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CONVERGENCE OF EMPIRICAL PROCESSES OF MIXING rv's ON [0, 1]

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Conditions are given for the weak convergence of weighted empirical cumulative processes of three types of mixing random variables (rv's) on [0, 1].

1. Introduction. In this section we define three types of mixing conditions and derive a basic lemma concerning them. Section 2 gives a central limit theorem for sums of uniformly bounded strongly-mixing rv's with some examples. Section 3 applies Section 1 and Section 2 to obtain the main results—the convergence of empirical processes to Gaussian processes. This work generalises a theorem of Koul [4] for independent rv's.

Let $\{X_{iN}, i = 1, 2, \cdots, n_N\}$ be a sequence of rv's defined on some probability space, $N = 1, 2 \cdots$. For $1 \leqq a \leqq b \leqq n_N$ let $M_{a,b}^N$ be the σ-algebra generated by $X_{aN}, \cdots, X_{bN}$. Let ψ_N, ϕ_N, α_N be functions on $\{0, 1, \cdots, n_N - 1\}$ such that $\psi_N(0) = \infty$, $\phi_N(0) = \alpha_N(0) = 1$. Suppose that for $1 \leqq k \leqq k + i \leqq n_N$, $A \in M_{1,k}^N$, $B \in M_{k+i,n_N}^N$

$$|P(AB) - P(A)P(B)| \leqq \psi_N(i)P(A)P(B) \tag{1}$$

then we call $\{X_{iN}\}$ ψ_N-*mixing*. If we replace (1) by

$$|P(AB) - P(A)P(B)| \leqq \phi_N(i)P(A) \tag{2}$$

or

$$|P(AB) - P(A)P(B)| \leqq \alpha_N(i) \tag{3}$$

we call $\{X_{iN}\}$ ϕ_N-*mixing* or α_N-*mixing* respectively.

This extension of the usual notions (e.g. see Phillip [5], [6] for $\psi_N(i) = \psi(i\gamma_N)$, $\phi_N(i) = \phi(i\gamma_N)$, $\alpha_N(i) = \alpha(i)$) allows us to obtain C.L.T. results even when $\psi_N(1) \to \infty$ or $\sum_i \alpha_N(i) \to \infty$ as $N \to \infty$.

LEMMA 1. *Suppose* $1 \leqq k \leqq k + i \leqq n_N$. *Let* X, Y *be real* rv's *measurable* $M_{1,k}^N$ *and* M_{k+i,n_N}^N, *respectively. Then each of the following is an upper bound* (*when finite*) *for* $|EXY - EXEY|$, *for* $\{X_{iN}\}$ ψ_N-*mixing*, (ϕ_N-*mixing*, α_N-*mixing respectively*).

(a) $\psi_N(i)E|X|E|Y|$
(b) $2\phi_N^{1/p}(i)E^{1/p}|X|^pE^{1/q}|Y|^q$, *for* $p^{-1} + q^{-1} = 1$, $1 < p < \infty$
(c) $2\phi_N(i)C_2E|X|$, *for* $|Y| \leqq C_2$
(d) $4\alpha_N(i)C_1C_2$, *for* $|X| \leqq C_1$, $|Y| \leqq C_2$

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(e) $10\alpha_N(i)^{1-1/p}C_2 2^{-1/p}E^{1/p}|X|^p$, for $|Y| \leqq C_2$, $1 \leqq p < \infty$
(f) $k\alpha_N(i)^{1-1/p-1/q}$, *for* $1 \leqq p < \infty$, $1 \leqq q < \infty$,

where $k = K(E|X|^p, E|Y|^q)$ *and* $K(x, y) = 8((x + y + x^{\frac{1}{2}}y^{\frac{1}{2}})/2)^{1/p+1/q}$.

Proof. For (a) to (d) see Lemmas 1, 2 of Phillip [5] and page 171 of Billingsley [1]. (f) is proved by an easy generalisation of Lemma 1.3 of Ibragimov [3], who gives $K(x, y) = 4 + 2(x + y + x^{\frac{1}{2}}y^{\frac{1}{2}})$ for $p = q$. An alternative value, $K(x, y) = 10x^{1/p}y^{1/q}$, is given by Lemma 1 of Deo [2], and thus implies (e). (These authors all consider the stationary case).

