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STRONG ADMISSIBILITY OF A SET OF CONFIDENCE INTERVALS FOR THE MEAN OF A FINITE POPULATION

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In a previous paper the admissibility was proved of a generalized version of the confidence sets, commonly used in practice, which are based on the sample mean and the sample standard deviation. A stronger definition of admissibility is obtained, if instead of the length of the interval for each individual observable sample s, only the expected length for all samples together for each $x \in R_N$, is taken into consideration for defining the permissible alternatives to the given set of confidence intervals. This stronger definition corresponds exactly to the definition of strong admissibility formulated by the author (1969) for confidence procedures for the parameter θ in a uni- or multivariate population. Using the stronger definition it is shown that confidence sets centered at the sample mean but having a fixed length are strongly admissible. The question of the strong admissibility of the usual confidence intervals with length proportional to the sample deviation remains open.

1. Introduction. In a previous paper (1967), a generalization of the usual confidence intervals, based on the sample mean and the sample standard deviation for the mean of a finite population, were shown to be admissible, whatever be the sampling design. In defining admissibility the alternative sets of confidence intervals are subject to the restriction that for any observable sample s and any point $x \in R_N$, the length of the alternative set does not exceed that in the given set.

In the case of confidence procedures for estimating the parameter θ in an infinite population, two concepts of admissibility, weak and strong, have been formulated by the author (1969). The weak concept is derived by considering the Lebesgue measures of the individual confidence sets and the strong concept by considering instead the expected Lebesgue measure for each θ, of all the confidence sets taken together.

Correspondingly for a finite population, we derive a stronger definition of admissibility, if instead of the length of the individual confidence interval for each observable sample s, we take into consideration the expected length for a given sampling design, of all the confidence intervals for a given point $x \in R_N$. Defining 'strong admissibility' in the above sense, we show in the following that if the sampling design is of fixed sample size, then the set of confidence intervals centered at the sample mean but having a fixed length, are 'strongly admissible'.

Such 'fixed length' confidence intervals are not often used in practice. However,

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the result is of theoretical interest. Also it may facilitate investigations regarding the 'strong admissibility' of the commonly used confidence intervals.

2. Notation and definitions. The argument in this paper follows very closely that in the previous paper (1967), which for brevity is referred to in the following as M.P. (short for Main Paper). We use the same notation as in the M.P. Equations in the M.P. are referred to by starred numbers to distinguish them from equations in this paper. Because of the limited nature of the result, and also because the argument has been set exhaustively in the M.P., in the following we shall give the argument only in outline.

In place of Definition 2.2 of the M.P. we definite 'strong admissibility' as follows.

DEFINITION 2.1. The set of confidence intervals $[e_1(s, x), e_2(s, x)]$ is 'strongly admissible' for the population mean if there exists no other set of confidence intervals $[e_1'(s, x), e_2'(s, x)]$, such that

(i) $$\sum_{s\in\bar{S}} p(s)[e_2'(s, x) - e_1'(s, x)] \leqq \sum_{s\in\bar{S}} p(s)[e_2(s, x) - e_1(s, x)]\,, \qquad \text{for all } x \in \mathbb{R}_N$$

and

(ii) $$\sum_{s\in\bar{S}_{e_1',e_2',x}} p(s) \geqq \sum_{s\in\bar{S}_{e_1,e_2,x}} p(s)\,,$$

for all $x \in R_N$, the strict inequality holding either in (i) or (ii) for at least one $x \in R_N$.

Note 2.1. We take this opportunity to insert a sort of 'corrigendum' to Definition 2.2 in the M.P. The strict inequality in clause (ii) of that definition was required to hold for at least one $x \in R_N$. The definition is 'improved' if, instead, we require that either the strict inequality in clause (i) holds for at least one $s \in \bar{S}$, and one $x \in R_N$, or that in (ii) holds for at least one $x \in R_N$. It is easily verified that the result in the M.P. holds for the definition so modified.

3. Bayes solution. We now take $[e_1(s, x), e_2(s, x)]$ to be the set of intervals given by

(1) $$e_1(s, x) = \bar{x}_s - c\,, \qquad e_2(s, x) = \bar{x}_s + c$$

where $c > 0$ is a known constant. The sampling design is of fixed size m, say, i.e.

(2) $$n(s) = m \qquad \text{for all} \quad s \in \bar{S}\,.$$

We assume the prior distribution as in the M.P. The Bayes confidence intervals $[b_1(s, x), b_2(s, x)]$ are subject to the restriction on lengths, in (i) of Definition 2.1, viz.

