2}$ then either $\min_{t \in I_3} m_n(t) \geqq M^*$ or $\max_{t \in I_4} m_n(t) \leqq -M^*$. Thus

$$\begin{aligned}&\{\sup_{t \leqq \frac{1}{2}} m_n(t) \geqq K\} \\ &\qquad \subset \{\min_{t \in I_3} m_n(t) \geqq M^*\} \cup \{\max_{t \in I_4} m_n(t) \leqq -M^*\} \\ &\qquad \subset \{\min_{t \in I_3} |m_n(t) - m(t)| \geqq M_1\} \cup \{\min_{t \in I_4} |m_n(t) - m(t)| \geqq M_1\} .\end{aligned}$$

Similarly

$$\begin{aligned}&\{\sup_{t \geqq \frac{1}{2}} m_n(t) \geqq K\} \\ &\qquad \subset \{\min_{t \in I_1} |m_n(t) - m(t)| \geqq M_1\} \cup \{\min_{t \in I_2} |m_n(t) - m(t)| \geqq M_1\} .\end{aligned}$$

It follows that

$$\limsup \{\sup_{t \in T} m_n(t) \geqq K\} \subset \bigcup_{k=1}^{4} \limsup \{\min_{t \in I_k} |m_n(t) - m(t)| \geqq M_1\}$$

so that $P(\limsup \{\sup_{t \in T} m_n(t) \geqq K\}) = 0$.

LEMMA 4. *If* $0 < \alpha < \beta < 1$ *then there exists a positive real number K such that* $P(\limsup \{\min_{t \in [\alpha,\beta]} m_n(t) \leqq -K\}) = 0$.

PROOF. It suffices to prove the lemma for $\alpha < \frac{1}{2}$ and $\beta > \frac{1}{2}$. Because m is continuous on T there exists a positive real number M_0 such that $\max_{t \in T} |m(t)| \leqq M_0$. Let $I_1 = [\alpha/2, \alpha]$ and $I_2 = [\beta, (1 + \beta)/2]$. From Lemma 2 there exists a positive real number M_1 such that for $k = 1$ and for $k = 2$ $P(\limsup \{|m_n(t) - m(t)| \geqq M_1 \text{ for all } t \text{ in } I_k\}) = 0$. Now let $K = M_0 + M_1$. Note that if $\alpha \leqq s \leqq \beta$ then

$$\begin{aligned}\{m_n(s) \leqq -K\} &= \{\sup_{t \leqq s} m_n(t) \leqq -K\} \cup \{\sup_{t \geqq s} m_n(t) \leqq -K\} \\ &\subset \bigcup_{i=1}^{2} \{\max_{t \in I_i} m_n(t) \leqq -K\} \\ &\subset \bigcup_{i=1}^{2} \{|m_n(t) - m(t)| \geqq M_1 \text{ for all } t \text{ in } I_i\} .\end{aligned}$$

It follows that

$$\limsup \{\min_{t\in[\alpha,\beta]} m_n(t) \leqq -K\} \subset \bigcup_{k=1}^{2} \limsup \{\min_{t\in I_k} |m_n(t) - m(t)| \geqq M_1\}$$

so that $P(\limsup \{\min_{t\in[\alpha,\beta]} m_n(t) \leqq -K\}) = 0$.

LEMMA 5. *Let* $T^* = \bigcup_{i=1}^{\infty} \{t_i\}$. *If* $0 < \alpha < \beta < 1$ *then there exists a constant* K *such that*

$$P(\limsup \{\min_{t\in[0,\beta]-T^*} m_n'(t) \leqq -K\}) = 0 \qquad \textit{and}$$
$$P(\limsup \{\max_{t\in[\alpha,1]-T^*} m_n'(t) \geqq K\}) = 0.$$

PROOF. Suppose $0 < \varepsilon < \min\{\alpha, 1-\beta\}$. From Lemmas 3 and 4 there exists a constant C such that $P(\limsup \{\max_{t\in[\alpha-\varepsilon,\beta+\varepsilon]} |m_n(t)| \geqq C) = 0$. Let $K \geqq 2C/\varepsilon$. Note that

$$\begin{aligned}\{\min_{t\in[0,\beta]-T^*} m_n'(t) \leqq -K\} &\subset \{\lim_{t\downarrow\beta} [m_n(t) - m_n(\beta)]/[t-\beta] \leqq -K\}\\ &\subset \{m_n(\beta) \geqq C\} \cup \{m_n(\beta+\varepsilon) \leqq -C\}\\ &\subset \{\max_{t\in[\alpha-\varepsilon,\beta+\varepsilon]} |m_n(t)| \geqq C\}\,.\end{aligned}$$

Thus $P(\limsup \{\min_{t\in[0,\beta]-T^*} m_n'(t) \leqq -K\}) \leqq P(\limsup \{\max_{t\in[\alpha-\varepsilon,\beta+\varepsilon]} |m_n(t)| \geqq C\}) = 0$. The proof of the second half of the lemma is similar.