2. A C.L.T. for uniformly bounded rv's.

Theorem 1. *Let* $\{X_{iN}\}$ *be real* rv's *satisfying* (3). *Suppose* $EX_{iN} = 0$, $|X_{iN}| \leqq C < \infty$, $1 \leqq i \leqq n_N$, *and* $t_N^2 = E(\sum_{j=1}^n X_{jN})^2$, $N \geqq 1$, *where* $n = n_N \to \infty$ *as* $N \to \infty$. *Let* $k = k_N$, $p = p_N$, $q = q_N$ *be sequences of positive integers such that* $k(p + q) \leqq n$, *and*

$$t_N^{-2}q^2k \textstyle\sum_1^{k-1} \alpha_N(jp) \to 0 \tag{4}$$

$$t_N^{-2}kq \textstyle\sum_0^{q-1} \alpha_N(j) \to 0 \tag{5}$$

$$t_N^{-2}q(p + q) \textstyle\sum_1^{k-1} \alpha_N(jp) \to 0 \tag{6}$$

$$t_N^{-2}(n - k(p + q)) \textstyle\sum_1^{n-k(p+q)} \alpha_N(j) \to 0 \tag{7}$$

$k\alpha_N(q) \to 0$, $t_N^{-4}k\{p \sum_0^{p-1} j^2\alpha_N(j) + p^2(\sum_0^{p-1} \alpha_N(j))^2\} \to 0$, *as* $N \to \infty$. *Then*

$$t_N^{-1} \textstyle\sum_{j=1}^{n_N} X_{jN} \to_{\mathscr{L}} N(0, 1)\,. \tag{8}$$

Proof. This follows from the proof of Theorem 1.6 of Ibragimov, [2]: define S_N'' analogously to S_N'' on line 4, page 359. Equations (4) and (5) deal with the cases $i \neq j$, $i = j$ respectively for the first term on the R.H.S. of line 4. Equations (6) and (7) ensure that the second and third terms on the R.H.S. of line 4 tend to zero. Hence $ES_N''^2 \to 0$.

Finally, in place of $E|\sum_1^p x_i|^4 \leqq MC^4p^3/1np$ we have used

$$E|\textstyle\sum_{1+j}^{p+j} X_{iN}|^4 \leqq 4!\, 4C^4\{3p \sum_0^{p-1} (i + 1)^2\alpha_N(i) + 4p^2(\sum_0^{p-1} \alpha_N(i))^2\}$$

(cf. Lemma 4, page 172 of [1].)

Corollary 1. *Let* $\{X_{iN}\}$ *be real uniformly bounded* rv's *with mean zero satisfying* (3). *Suppose* $E(\sum_1^n X_{jN})^2/n \to T < \infty$ *and*

$$n = n_N \to \infty \quad \text{as} \quad N \to \infty\,. \tag{9}$$

Then any of the following sets of conditions are sufficient to ensure that $n^{-\frac{1}{2}} \sum_{j=1}^n X_{jN} \to_{\mathscr{L}} N(0, T)$.

(a) $\max_{1\leq i\leq n} \alpha_N(i) = o(n^{-\frac{1}{2}})$
(b) $\max_{1\leq i<n} i \log i\alpha_N(i) = o(e^{\frac{3}{2}}/\log^{\frac{1}{2}} e)$, *where* $e = \log n_N$.
(c) $\sum_1^{n-1} \alpha_N(i) \leqq k < \infty$ *and* $n_N^{1-a}\alpha_N([n_N^b]) \to 0$, *where* $0 < 2b < a < 1 - b$, $n = n_N$.

(d) $\max_{1 \leq i < n} i^2 \alpha_N(i) = o(n^{\frac{1}{3}})$.
(e) $\sum_1^{n-1} i^2 \alpha_N(i) = o(n_N^{\frac{1}{2}})$, *and* $\max_1^{n-1} \alpha_N(i) \leq K < \infty$.
(f) $\sum_1^j i^2 \alpha_N(i) \leq K j^r$, $1 \leq j < n_N$, *where* $r < \frac{3}{2}$.
(g) $\sum_1^{n-1} i^2 \alpha_N(i) \leq K n^r$ *and* $\sum_1^{n-1} \alpha_N(i) < K n^d$,

where either $0 \leq d \leq \frac{1}{12}$, $r < \frac{4}{3} - 2d$ *or* $\frac{1}{12} \leq d \leq \frac{3}{10}$, $r < \frac{3}{2} - 4d$.

