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ANALYSIS OF NONORTHOGONAL n-WAY CLASSIFICATIONS[1]

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Four problems associated with the use of Zelen's calculus of factorials in the statistical analysis of nonorthogonal n-way classification data are solved. These are for the situations for which (i) some effect parameters are equated to zero, (ii) some combinations (subclasses) contain no observations, (iii) expected values of mean squares under fixed, mixed, and random models are desired, and (iv) expected values of single degree of freedom sums of squares are wanted. A unified approach to these problems was developed. Relationships to previous work, to blocked experiments, to fractional replication, and to "messy data" situations are discussed. The various analyses are first described for a nonorthogonal two-way classification and then generalized to an n-way classification in the final section of the paper. Numerical examples are presented to illustrate the various procedures.

0. Introduction and summary. The calculus of factorials was introduced by Kurkjian and Zelen [7] and was subsequently extended and applied in a number of published papers [3], [6], [8], [14], [15]; two of these, [3], [15], dealt with analyses for nonorthogonal n-way classifications (factorials) with all main effects and interactions present in the model and with at least one observation associated with every combination of the factorial, that is, no missing subclasses. In this paper we consider the following four extensions associated with the use of the calculus of factorials in the analysis of data from nonorthogonal n-way classifications:

(i) analysis when some of the effect parameters are equal to zero,

(ii) analysis when some of the combinations (subclasses) contain no observations,

(iii) the expected values of sums of squares in the analysis of variance, and

(iv) the expected value of the sum of squares for a single degree of freedom contrast or of several single degree of freedom contrasts.

It should be noted under (i) that one or more factors may relate to blocking effects and the remaining factor(s) to the treatments; also, from Paik's [9] results, one can include the case of nonconnected designs under singular fractional replicates; Kurkjian and Woodall [6] present results on some aspects of nonconnected

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block designs. Some other aspects of (i), (ii), and (iii) have been treated in previously published literature (see [2], [4], [5], [10], [11], [12], [13]).

In order to obtain a unified approach to the four problems listed above, we make use of the vector of subclass means and of an orthogonal transformation of the levels of any given factor into single degree of freedom contrasts. As illustrated in the next section, use of the vector of subclass means does not always result in the same estimates of effects and of expected values of sums of squares as a number of other methods discussed in the literature (see [2] and [4], for example). Some of these procedures result in the same estimates and expectation of sums of squares when all subclasses contain one or more observations and/or all subclasses with observations have the same number of observations. The use of an orthogonal transformation of levels of effects into single degree of freedom contrasts allows us to relate the concepts of fractional replication from complete factorials to any n-way classification with missing subclasses (that is, no observations), to illustrate the biases in estimates of effect parameters through the aliasing matrix, and to illustrate the nature of the expected value of the sums of squares (Sections 3 and 6) when the contrast parameters associated with the missing subclasses are not zero (Section 5). Since the preceding was not the concern of previous workers, there was no need for them to utilize the orthogonal transformation matrix $(P_s'P_s)^{-\frac{1}{2}}P_s$ and associated ideas as used in this paper (see [6], for example).

1. Preliminaries and a numerical example. Consider a simple asymmetric factorial experiment with two factors $\{A_s: s = 1, 2\}$ such that the sth factor A_s has m_s levels. The method of analysis of this case is easily extended to the general asymmetric factorial experiment. For the case of two factors, the number of treatments is $t = m_1 m_2$ and the space of treatments, Z, is represented by the set $Z = \{(i_1, i_2): i_s = 0, 1, \cdots, m_s - 1 \text{ for all } s = 1, 2\}$ which clearly contains t points. The order of the points in Z is given by the relationship between the coordinate of the point $Z_v = (i_1, i_2)$, $v = 0, 1, \cdots, t - 1$, and order subscript

$$v = m_2 i_1 + i_2 \,. \tag{1.1}$$

Let y_{vj} be the jth observation made on the vth treatment combination (i_1, i_2), where $j = 1, 2, \cdots, r_v$, $(r_v \geqq 1)$, and let N be the total number of observations. Then y_{vj} may be written as

$$y_{vj} = \eta_v + \varepsilon_{vj} \,, \tag{1.2}$$

where

$$\eta_v = \mu + \alpha_1(i_1) + \alpha_2(i_2) + \alpha_{12}(i_1, i_2) \,, \tag{1.3}$$

$\alpha_s(i_s)$ and $\alpha_{12}(i_1, i_2)$ denote the main effect and two-factor interaction parameters, respectively, $E(\varepsilon_{vj}) = 0$ for all v and j, and $E(\varepsilon_{vj}\varepsilon_{v'j'}) = \sigma^2$ if $v = v'$, $j = j'$, and zero otherwise.