In the proofs which follow we occasionally use an "expression number" instead of the expression written out in symbols. This is done in an attempt to reduce the amount of notation necessary. For example, we use $(3.1)/n_\nu$ instead of

$$\frac{1}{n_\nu}\left[\sum_{k=1}^{n_\nu} (m_{n_\nu}(t_k) - Y_k)^2 - \sum_{k=1}^{n_\nu} (g_{n_\nu}(t_k) - Y_k)^2\right]$$

or even (expression (3.1))$/n_\nu$ in the proof of Lemma 6. As another example, we refer to expression (3.3) by number in the expression immediately following its definition.

LEMMA 6. *If* $0 \leqq \alpha < \beta \leqq 1$ *and* $\varepsilon > 0$ *then*

$$P(\limsup \{\min_{t\in[\alpha,\beta]} [m_n(t) - m(t)] \geqq \varepsilon\}) = 0\,.$$

PROOF. If $\min_{t\in[\alpha,\beta]} [m_n(t) - m(t)] \geqq \varepsilon$ let A_n be the largest interval containing $[\alpha, \beta]$ such that $m_n(t) \geqq m(t) + \varepsilon/2$ for all t in the interval. Otherwise let $A_n = \emptyset$. Note that A_n depends on ω. Now define

$$g_n(t) = (m(t) + \varepsilon/2) I_{A_n}(t) + m_n(t) I_{A_n^c}(t)\,.$$

Observe that g_n is a continuous concave function and hence that

$$\sum_{k=1}^{n} (m_n(t_k) - Y_k)^2 - \sum_{k=1}^{n} (g_n(t_k) - Y_k)^2 \tag{3.1}$$

is nonpositive for *every* ω in the underlying probability space by the minimizing property of m_n. We will use $\sum^{(n)}$ to denote $\sum_{\{k:1\leqq k\leqq n \text{ and } t_k\in A_n\}}$. Then, using the notation $Z_k = Y_k - m(t_k)$, we can rewrite (3.1) as

$$\sum^{(n)} [(Y_k - m_n(t_k))^2 - (Y_k - g_n(t_k))^2]$$
$$= \sum^{(n)} [2Y_k - m_n(t_k) - g_n(t_k)][g_n(t_k) - m_n(t_k)]$$
$$= 2 \sum^{(n)} Z_k[g_n(t_k) - m_n(t_k)] \tag{3.2}$$
$$+ \sum^{(n)} [m_n(t_k) - g_n(t_k)][m_n(t_k) + g_n(t_k) - 2m(t_k)] . \tag{3.3}$$

Now

$$\begin{aligned}(3.3) &\geqq \varepsilon \sum^{(n)} [m_n(t_k) - g_n(t_k)] \\ &\geqq \varepsilon[(\varepsilon/2)\#\{k : 1 \leqq k \leqq n \text{ and } t_k \text{ is in } A_n \cap [\alpha, \beta]\}] \\ &= (\varepsilon^2/2)N_n[\alpha, \beta] \quad \text{if } A_n \neq \varnothing \\ &= 0 \quad \text{if } A_n = \varnothing .\end{aligned}$$

We will use Lemma 1 with $r = 1$ on the random variables $Z_1, Z_2, \cdots$. Produce an ordering $\overset{*}{<}$ of $1, 2, \cdots$ so that $t_i < t_j$ implies $i \overset{*}{<} j$. If $t_i = t_j$ any arbitrary ordering will do. We choose to set $i \overset{*}{<} j$ if $t_i = t_j$ and $i < j$. Now let $k = i_{\nu,n}$ if $i \overset{*}{<} k$ for exactly $\nu - 1$ of the integers $i = 1, \cdots, n$. The permutations $i_{1,n}, \cdots, i_{n,n}$ are obtained in the manner specified in Lemma 1 and (3.2) can be rewritten as

$$2 \sum_{k=1}^{n} Z_{i_{k,n}}[g_n(t_{i_{k,n}}) - m_n(t_{i_{k,n}})]I_{A_n}(t_{i_{k,n}}) . \tag{3.4}$$