PROOF. If $T = 0$ conditions (a)—(g) are redundant. If $T > 0$ this is just a matter of checking the conditions of Theorem 1 with $k = [n/(p+q)]$, and $p = [n^a]$, $q = [n^b]$ where

(a) $a = \frac{1}{2}$, $b = \frac{1}{4}$
(d) $a = \frac{5}{9}$, $b = \frac{1}{3}$
(e) $a = \frac{1}{2}$, $b = \frac{1}{4}$
(f) $a = \frac{2}{3}$, $6b = (4+\varepsilon)/(1+\varepsilon)$ where $\varepsilon = 3/r - 2$
(g) $a = 1 - \varepsilon - 2d$, $b = \min(\frac{2}{3}, \frac{3}{4} - d)$ where $\varepsilon > 0$ is small.

For (b), use $p = [n^{\frac{1}{2}} \cdot e^{\frac{1}{2}} \cdot \log^{\frac{1}{2}} e]$, $q = [n^{\frac{1}{2}} \cdot e^{-\frac{1}{2}} \cdot \log^{-\frac{1}{2}} e]$, and in (e)—(g) apply inequalities such as $\sum_1^{n-1} \alpha_N(i) \leq K$, K an integer $\Rightarrow \sum_1^p i^2 \alpha_N(i) \leq K \sum_{p-K+1}^p i^2 = O(p)$.

For (f) one uses $\sum_1^k \alpha_N(jp) \leq \sup \sum_1^k \beta_j = \sum_1^k K j^{-2} p^{r-2}(j^r - (j-1)^r)$, where the sup is taken over $\{\sum_1^j i^2 \beta_i \leq K j^r p^{r-2}, \beta_j \geq 0, j = 1, \cdots, k\}$.

NOTE. Of course there is no loss in assuming $K = 1$ in (e). By (b), M in Theorem 1.6 of [2] can be improved to $o(e^2/\ln e)^{\frac{1}{2}}$ where $e = \ln n$.

As an example we show that Theorem 3.2 of Serfling [8] holds with the assumption of strict stationarity removed provided that A^2 given by (3.8) of [8] is well defined. This follows from

COROLLARY 2. *Suppose for* $j = 1, 2$, $\{X_{iN}^{(j)}, i = 1, \cdots, n^{(j)}\}$ *are* α_N*-mixing sequences of real* rv's *with* $\alpha_N = \alpha_N^{(j)}$, $n^{(j)} = n_N^{(j)}$ *and* $n^{(1)}/n^{(2)} \to C < \infty$ *as* $N \to \infty$. *Suppose that the two sequences are independent, that for each i,* $X_{iN}^{(j)}$, *has continuous* cdf $F_N^{(j)}$, *that*

$$\sum_1^{n(j)-1} \alpha_N^{(j)} \leq K < \infty ,$$

and $\alpha_N^{(j)}$ *satisfies one of the sets of conditions* (a)—(e) *of Corollary* 1, $j = 1, 2$. *Let* U_N *be the two-sample Wilcoxon statistic*

$$U_N = (n^{(1)} n^{(2)})^{-1} \sum_{i=1}^{n^{(1)}} \sum_{j=1}^{n^{(2)}} s(X_{jN}^{(2)} - X_{iN}^{(1)})$$

where $s(u) = -1, 0, 1$ *according as* $u < 0, = 0, > 0$. *Let* $\gamma_N = 2 \int F_N^{(1)} dF_N^{(2)} - 1$. *Then*

$$n^{(1)}(U_N - \gamma_N) \to_{\mathscr{L}} N(0, 4V)$$

where

$$V = \lim_{N\to\infty} n^{(1)-1} \operatorname{Var} \sum_1^{n^{(1)}} x_i + C \lim_{N\to\infty} n^{(2)-1} \operatorname{Var} \sum_1^{n^{(2)}} y_i ,$$
$$x_i = F_N^{(2)}(X_{iN}^{(1)}) ,$$
$$y_i = F_N^{(1)}(X_{iN}^{(2)})$$