Using matrix notation,

$$\mathbf{y} = X\boldsymbol{\eta} + \boldsymbol{\varepsilon} \,, \tag{1.4}$$

where $\mathbf{y}$ and $\boldsymbol{\varepsilon}$ are the $N \times 1$ observation vector and the corresponding error vector, respectively, $\boldsymbol{\eta}$ is a $t \times 1$ treatment vector, i.e., $\boldsymbol{\eta} = (\eta_0, \eta_1, \cdots, \eta_{t-1})'$, and X is an $N \times t$ design matrix. We shall not be concerned about whether or not the parameters are random or fixed until Section 3.

Let $P_s = ||p_s(i,j)||$, $i, j = 0, 1, \cdots, m_s - 1$, be an $m_s \times m_s$ matrix such that $p_s(i, 0) = 1$ for $i = 0, 1, \cdots, m_s - 1$, and $P_s'P_s = D_s$, D_s is an $m_s \times m_s$ diagonal matrix, and $P_{12} = P_1 \otimes P_2$, $D_{12} = P_{12}'P_{12}$, where the symbol $\otimes$ refers to the Kronecker product. Consider the following orthogonal transformation of the parameters in $\boldsymbol{\eta}$:

$$(1.5) \qquad \begin{bmatrix} \mathbf{a}_s^* \\ \mathbf{a}_s \end{bmatrix} = D_s^{-\frac{1}{2}} P_s' \boldsymbol{\alpha}_s \qquad \text{and} \qquad \begin{bmatrix} \mathbf{a}_{12}^* \\ \mathbf{a}_{12} \end{bmatrix} = D_{12}^{-\frac{1}{2}} P_{12}' \boldsymbol{\alpha}_{12} ,$$

where $\boldsymbol{\alpha}_s = [\alpha_s(0), \alpha_s(1), \cdots, \alpha_s(m_s - 1)]'$, $s = 1, 2$, $\boldsymbol{\alpha}_{12} = [\alpha_{12}(0, 0), \alpha_{12}(0, 1), \cdots, \alpha_{12}(m_1 - 1, m_2 - 1)]'$, and $\mathbf{a}_s^* = a_s(0)$, $\mathbf{a}_s = [a_s(1), a_s(2), \cdots, a_s(m_s - 1)]'$, $\mathbf{a}_{12}^* = [a_{12}(0, 0), a_{12}(0, 1), \cdots, a_{12}(0, m_2 - 1), a_{12}(1, 0), \cdots, a_{12}(m_1 - 1, 0)]'$ and $\mathbf{a}_{12} = [a_{12}(1, 1), a_{12}(1, 2), \cdots, a_{12}(m_1 - 1, m_2 - 1)]'$.

Let $W = P_{12}(P_{12}'P_{12})^{-\frac{1}{2}}$; we describe the column vectors $W(\alpha_1, \alpha_2)$ in the $t \times t$ orthogonal matrix W by considering the space of the t points where $\{(\alpha_1, \alpha_2)\colon \alpha_s = 0, 1, \cdots, m_s - 1, \text{ for } s = 1, 2\}$. The correspondence between the column order $0, 1, \cdots, t - 1$ and the points (α_1, α_2) is given by the order relation specified by order number $= m_2\alpha_1 + \alpha_2$.

Let W^* be the column order rearranged matrix from the matrix W in the following way, i.e.,

$$(1.6) \qquad \begin{aligned} W^* = [&W(0, 0), W(1, 0), \cdots, W(m_1 - 1, 0), W(0, 1), \cdots, \\ &W(0, m_2 - 1), W(1, 1), \cdots, W(m_1 - 1, m_2 - 1)] , \end{aligned}$$

and let

$$(1.7) \qquad A = W^*K ,$$

where

$$(1.8) \qquad K = \operatorname{diag}\left((m_1 m_2)^{\frac{1}{2}}, m_2^{\frac{1}{2}} I_{(m_1-1)}, m_1^{\frac{1}{2}} I_{(m_2-1)}, I_{(m_1-1)(m_2-1)}\right) .$$

Let

$$(1.9) \qquad \mathbf{b} = (b_0, \mathbf{b}_1', \mathbf{b}_2', \mathbf{b}_{12}')' ,$$

where

$$(1.10) \qquad \begin{aligned} b_0 &= \mu + \textstyle\sum_{s=1}^{2} m_s^{-\frac{1}{2}} a_s(0) + (m_1 m_2)^{-\frac{1}{2}} a_{12}(0, 0) , \\ \mathbf{b}_1 &= \mathbf{a}_1 + m_2^{-\frac{1}{2}} \mathbf{a}_{12}(i_1, 0) , \\ \mathbf{b}_2 &= \mathbf{a}_2 + m_1^{-\frac{1}{2}} \mathbf{a}_{12}(0, i_2) , \\ \mathbf{b}_{12} &= \mathbf{a}_{12} , \end{aligned}$$

where $\mathbf{a}_{12}(i_1, 0) = [a_{12}(1, 0), \cdots, a_{12}(m_1 - 1, 0)]'$ and $\mathbf{a}_{12}(0, i_2) = [a_{12}(0, 1), \cdots, a_{12}(0, m_2 - 1)]'$.