Recall that $t_{i_{1,n}} \leqq t_{i_{2,n}} \leqq \cdots \leqq t_{i_{n,n}}$. Define $a_{n,0} = 0$ and for $k = 1, \cdots, n$ define the random variables $a_{n,k}$ by

$$\begin{aligned}a_{n,k} &= [g_n(t_{i_{k,n}}) - m_n(t_{i_{k,n}})]I_{A_n}(t_{i_{k,n}}) \\ &= [\varepsilon/2 - (m_n(t_{i_{k,n}}) - m(t_{i_{k,n}}))]I_{A_n}(t_{i_{k,n}})\end{aligned}$$

so that (3.4), and hence (3.2), becomes

$$2 \sum_{k=1}^{n} a_{n,k} Z_{i_{k,n}} . \tag{3.5}$$

Since m is continuous $\max_{t \in [0,1]} |m(t)| = M_0$ for some real M_0. For a function f defined on $[0, 1]$ let $V(f)$ denote the variation of f over $[0, 1]$. Since m is concave $V(m) \leqq 4M_0$ and $V(mI_{A_n}) \leqq 6M_0$. Clearly $V((\varepsilon/2)I_{A_n}) \leqq \varepsilon$. We also have

$$-M_0 \leqq mI_{A_n} \leqq (m + \varepsilon/2)I_{A_n} \leqq m_nI_{A_n} \leqq [\max_{s \in T} m_n(s)]I_{A_n}$$

so $V(m_nI_{A_n}) \leqq 4M_0 + 2 \max\{0, \max_{s \in T} m_n(s)\}$. From Lemma 3 there thus exists a constant K such that $P\{V(m_nI_{A_n}) \geqq K \text{ i.o.}\} = 0$ (where i.o. = "infinitely often"). It follows that

$$P\{V([\varepsilon/2 - (m_n - m)]I_{A_n}) \geqq \varepsilon + K + 6M_0 \text{ i.o.}\} = 0$$

and that therefore

$$P\{\limsup \sum_{i=1}^{n} |a_{n,i} - a_{n,i-1}| \leqq \varepsilon + K + 6M_0\} = 1 .$$

We now apply Lmma 1 setting $B = \varepsilon + K + 6M_0$ and $X_{i_{k,n}} = Z_{i_{k,n}}$. If we set

$$A = \{n^{-1} \sum^{(n)} Z_k[g_n(t_k) - m_n(t_k)] \to 0\} \tag{3.6}$$

it follows from Lemma 1, and from the equality of (3.2) and (3.5), that $P(A) = 1$.

Let $C = \limsup \{\min_{t \in [\alpha,\beta]} [m_n(t) - m(t)] \geqq \varepsilon\}$. We will argue that AC is empty so that $P(C) = 0$. Suppose not. Fix ω in AC and for that ω let $\{n_\nu\}$ be a subsequence such that for all ν we have $\min_{t \in [\alpha,\beta]} [m_{n_\nu}(t) - m(t)] \geqq \varepsilon$ so that $A_{n_\nu} = A_{n_\nu}(\omega) \neq \emptyset$. Then

$$\begin{aligned}\frac{(3.1)}{n_\nu} &= \frac{2}{n_\nu} \sum^{(n_\nu)} Z_k[g_{n_\nu}(t_k) - m_{n_\nu}(t_k)] + \frac{(3.3)}{n_\nu} \\ &\geqq \frac{2}{n_\nu} \sum^{(n_\nu)} Z_k[g_{n_\nu}(t_k) - m_{n_\nu}(t_k)] + \varepsilon^2 N_{n_\nu}[\alpha, \beta]/2n_\nu\end{aligned}$$

so that $\liminf [(3.1)/n_\nu] \geqq 0 + [\varepsilon^2/2][\liminf (N_{n_\nu}[\alpha, \beta]/n_\nu)] > 0$. Thus for some large values of n the expression (3.1) is positive contradicting the definition of m_n which requires (3.1) to be nonpositive. It follows that AC is empty and the lemma is proved.