provided V exists. Further, if $\{x_i\}$, $\{y_i\}$ do not depend on N and for all i, j

$$\operatorname{Cov}(x_i, x_{i+j}) = \operatorname{Cov}(x_1, x_{i+j})$$
$$\operatorname{Cov}(y_i, y_{i+j}) = \operatorname{Cov}(y_1, y_{1+j})$$

then V exists and equals

$$\operatorname{Var} x_1 + 2 \textstyle\sum_1^\infty \operatorname{Cov}(x_1, x_{1+i}) + C(\operatorname{Var} y_1 + 2 \sum_1^\infty \operatorname{Cov}(y_1, y_{1+i}))\,.$$

PROOF. This is immediate on examining that of Theorem 3.2; the requirement $C \neq 0$ is unnecessary.

3. Empirical processes. Theorems 2, 3, 4, 5, respectively, concern independent, α_N-dependent, ϕ_N-dependent, and ψ_N-dependent samples. For definition of C, $(D, \mathscr{D})$ see [1].

Let $(C_{iN}, \cdots, C_{n_N N})$ be constants and let

$$\sigma_N^2 = n_N^{-1} \textstyle\sum_{i=1}^{n_N} C_{iN}^2\,.$$

Suppose $\{X_{iN}\}$ have cdfs $\{F_{iN}\}$ on $[0, 1]$, and $n_N \to \infty$ as $N \to \infty$. Let

$$g_{Nq}(t) = n_N^{-1} \sum_{i=1}^{n_N} \left|\frac{C_{iN}}{\sigma_N}\right|^q F_{iN}(t)\,, \qquad 0 \leqq t \leqq 1\,,$$

and $L_N(t) = \sigma_N^{-1} n_N^{-\frac{1}{2}} \sum_{i=1}^{n_N} C_{iN}(I(X_{iN} \leqq t) - F_{iN}(t))$. Let r, $\{t_i\}$ be numbers depending on $\delta > 0$ such that

$$0 = t_0 < t_1 \cdots < t_r = 1\,, \qquad (10)$$
$$t_i - t_{i-1} \geqq \delta,\ i = 2, \cdots, r-1\,.$$

THEOREM 2. (*Koul, Theorem* 2.2 *of* [2]). *Suppose for $N \geqq 1$ $\{X_{iN}\}$ are independent for $N \geqq 1$,*

$$\max_{i=1}^{n_N} C_{iN}^2/(n_N \sigma_N^2) \to 0\,, \qquad (11)$$

$$EL_N(s)L_N(t) \to K(s, t)\,, \qquad 0 \leqq s, t \leqq 1, \ \text{as } N \to \infty\,, \qquad (12)$$

and

$$\limsup_{N\to\infty} \sup_{0 \leqq t \leqq 1-\delta} (g_{N2}(t + \delta) - g_{N2}(t)) \to 0 \qquad \text{as } \delta \to 0\,. \qquad (13)$$

Then

$$L_N \to_{\mathscr{D}} L \quad \text{in} \quad (D, \mathscr{D}) \qquad (14)$$

where L is a zero-mean Gaussian process such that $P(L \in C[0, 1]) = 1$ and $EL(s)L(t) = K(s, t)$.

NOTE 1. Koul gives (incorrectly) $t_i - t_{i-1} \leqq \delta$ in (10).

Koul required F_{iN} continuous (which is not necessary from his proof), $n_N = N$ and replaced (13) by the stronger condition

$$\limsup_{N\to\infty} \max_{1 \leqq i \leqq N_n} \max_{1 \leqq j \leqq r} (F_{iN}(t_j) - F_{iN}(t_{j-1})) \to 0 \qquad \text{as } \delta \to 0\,,$$

which is equivalent to

(15) $$\limsup_{N\to\infty} \max_{1\le i\le n} \sup_t (F_{iN}(t+\delta) - F_{iN}(t)) \to 0 \quad \text{as } \delta \to 0 .$$

(For example if $C_{iN} \equiv 1$, and

$$F_{iN}(t) = 1 \qquad t = 1$$
$$= (1 - k_{iN})t , \quad 0 \leqq t < 1 ,$$

where $n_N^{-1} \sum_{i=1}^{n_N} k_{iN} \to 0$, $0 \leqq k_{iN} \leqq 1$, but $\max_{1\leqq i\leqq N_n} k_{iN} \not\to 0$ then (13) holds but not this stronger condition.)