(3) $$\sum_{s\in\bar{S}} p(s)[b_2(s, x) - b_1(s, x)] \leqq 2c\,.$$

By (32)*, the Bayes intervals are (in this case also) centered at $\bar{x}_s g/g_s$. We next show that they are of fixed lengths $2c$. Let the length of the Bayes interval for sample s be $2u(s, x)$. Put

$$K = (N - m)mg_s/Ng , \tag{4}$$

where the right-hand side is the constant in the expression for F_2 in (27)*, after putting $n(s) = m$, by (2). Note also that, by (2), g_s in (26)* has the same value for all s. Substituting for F_2 in (32)* by (27*) and (4), and summing over $s \in \bar{S}$, we obtain

$$\sum_{s\in\bar{S}} p(s)B_{\tau,s} = \sum_{s\in\bar{S}} p(s) \int_{R_{n(s)}} L_1F_1\, dx_s \left\{\left(\frac{K}{2\pi}\right)^{\frac{1}{2}} \int_{-u(s,x)}^{u(s,x)} \exp\left[-\frac{K}{2} z^2\right] dz\right\} . \tag{5}$$

Now for each $s \in \bar{S}$, substitute for x_i, $i \in s$, taken in some particular order, x_1, $x_2, \cdots, x_m$. Let $L_1{}^*$, $F_1{}^*$ and $u^*(s, x')$ be the resulting functions derived from L_1, F_1 and $u(s, x)$ respectively. Because the prior distributions of x_i, $i = 1, 2, \cdots, N$, are identical, the expressions $L_1{}^*$, $F_1{}^*$ are identical for all $s \in \bar{S}$, because of (2). Let

$$u^*(s, x') = c + h^*(s, x') \tag{6}$$

where $x' = (x_1', x_2', \cdots, x_m')$ is a generic point of R_m. Since $\exp[-Kz^2/2]$ decreases strictly as $|Z|$ increases, we obtain from (5), by transforming the variables to $x_1, \cdots, x_m$, for each s and using (6), that

$$\begin{aligned}\sum_{s\in\bar{S}} p(s)B_{\tau,s} \leqq \left(\frac{K}{2\pi}\right)^{\frac{1}{2}} \int_{R_m} L_1{}^*F_1{}^*\, dx' \Bigg\{\int_{-c}^{c} \exp\left(-\frac{Kz^2}{2}\right) dz \\ + 2\exp\left(-\frac{Kc^2}{2}\right) \sum_{s\in\bar{S}} p(s)h^*(s, x')\Bigg\} .\end{aligned} \tag{7}$$

The Bayes intervals satisfy the restriction on expected length, viz.

$$\sum_{s\in\bar{S}} p(s)u(s, x) \leqq c \qquad \text{for all} \quad x \in R_N .$$

Hence taking expectations, and then transforming variables for each s to $x_1, \cdots, x_m$, we obtain

$$\int_{R_m} L_1{}^*F_1{}^*\, dx' \{\textstyle\sum_{s\in\bar{S}} p(s)u^*(s, x')\} \leqq c ,$$

so that by (6),

$$\int_{R_m} L_1{}^*F_1{}^*\, dx' \{\textstyle\sum_{s\in\bar{S}} p(s)h^*(s, x')\} \leqq 0 . \tag{8}$$

Also in (7), the strict inequality holds unless $h^*(s, x') = 0$ for all $s \in \bar{S}$. From (7) and (8), it follows that the inclusion probability is maximized by taking for all $s \in \bar{S}$, $h^*(s, x) = 0$, which by (6) implies $u(s, x) = c$ for all $s \in \bar{S}$. The Bayes intervals are thus centered at $\bar{x}_s g/g_s$ and are of fixed length $2c$. The Bayes improvement is now worked out as in the M.P. and is bounded by $1/2\tau^2$ by (48)*.

Using this upper bound we next show by an argument similar to that in Section 3-II of the M.P. that the set of confidence intervals in (1) is 'almost strongly admissible', which is a stronger version of the weak admissibility proved in Section 3-II of the M.P. The main change in the argument is that we cannot now introduce the set of confidence intervals $[e_1''(s, x), e_2''(s, x)]$ defined in (9)*.

Hence in place of $U''(s, x)$ in (54)*, we put

$$(9) \qquad U'(s, x) = \textstyle\int_{I_{e',s}} f_s \, d\bar{X}_{N-n(s)} \, .$$

But we introduce two functions $w_i(s, x)$ $i = 1, 2$, by putting

$$(10) \qquad \begin{aligned} e_1'(s, x) &= \bar{x}_s - c - w_1(s, x) \, , \\ e_2'(s, x) &= \bar{x}_s + c + w_2(s, x) \, . \end{aligned}$$

Let

$$(11) \qquad K_1 = (N - m) \cdot m \cdot N^{-1} \, .$$

Here K_1 is the constant in the expression for f_s in (53)*. Then again since $\exp(-K_1 z^2/2)$ decreases strictly as $|z|$ increases, we have

$$(12) \qquad U'(s, x) \leqq U(s, x) + \left(\frac{K_1}{2\pi}\right)^{\frac{1}{2}} \exp\left(-\frac{K_1 c^2}{2}\right) [w_1(s, x) + w_2(s, x)] \, .$$

In (12) the strict inequality holds unless $w_1(s, x) = w_2(s, x) = 0$.