LEMMA 7. *If* $\varepsilon > 0$ *and* $0 < \alpha < \beta < 1$ *then*

$$P(\limsup \{\max_{t \in [\alpha,\beta]} [m_n(t) - m(t)] \geqq \varepsilon\}) = 0 .$$

PROOF. Use Lemma 5 to get K such that if $A_n = \{\max_{t \in [\alpha,\beta] - T^*} |m_n'(t)| \geqq K\}$ then $P(\limsup A_n) = 0$. m is continuous (so uniformly continuous) on $[0, 1]$. Let $\delta_1 > 0$ be such that if $|s - t| < \delta_1$ then $|m(t) - m(s)| < \varepsilon/3$. Let $\delta = \min\{\delta_1, \varepsilon/3K\}$. Let $\alpha = a_0 < a_1 < \cdots < a_\eta = \beta$ be such that $\max_{k=1,\cdots,\eta} \{a_k - a_{k-1}\} < \delta$ and let $B_k = [a_{k-1}, a_k]$. Let $\Omega^* = (\limsup A_n)^c$ and for ω in Ω^* let $N(\omega)$ be such that $n \geqq N(\omega)$ implies ω in A_n^c. Now suppose ω is in $\Omega^* \cap \limsup \{\max_{t \in [\alpha,\beta]} [m_n(t) - m(t)] \geqq \varepsilon\}$. Then there exist sequences $\{s_\nu\}$ and $\{n_\nu\}$ of real numbers and positive integers, respectively, such that

(i) s_ν is in $[\alpha, \beta]$ for all ν,
(ii) $N(\omega) \leqq n_1 < n_2 < \cdots$, and
(iii) $m_{n_\nu}(s_\nu) - m(s_\nu) \geqq \varepsilon$.

Infinitely many of the s_ν's will be in some B_k so suppose we have chosen the subsequence so that

(iv) s_ν is in B_k for all ν (k fixed).

Then if s is in B_k

$$\begin{aligned}m_{n_\nu}(s) - m(s) &= [m_{n_\nu}(s) - m_{n_\nu}(s_\nu)] + [m_{n_\nu}(s_\nu) - m(s_\nu)] + [m(s_\nu) - m(s)] \\ &\geqq -|s - s_\nu|K + \varepsilon - \varepsilon/3 \\ &\geqq -\delta K + 2\varepsilon/3 \geqq \varepsilon/3 .\end{aligned}$$

Thus $\min_{s \in B_k} [m_{n_\nu}(s) - m(s)] \geqq \varepsilon/3$. It follows that ω is in

$$\bigcup_{k=1}^{\eta} \limsup \{\min_{t \in B_k} [m_n(t) - m(t)] \geqq \varepsilon/3\} .$$

Lemma 6 now shows that

$$\begin{aligned}&P(\limsup \{\max_{t \in [\alpha,\beta]} [m_n(t) - m(t)] \geqq \varepsilon\}) \\ &\quad = P(\Omega^* \cap \limsup \{\max_{t \in [\alpha,\beta]} [m_n(t) - m(t)] \geqq \varepsilon\}) \\ &\quad \leqq \sum_{k=1}^{\eta} P(\limsup \{\min_{t \in B_k} [m_n(t) - m(t)] \geqq \varepsilon/3\}) = 0 .\end{aligned}$$

LEMMA 8. *If $\varepsilon > 0$ then there exist $\alpha, \beta \in (0, 1)$ such that*

(3.7) $$P(\limsup \{\max_{t \in [0,\alpha]} [m_n(t) - m(t)] \geqq \varepsilon\}) = 0 \quad \textit{and}$$

(3.8) $$P(\limsup \{\max_{t \in [\beta,1]} [m_n(t) - m(t)] \geqq \varepsilon\}) = 0 .$$