NOTE 2. If (16) holds (13) is equivalent to (13) with g_{Nq} replacing g_{N2} for any $q > 0$. The latter holds if $g_{Nq}(t) \to g(t)$, $0 \leqq t \leqq 1$, where g is continuous.

THEOREM 3. *Suppose for* $N \geqq 1$ $\{X_{iN}\}$ *are* α_N*-mixing*, (12), (13),

(16) $$\max_{1\leqq i\leqq n} |C_{iN}| \leqq k_0\sigma_N \qquad \textit{where } k_0 < \infty ,$$

for some b *in* $(\frac{1}{2}, 1)$ *and some* $d < (1 + b)/2$

(17) $$\sum_0^{n-1} (j+1)^2\alpha_N(j)^{1-b} \leqq k_3 n^d , \qquad N \geqq 1 \ \textit{where } k_3 < \infty$$

(18) $$\sum_0^{n-1} \alpha_N(j)^{1-b} < k_2 , \qquad N \geqq 1 \ \textit{where } k_2 < \infty ,$$

and

(19) $$g_{N1} \ \textit{is a continuous, strictly increasing function}, \qquad N \geqq 1 .$$

Then (14) *holds.*

NOTE 3. (19) can be removed if (13) is strengthened to $\sup_t |H_N(t) - g(t)| \to 0$ as $N \to \infty$ where $H_N = g_{N1}$ or $n^{-1} \sum_1^n F_{iN}$, and $g \in C$.

Since our draft report [10], Deo [2] has published a special case of Theorem 3, viz $\alpha_N = \alpha$, $C_{iN} = 1$, $F_{iN} = F$, $d = 0$.

PROOF. For convenience we suppress N. Using $d = 0$ in Corollary 1(g), we conclude that the finite-dimensional distributions of L_N converge to those of L. (One can also prove this under stronger conditions by adapting Theorem 19.4 of [1], based on Rosen [7]). By Theorems 15.1, 15.5 of [1] it suffices to prove (19.51) of [1] for L_N. By Note 2 with $q = 1$ this is so if (19.51) holds for $\bar{L}_N = L_N(g_1^{-1})$. For $0 \leqq s < t \leqq 1$ set

$$\Delta_i = F_i(g_1^{-1}(t)) - F_i(g_1^{-1}(s))$$
$$Z_i = I(s < g_1(X_i) \leqq t) - \Delta_i,\ 1 \leqq i \leqq n .$$

By Lemma 1(e) with $i_1 \leqq i_2 = i_1 + i \leqq i_3 = i_2 + j \leqq i_4 = i_3 + k$,

(20) $$|EZ_{i_1} \cdots Z_{i_4}|/10\Delta_{i_1}^b \leqq \min \{\alpha(i)^{1-b}, \alpha(k)^{1-b}, \alpha(j)^{1-b} + 10\alpha(i)^{1-b}\alpha(k)^{1-b}\Delta_{i_3}^b\} .$$

Therefore

$$E|\bar{L}_N(t) - \bar{L}_N(s)|^4 \leqq 10.4!\,(hk_0^3 \cdot 3k_3n^{d-1} + 10k_0^2k_2^2h^2)$$

where $h = k_0^{1-b}(t-s)^b$. Let $g = \min(2b, b+2-2d)$, and

$$R_\varepsilon = 10.4!\,(3k_0^3k_3(2\varepsilon^{-1})^{2-2d} + 10k_0^2k_2^2)k_0^{1-b}\,.$$

If

$$\varepsilon n^{-\frac{1}{2}} < 2(t-s) \tag{21}$$

then

$$E|\bar{L}_N(t) - \bar{L}_N(s)|^4 \leqq R_\varepsilon(t-s)^g\,. \tag{22}$$

Therefore (21) $\Rightarrow$ (22) with $\bar{L}_N$ replaced by $\bar{L}_{Ni} = L_{Ni}(g_1^{-1})$, $i = 1, 2$ where $L_{N1}(t) = \sigma_N^{-1}n^{-\frac{1}{2}}\sum_{C_i\geqq 0} C_i(I(X_i \leqq t) - F_i(t))$, and $L_{N2}(t) = L_N(t) - L_{N1}(t)$. Hence by Theorem 12.2 of [1], for $m = 1, 2, \cdots$