Summing up (12) over all samples s after multiplying both sides by $p(s)$, and noting that by (i) in Definition 2.1

$$(13) \qquad \textstyle\sum_{s \in \bar{S}} p(s)[w_1(s, x) + w_2(s, x)] \leqq 0 \, ,$$

we obtain that

$$(13)' \qquad \textstyle\sum_{s \in \bar{S}} p(s) U'(s, x) \leqq \sum_{s \in \bar{S}} p(s) U(s, x)$$

where the strict inequality holds at any point x, at which $w_1(s, x) \neq 0$ or $w_2(s, x) \neq 0$. Hence in place of (59)*, we now have

$$(14) \qquad \textstyle\sum_{s \in \bar{S}} p(s) \int_{T_a} L_1 \, dx_s \, [U(s, x) - U'(s, x)] = k \, .$$

The argument now proceeds as in Section 3-II of the M.P., leading to the 'almost strong admissibility' of the set of confidence intervals in (1). Corresponding to Corollary 3.1 in the M.P. we obtain the following. Let E be the subset of R_N consisting of all points x at which at least one of the functions $w_1(s, x)$, $i = 1, 2$, $s \in \bar{S}$, does not vanish. Then

$$(15) \qquad E \quad \text{is a null set.}$$

4. Almost strong admissibility under restraints. We next prove a result corresponding to that in Section 4 of the M.P. The estimates $e_1'(s, x)$, $e_2'(s. x)$ are assumed to satisfy (77)*, and in addition the following,

$$(16) \qquad \textstyle\sum_{s \in \bar{S}} p(s)[e_2'(s, x) - e_1'(s, x)] \leqq 2c \qquad \text{for a.a.} \quad (\mu_{N-k}) x \in Q_{N-k}^a \, .$$

In place of (78)*, we define E_{N-k}^a by,

$$(17) \qquad x \in E_{N-k}^a \, , \qquad \text{if} \quad x \in Q_{N-k}^a \, ,$$

and $w_i(s, x) \neq 0$ for at least one i, $i = 1, 2$, for at least one $s \in \bar{S}$.

The slight change in the argument is that we do not introduce the new set of confidence intervals $[e_1^*(s, x), e_2^*(s, x)]$ but work with the original set in (1). The values of $e_i'(s, x)$, $i = 1, 2$ are defined by for $x' \in Q^{a'}_{N-k}$, $s \in \bar{S}_k$,

$$e_i'(s, x') = e_i'(s, x) , \qquad i = 1, 2, \cdots ; \tag{18}$$

and for $s \notin S_k$, $x \in R_N$

$$e_i'(s, x) = e_i(s, x) , \qquad i = 1, 2, \cdots .$$

If the set E^a_{N-k} in (17) is non-null (μ_{N-k}), we obtain for the confidence intervals defined by (18), a non-null (μ_N) set E of R_N which contradicts (15). This proves that the set E^a_{N-k} in (17) is a null (μ_{N-k}) set.

5. Strong admissibility. The proof is completed by an argument similar to that in Section 5 of the M.P. In place of (104)*, the set E is now defined as in (15). Suppose the set E is not empty. Then it contains a point $a = (a_1, a_2, \cdots, a_N)$ and there exists an $s_0 \in \bar{S}$, such that at least one of $w_i(s_0, a)$ $i = 1, 2$ is non-vanishing. As in the M.P., without loss of generality, we take s_0 to consist of the first m units. Then two alternatives arise, viz. (A_1) at least one of the values $w_1(s_0, a)$, $w_2(s_0, a)$ is negative; (B_1) $w_1(s_0, a)$, $w_2(s_0, a)$ are both nonnegative, so that one of them is positive.

Suppose (A_1) holds. Suppose $w_1(s_0, a) < 0$. Then in place of (109)*, we define the set T^a_{N-m} by

$$\bar{a}_0\left(1 - \frac{m}{n}\right) - c \leqq N^{-1} \textstyle\sum_{i=m+1}^N x_i \leqq \bar{a}_0\left(1 - \frac{m}{n}\right) - c - w_1(s_0, a) . \tag{19}$$

The set of T^a_{N-m} is of infinite measure (μ_{N-m}) and because of (ii) in Definition 2.1, for every $x \in T^a_{N-m}$, for at least one, $s \neq s_0$, $s \in \bar{S}$, at least one of the functions $w_i(s, x)$ is non-vanishing.

If (B_1) holds, then because of (i) in Definition 2.1, for every $x \in P^a_{N-m}$, at least one of the functions $w_i(s, x)$, $i = 1, 2$, $s \in \bar{S}$, $s \neq s_0$ is non-vanishing. Thus, under this alternative, we put

$$T^a_{N-m} = P^a_{N-m} .$$

Thus, under either alternative, we obtain a set T^a_{N-m} of infinite measure (μ_{N-m}). The rest of the argument proceeds as in the M.P. Note that alternative (A) [page 1201 of M.P.] cannot arise, as all samples are of the same size.

The process can end either with the alternative (A′) or with alternative (B′), in the M.P., substituting in them, the definition of E_N in (15) and a definition of E_{N-j} corresponding to (17). (A′) thus contradicts (15), and (B′) contradicts (19). Hence the set E must be empty, thus proving the strong admissibility of the set of intervals in (1).

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