PROOF. Let $\delta_1 \in (0, 1)$ be such that $|s - t| \leqq \delta_1$ implies $|m(t) - m(s)| \leqq \varepsilon/3$. From Lemma 5 we get $K > 0$ such that $P(\limsup\{\min_{t \in [0,\delta_1]-T^*} m_n'(t) \leqq -K\}) = 0$. Let $\delta = \min\{\delta_1, \varepsilon/3K\}$ and let $\alpha = \delta/2$. Let $\Omega^* = (\limsup\{\min_{t \in [0,\delta_1]-T^*} m_n'(t) \leqq -K\})^c$ and for ω in Ω^* let $N(\omega)$ be such that if $n \geqq N(\omega)$ then $m_n'(t) \geqq -K$ for all t in $[0, \delta_1] - T^*$. Suppose $\omega \in \Omega^*$, $n \geqq N(\omega)$, $t \in [0, \alpha]$, and $m_n(t) - m(t) \geqq \varepsilon$. Then if $s \in [\alpha, 2\alpha]$

$$\begin{aligned} m_n(s) - m(s) &= [m_n(s) - m_n(t)] + [m_n(t) - m(t)] + [m(t) - m(s)] \\ &\geqq -K|t - s| + \varepsilon - \varepsilon/3 \geqq \varepsilon/3 . \end{aligned}$$

Thus $\Omega^* \cap \limsup\{\max_{t \in [0,\alpha]} [m_n(t) - m(t)] \geqq \varepsilon\} \subset \limsup\{\min_{s \in [\alpha,2\alpha]} [m_n(s) - m(s)] \geqq \varepsilon/3\}$. An application of Lemma 6 proves (3.7). The proof of (3.8) is similar.

LEMMA 9. *If $\varepsilon > 0$ then* $P(\limsup\{\max_{t \in [0,1]} [m_n(t) - m(t)] \geqq \varepsilon\}) = 0$.

PROOF. This is an immediate consequence of Lemmas 7 and 8.

LEMMA 10. *If $0 \leqq \alpha < \beta \leqq 1$ and $\varepsilon > 0$ then*

$$P(\limsup\{\min_{t \in [\alpha,\beta]} [m(t) - m_n(t)] \geqq \varepsilon\}) = 0 .$$

PROOF. Define $g_n(t) = \min\{m_n(t) + \varepsilon, m(t)\}$. Let

$$\Omega^* = (\limsup\{\max_{t \in [0,1]} [m_n(t) - m(t)] \geqq \varepsilon\})^c$$

and recall that $P(\Omega^*) = 1$ from Lemma 9. For $\omega \in \Omega^*$ let $N(\omega)$ be such that if $n \geqq N(\omega)$ then $m_n(t) \leqq m(t) + \varepsilon$ for all $t \in [0, 1]$. From the minimizing property of m_n we see that

(3.9) $$\sum_{k=1}^n (Y_k - m_n(t_k))^2 - \sum_{k=1}^n (Y_k - g_n(t_k))^2$$

is nonpositive for all n and all ω (in particular for all $\omega \in \Omega^*$ and all $n \geqq N(\omega)$). Using the notation $Z_k = Y_k - m(t_k)$ we can rewrite (3.9) as

(3.10) $$2 \sum_{k=1}^n Z_k[g_n(t_k) - m_n(t_k)]$$

(3.11) $$+ \sum_{k=1}^n [2m(t_k) - m_n(t_k) - g_n(t_k)][g_n(t_k) - m_n(t_k)] .$$

Now for $\omega \in \Omega^*$ and $n \geqq N(\omega)$ we have

$$\begin{aligned} (3.11) &\geqq \sum_{g_n = m_n + \varepsilon} [2m(t_k) - 2m_n(t_k) - \varepsilon]\varepsilon + \sum_{g_n \neq m_n + \varepsilon} [m(t_k) - m_n(t_k)]^2 \\ &\geqq \sum_{g_n = m_n + \varepsilon} \varepsilon^2 . \end{aligned}$$

The last inequality comes from the fact that if $g_n = m_n + \varepsilon$ then $m_n + \varepsilon \leqq m$. Define the random variables $Z_{i_{k,n}}$ as in Lemma 6. Let $A_n = \{\omega : \omega \in \Omega^* \text{ and } n \geqq N(\omega)\}$. Define $a_{n,0} = 0$ and for $k = 1, \cdots, n$ define the random variables $a_{n,k}$ by $a_{n,k} = [g_n(t_{i_{k,n}}) - m_n(t_{i_{k,n}})]I_{A_n}$. If ω is not in A_n then $a_{n,k} = 0$ so that