$$P(M_{mj} \geqq \varepsilon) \leqq K_\varepsilon \delta^g$$

where $K_\varepsilon = R_\varepsilon \cdot K_{4,g}^1 \cdot \varepsilon^{-4}$ and

$$M_{mj} = \max_{i=1}^m |\bar{L}_{Nj}(s + i\delta/m) - \bar{L}_{Nj}(s)|\,, \qquad j = 1, 2,$$

whenever

$$\varepsilon n^{-\frac{1}{2}} \leqq 2\delta/m\,. \tag{23}$$

For $s \leqq t \leqq s + p$,

$$\begin{aligned} |L_N(t) - L_N(s)| &\leqq |L_{N1}(s+p) - L_{N1}(s)| \\ &\quad + |L_{N2}(s+p) - L_{N2}(s)| + n^{\frac{1}{2}}(g_1(s+p) - g_1(s))\,. \end{aligned} \tag{24}$$

Hence if $V_N = \sup_{s\leqq t\leqq s+\delta} |\bar{L}_N(t) - \bar{L}_N(s)|$ then $V_N \leqq 3M_{m1} + 3M_{m2} + n^{\frac{1}{2}}\delta/m$ (c.f. (22.17), (22.18) of [1].) Hence if (23) holds and $n^{\frac{1}{2}}\delta < \varepsilon m$ then

$$P(V_N \geqq 7\varepsilon) \leqq 2K_\varepsilon \delta^g\,. \tag{25}$$

Choosing m satisfying $r^{-1} < m\varepsilon n^{-\frac{1}{2}} \leqq 2r^{-1}$ where $r = [\delta]^{-1}$, and using Corollary 8.3 of [1] with $t_i = i/r$, (19.51) for $\bar{L}_N$ now follows.

THEOREM 4. *If for some $d < 1$ the conditions of Theorem 3 hold with "α_N-mixing" replaced by "ϕ_N-mixing" and $\alpha_N(j)^{1-b}$ is replaced by $\phi_N(j)$ in (17) and (18), then (14) holds.*

PROOF. Instead of (20) one uses

$$|EZ_{i_1} \cdots|/(4\Delta_{i_1}) \leqq \min\{\phi(i), \phi(k), \phi(j) + 4\phi(i)\phi(k)\Delta_{i_3}\}\,.$$

COROLLARY 3. *The condition $\sum_0^\infty i^2\phi(i)^{\frac{1}{2}} < \infty$ of Theorems 22.1, 22.2 of [1] can be weakened to*

$$\sum_0^n i^2\phi(i) = O(n^d) \qquad \textit{for some} \quad d < 1\,.$$

(This improves Sen [8] who showed that $\sum i\phi(i)^{\frac{1}{2}} < \infty$ was sufficient.)

THEOREM 5. *Suppose for $N \geqq 1$ $\{X_{iN}\}$ are ϕ_N-mixing. Suppose for $N \geqq 1$ (12), (13), (16) and*

$$\sum_1^{n-1} i\phi_N(i) \leqq k_2 < \infty \qquad \textit{for} \quad N \geqq 1\,.$$

Then (14) holds.

PROOF. Here we avoid assuming (19) by proving for $s \leqq t \leqq u$

$$E|L_N(t) - L_N(s)|^2|L_N(u) - L_N(t)|^2 \leqq k(g_{N1}(u) - g_{N1}(s))^2, \qquad N \geqq 1$$

where $k < \infty$. This is done by breaking the L.H.S. into 29 separate sums and applying Lemma 1(a). The proof now proceeds as for Theorem 2.

NOTE 4. With obvious changes (e.g. replacing $\sum_i^{n-1}$ by $\int_0^n di$) the results in this paper apply to processes $\{X_{iN}, 1 \leqq i \leqq n_N\}$ where i, N, n vary continuously.