$\sum_{k=1}^{n} |a_{n,k} - a_{n,k-1}| = 0$. If $\omega \in A_n$ and $g_n(t) = m_n(t) + \varepsilon$ for all t then $a_{n,k} = \varepsilon$ for $k \neq 0$ so $\sum_{k=1}^{n} |a_{n,k} - a_{n,k-1}| = \varepsilon$. If $\omega \in A_n$ and there is at least one t such that $g_n(t) \neq m_n(t) + \varepsilon$, then let $\alpha^* = \min\{t : g_n(t) = m(t)\}$ and let $\beta^* = \max\{t : g_n(t) = m(t)\}$. Note that $g_n(t) - m_n(t) = \varepsilon$ for $t \in [0, \alpha^*) \cup (\beta^*, 1]$, that $g_n(\alpha^*) = m(\alpha^*)$, and that $g_n(\beta^*) = m(\beta^*)$. Let $V_I(f)$ denote the variation of the function f over the interval I. Then

$$\begin{aligned} V[(g_n - m_n)I_{A_n}] &= V_{[0,1]}[(g_n - m_n)I_{A_n}] \\ &= V_{[0,\alpha^*]}(\cdots) + V_{[\alpha^*,\beta^*]}(\cdots) + V_{[\beta^*,1]}(\cdots) \\ &\leqq V_{[\alpha^*,\beta^*]}(g_n I_{A_n}) + V_{[\alpha^*,\beta^*]}(m_n I_{A_n}) . \end{aligned}$$

Let M be a bound on $|m|$. Then since g_n is concave, since $g_n(\alpha^*) = m(\alpha^*)$ and $g_n(\beta^*) = m(\beta^*)$, and since $g_n(t) \leqq m(t)$ for all t, we see that $|g_n(t)| \leqq M$ for $t \in [\alpha^*, \beta^*]$ and further that $V_{[\alpha^*,\beta^*]}(g_n I_{A_n}) \leqq 4M$. Similarly, since m_n is concave, since $m_n(\alpha^*) \geqq m(\alpha^*) - \varepsilon$ and $m_n(\beta^*) \geqq m(\beta^*) - \varepsilon$, and since for $\omega \in A_n$ we have $m_n(t) \leqq m(t) + \varepsilon$ for all t, we see that $|m_n(t)| \leqq M + \varepsilon$ for $t \in [\alpha^*, \beta^*]$ and further that $V_{[\alpha^*,\beta^*]}(m_n I_{A_n}) \leqq 4(M + \varepsilon)$. Thus in this case $V[(g_n - m_n)I_{A_n}] \leqq 8M + 4\varepsilon$ so that $\sum_{k=1}^{n} |a_{n,k} - a_{n,k-1}| \leqq 8M + 4\varepsilon + |g_n(0) - m_n(0)| \leqq 8M + 5\varepsilon$. An application of Lemma 1 now shows that if $\Omega_0 = \{n^{-1} \sum_{k=1}^{n} Z_{i_{k,n}} a_{n,k} \to 0\}$ then $P(\Omega_0) = 1$. We will argue that $\Omega_0 \cap \Omega^* \cap \limsup \{\min_{t \in [\alpha,\beta]} [m(t) - m_n(t)] \geqq \varepsilon\} = \phi$ so that $P(\limsup \{\min_{t \in [\alpha,\beta]} [m(t) - m_n(t)] \geqq \varepsilon\}) = 0$. Suppose not, that ω is in the set, and that $\{n_\nu\}$ is an increasing sequence of positive integers such that $N(\omega) \leqq n_1$ and such that for every ν we have $\min_{t \in [\alpha,\beta]} [m(t) - m_{n_\nu}(t)] \geqq \varepsilon$. Then since $\omega \in \Omega_0$

$$(3.10)/n_\nu = (2/n_\nu) \textstyle\sum_{k=1}^{n_\nu} Z_{i_{k,n_\nu}} a_{n_\nu,k} \to 0 .$$

In addition, since $\omega \in \Omega^*$ and $n_\nu \geqq N(\omega)$ for all ν

$$\liminf \frac{(3.11)}{n_\nu} \geqq \varepsilon^2 \liminf (N_{n_\nu}[\alpha, \beta]/n_\nu) > 0 .$$

It follows that for large enough values of ν

$$(3.9) = n_\nu[(3.10)/n_\nu + (3.11)/n_\nu] > 0 ,$$

giving the desired contradiction.

LEMMA 11. *If* $0 < \alpha < \beta < 1$ *and* $\varepsilon > 0$ *then*

$$P(\limsup \{\max_{t \in [\alpha,\beta]} [m(t) - m_n(t)] \geqq \varepsilon\}) = 0 .$$

PROOF. Follows from Lemma 10 and is similar to the proof of Lemma 7.

PROOF OF THE MAIN THEOREM. An immediate consequence of Lemmas 9 and 11.

4. Derivatives. We will use $f'(t+)$ to denote the right derivative of f at t and $f'(t-)$ to denote the left derivative of f at t.

COROLLARY 1. *Suppose* $T = [0, 1]$, $0 \leqq t < 1$, *and* $0 < s \leqq 1$. *Then*

$$P\{\liminf m_n'(t+) \geqq m'(t+)\} = 1 \quad \textit{and} \tag{4.1}$$

$$P\{\limsup m_n'(s-) \leqq m'(s-)\} = 1 . \tag{4.2}$$

PROOF. Suppose $\varepsilon = (1 - t)/2$ and define $B = \{\liminf m_n'(t+) \geqq m'(t+)\}^c$. Let $\{a_\nu\}$ be a decreasing sequence of positive real numbers such that $a_1 < \varepsilon$ and $a_\nu \to 0$. For each ν define

$$A_\nu = \{\limsup \max_{x \in T} [m_n(x) - m(x)] \leqq 0; \\ \liminf \min_{t+a_\nu \leqq x \leqq t+\varepsilon} [m_n(x) - m(x)] \geqq 0\} .$$

From our main theorem $P(A_\nu) = 1$ for each ν. We will argue that $B \cap [\bigcap_{\nu=1}^\infty A_\nu] = \emptyset$ so that $P(B) = 0$ and (4.1) is true. A similar proof would give (4.2).

Suppose ω is in $B \cap [\bigcap_{\nu=1}^\infty A_\nu]$. Because ω is in B there exist a real number $C < m'(t+)$ and an increasing sequence $\{n_k\}$ of positive integers such that $m'_{n_k}(t+) \leqq C$ for all k. Because $m'(t+)$ exists there are real numbers δ and D such that $0 < \delta < \varepsilon$, $C < D < m'(t+)$, and if $t \leqq x \leqq t + \delta$ then $m(x) \geqq m(t) + (x - t)D$. We set $x = t + \delta$ so that

$$m(t + \delta) \geqq m(t) + \delta D . \tag{4.3}$$

Since m_{n_k} is concave we have

$$m_{n_k}(t + \delta) \leqq m(t) + \delta m'_{n_k}(t+) \leqq m(t) + \delta C . \tag{4.4}$$

Subtracting (4.4) from (4.3) gives for all k

$$m(t + \delta) - m_{n_k}(t + \delta) \geqq \delta(D - C) > 0 . \tag{4.5}$$

Now choose ν large enough that $a_\nu < \delta$. Then, since $\omega \in A_\nu$ for all ν, we have $m_n(t + \delta) \to m(t + \delta)$. This combined with (4.5) gives the desired contradiction.

COROLLARY 2. *Suppose* $T = [0, 1]$ *and let* $T^* = \{t_i : i = 1, 2, \cdots\} \cup \{t : m'(t)$ *does not exist*$\}$. *If* $t \in (0, 1) - T^*$ *then* $P\{m_n'(t) \to m'(t)\} = 1$.

PROOF. An immediate consequence of Corollary 1.

COROLLARY 3. *Suppose* $T = [0, 1]$, $0 < \alpha < \beta < 1$, *and that* $m'(t)$ *exists on* $(0, 1)$. *Then*

$$P\{\sup_{\alpha \leqq t \leqq \beta} |m_n'(t+) - m'(t)| \to 0\} = 1$$

and

$$P\{\sup_{\alpha \leqq t \leqq \beta} |m_n'(t-) - m'(t)| \to 0\} = 1 .$$

PROOF. Is standard and follows from Corollary 1 or Corollary 2 and the concavity of m_n and m. Note that if $m'(t)$ exists for t in $(0, 1)$ then, because m is concave, m' is automatically continuous on $(0, 1)$.