EXAMPLE 1. Let F_0 be a cdf on [0, 1], and $\psi \geqq 0$, a function on [0, 1] such that $0 < x < 1 \Rightarrow \psi(x) < \infty$. Suppose the conditions of Theorem 2 or 3 or 4 or 5 hold and

$$\int_0^1 (x - x^2)^b \psi(x)\, dx < \infty, \quad (\text{with } b = 1 \text{ for Theorems 2, 4, 5}),$$

$$\limsup_N \max_i \sup_t \frac{dF_{iN}}{dF_0}(t) < \infty,$$

and

$$\limsup_N \int \delta_N^2 \psi(F_0)\, dF_0 < \infty$$

where $\delta_N = n^{\frac{1}{2}}(n^{-1}\sigma_N^{-1} \sum_1^n C_{iN} F_{iN} - F_0) \rightarrow \delta$ uniformly as $N \rightarrow \infty$. Let $F_N(x) = n^{-1}\sigma_N^{-1} \sum_1^n C_{iN} 1(X_{iN} \leqq x)$. Then $A_N = n \int (F_N - F_0)^2 \psi(F_0)\, dF_0 \rightarrow_{\mathscr{L}} A = \int (L + \delta)^2 \psi(F_0)\, dF_0$ where L is given by (14).

PROOF. The condition on $\{dF_{iN}/dF_0\}$ implies for some $C < \infty$ $EL_N^2 \leqq C(F_0 - F_0^2)^b$ in [0, 1], N large (with $b = 1$ if using Theorems 2, 4, 5.) Hence, for $\varepsilon > 0$ there exists $u \in (0, \frac{1}{2})$ and N_0 such that

$$\left(\int_0^u + \int_{1-u}^1\right) E(L_N + \delta_N)^2 \psi(F_0)\, dF_0 < \varepsilon^2, \qquad N_0 \leqq N \leqq \infty$$

where $L_\infty = L$, $\delta_\infty = \delta$. Therefore by Theorem 5.5 of [1],

$$B_n = \int_u^{1-u} (L_N + \delta_N)^2 \psi(F_0)\, dF_0 \rightarrow_{\mathscr{L}} B = \int_u^{1-u} (L + \delta)^2 \psi(F_0)\, dF_0.$$

Hence,

$$\begin{aligned} |P(A_N \leqq x + \varepsilon) - P(A \leqq x + \varepsilon)| &\leqq |P(A_N \leqq x + \varepsilon) - P(B_N \leqq x)| + |P(B_N \leqq x) - P(B \leqq x)| \\ &\quad + |P(B \leqq x) - P(A \leqq x + \varepsilon)| \\ &\leqq \varepsilon + P(x < A_N \leqq x + \varepsilon) + \varepsilon + P(x < A \leqq x + \varepsilon). \end{aligned}$$

Hence $A_n \rightarrow_{\mathscr{L}} A$.

EXAMPLE 2. Let $a(\cdot)$ be continuous and nondecreasing on [0, 1] such that $a(s) > 0$ for $s > 0$. Consider the cdf

$$\begin{aligned} F(s, u) &= a(s)/a(u), \quad 0 \leqq s < u \leqq 1 \\ &= 1, \qquad\qquad 0 < u \leqq s \leqq 1. \end{aligned}$$

When testing $H_{0N}: \{F_{iN}(s) \equiv F(s, i/n_N)\}$ for $\{X_{iN}\}$ independent, an asymptotically α-level test is to reject $H_0 \Leftrightarrow$

$$\int_0^1 L_{0N}^2(s) \cdot \frac{(\int_s^1 a^{-1})}{(k^{-1}A(s) + ka(s))^4}\, da(s) > t_\alpha,$$

where L_{0N} denotes L_N with expectations under H_0, and $P(\int_0^1 (W^0)^2 > t_\alpha) = \alpha$. W^0, W are the Brownian-Bridge and Wiener process, $\{C_{iN}\}$ satisfy (16), $A(s) = \int_s^1 a^{-1} - a(s) \int_s^1 a^{-2}$, and $k > 0$ is an arbitrary constant.

PROOF. $n^{-1} \sum_1^n F(s, i/n) \to s + a(s) \int_s^1 a^{-1} \Rightarrow$ (13) and (12) holds with $K(s, t) = a(s)A(t)$, $s \leqq t$ so that by Theorem 2 under H_0, $L_{0N} \to_{\mathscr{L}} A \cdot W(a/A)$. Finally on uses Theorem 5.1 of [1] and expresses W in terms of W^0.

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