5. Concluding remarks. One might wonder whether the assumption $\liminf [N_n(I)/n] > 0$ is necessary. It is easy to see that one cannot simply eliminate the assumption, that some restriction, in addition to its being dense in T, must be placed on the sequence $\{t_i\}$ of observation points. Suppose, for example, that observations have been taken at $t_1, \cdots, t_n$; that $t_j - t_i > 0$; and that there are no observation points between t_i and t_j. Suppose further that there is a $\delta > 0$ such that for all t in T we have $F_t[m(t) - \delta] > \delta$. Now suppose $t_i < t_{n+1} < t_{n+2} < \cdots < t_j$. Let N be any positive integer. With probability one (eventually) we will have $Y_k - m(t_k) \leqq -\delta$ for some block of N consecutive integers $k = \nu + 1, \cdots, \nu + N$. Because of the huge number of observations taken to the left of this block and the stabilizing effect of these observations on the estimates of m to the left, it is likely that the fact that $Y_{\nu+1}, \cdots, Y_{\nu+N}$ are all too small will have little effect on $m_{\nu+N}(t)$ for $t < t_{\nu+1}$. On the other hand, at most $n - 1$ of the observation points $t_1, \cdots, t_{\nu+N}$ are to the right of $t_{\nu+N}$ so if N is large enough relative to n, and if $t_{n+1}, t_{n+2}, \cdots$ are close enough to t_j, then we ought to be able to pull the value of $m_{\nu+N}(t_j)$ down at least to $m(t_j) - \delta/2$ with high probability. A construction like that used in Theorem 2 of [2] can be used and should give an example in which for some $\varepsilon > 0$

$$P\{\sup_{0 \leq t \leq 1} \liminf [m_n(t) - m(t)] \leqq -\varepsilon\} = 1 .$$

The sequence $\{t_i\}$ used in the construction would be made dense in [0, 1] but (of course) would not satisfy (1.2) of Assumption 2. Note that we have not suggested the possibility of an example in which

$$P\{\inf_{0 \leq t \leq 1} \limsup [m_n(t) - m(t)] \geqq \varepsilon\} = 1 .$$

One can bend one side of a concave curve (say for $t > t_0$) downward at an arbitrarily steep angle without changing the curve for $t \leqq t_0$ and without destroying concavity. One cannot perform the corresponding "upward" bending.

It is easy to understand why we are able to bound $m_n(t)$ from above uniformly "at the ends of the interval T," but are unable to bound $m_n(t)$ from below uniformly "at the ends of the interval T." Suppose $t_i < t_j$ for $j \in \{1, \cdots, n\} - \{i\}$. Then $m_n(t_i) \leqq Y_i$ since if $m_n(t_i) > Y_i$ we could reduce the sum of squares $\sum_{k=1}^{n} (Y_k - m_n(t_k))^2$ without violating concavity by letting $m_n(t_i) = Y_i$. Under any sort of reasonable assumptions on the distributions of the error terms $Y_k - m(t_k)$ there will be a $\delta > 0$ such that $P(Y_k \leqq m(t_k) - \delta) \geqq \delta$ for all k. Thus we always have $P\{m_n(t_i) \leqq m(t_i) - \delta\} > \delta$ for that δ.

It is clear that our main theorem holds for any compact interval $[a, b]$, that T need not be [0, 1]. In fact, if T is a finite interval it need not be closed. The fact that T is closed was used to give boundedness (and hence uniform continuity) of m on T. If T is finite, but not necessarily closed, it suffices simply to add the additional assumption that m is bounded. If $t = (0, 1)$ and if $\lim_{t \downarrow 0} m(t) = -\infty$ ($\lim_{t \downarrow 0} m(t) = +\infty$ is impossible for concave m), then, since $m_n(t)$ is always bounded on finite intervals, $\sup_{t \in T} [m_n(t) - m(t)] = \infty$ for every n. If $\lim_{t \uparrow 1} m(t) = -\infty$ as well, then (1.6) would have to be weakened to $P\{\lim \max_{\alpha \leq t \leq \beta} |m_n(t) - m(t)| = 0\} = 1$. If T is the real line then one can get uniform strong consistency over finite subintervals but one cannot get any sort of uniform consistency on an unbounded end of the interval because a small error in the slope of $m_n(t)$ at the rightmost or leftmost observation point can lead to arbitrarily large errors $|m_n(t) - m(t)|$ as $t \to \infty$ or $t \to -\infty$.

It is clear that similar results can be obtained if any or all of $m_n(s)$,

$$\lim_{t \downarrow s} \frac{m_n(t) - m_n(s)}{t - s}, \qquad \lim_{t \uparrow s} \frac{m_n(t) - m_n(s)}{t - s}$$

are specified or bounded (correctly) at a finite number of points. One might, in particular, want to specify $m_n(s)$ at the endpoints of T.

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