Then, the vector $\boldsymbol{\eta}$ may be written as

$$(1.11) \qquad \boldsymbol{\eta} = A\mathbf{b} ,$$

and from (1.4)

$$\mathbf{y} = XA\mathbf{b} + \boldsymbol{\varepsilon} . \tag{1.12}$$

We now present two methods for estimating the parameter vector $\mathbf{b}$ in equation (1.12).

Method I. (Sometimes denoted as F. Yates's "fitting constants" procedure): Using the least squares method, from (1.12)

$$\begin{aligned} \hat{\mathbf{b}} &= (A'SA)^{-1}A'X'\mathbf{y} , \qquad \text{where} \quad S = X'X \\ &= A^{-1}S'X'\mathbf{y} , \qquad \text{because} \quad A \quad \text{is a nonsingular matrix,} \\ &= K^{-1}W^{*\prime}S^{-1}X'\mathbf{y} . \end{aligned} \tag{1.13}$$

Method II. (Sometimes denoted as F. Yates's "weighted squares of means" procedure or S. N. Roy's "sum of squares for the hypothesis" method): From (1.11) we obtain

$$\mathbf{b} = A^{-1}\boldsymbol{\eta}$$

and from (1.4), by the least squares method, $\hat{\boldsymbol{\eta}} = S^{-1}X'\mathbf{y}$, i.e., $\hat{\eta}_v = \sum_j y_{vj}/r_u$, and consequently

$$\begin{aligned} \hat{\mathbf{b}} &= A^{-1}\hat{\boldsymbol{\eta}} \\ &= K^{-1}W^{*\prime}\hat{\boldsymbol{\eta}} \\ &= K^{-1}W^{*\prime}S^{-1}X'\mathbf{y} . \end{aligned} \tag{1.14}$$

Numerical example 1.1. A 3 x 4 factorial experiment with unequal numbers of observations for the combinations of three levels of oven, A_1, and four levels of temperature, A_2, was conducted to ascertain the strength of the final product. The data were selected from Table 12.5 of Graybill [5] with some observations arbitrarily deleted.

TABLE 1.1
Data for example

		Temperature, A_2 (4 levels)			
		0	1	2	3
		Observations			
Oven A_1 (3 levels)	0	3, 3, 2	6, 4	3, 4, 4	4, 5
	1	4, 3	6, 2, 7	6	3, 7, 9
	2	4, 6	8, 5, 9	5, 6	7, 8, 9

We shall assume the statistical model to be

$$\begin{aligned} & y_{vj} = \eta_v + \varepsilon_{vj} , \qquad v = 0, 1, \cdots, t-1 ; \quad j = 1, 2, \cdots, r_v , \\ & \eta_v = \mu + \alpha_1(i_1) + \alpha_2(i_2) + \alpha_{12}(i_1, i_2) , \\ & \qquad i_1 = 0, 1, 2 ; \quad i_2 = 0, 1, 2, 3 ; \quad v = 4i_1 + i_2 , \end{aligned}$$

where y_{vj} is the yield of the jth observation for the vth treatment combination,

μ is the overall effect, $\alpha_1(i_1)$ is the effect of i_1 level of the oven, $\alpha_2(i_2)$ is the effect of the i_2 level of the temperature, and $\alpha_{12}(i_1, i_2)$ is the interaction between them.

In this case, using orthogonal polynomials, let

$$P_1 = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 1 & -3 & 1 & -1 \\ 1 & -1 & -1 & 3 \\ 1 & 1 & -1 & -3 \\ 1 & 3 & 1 & 1 \end{bmatrix}. \tag{1.15}$$

Then $D_1 = P_1'P_1 = \operatorname{diag}(3, 2, 6)$ and $D_2 = P_2'P_2' = \operatorname{diag}(4, 20, 4, 20)$, and we obtain the matrix W^* using the following table:

TABLE 1.2

	$W(0, i_2)$				$W(1, i_2)$				$W(2, i_2)$			
i_2:	0	1	2	3	0	1	2	3	0	1	2	3
$P_1 \otimes P_2$:	1	−3	1	−1	−1	3	−1	1	1	−3	1	−1
	1	−1	−1	3	−1	1	1	−3	1	−1	−1	3
	1	1	−1	−3	−1	−1	1	3	1	1	−1	−3
	1	3	1	1	−1	−3	−1	−1	1	3	1	1
	1	−3	1	−1	0	0	0	0	−2	6	−2	2
	1	−1	−1	3	0	0	0	0	−2	2	2	−6
	1	1	−1	−3	0	0	0	0	−2	−2	2	6
	1	3	1	1	0	0	0	0	−2	−6	−2	−2
	1	−3	1	−1	1	−3	1	−1	1	−3	1	−1
	1	−1	−1	3	1	−1	−1	3	1	−1	−1	3
	1	1	−1	−3	1	1	−1	−3	1	1	−1	−3
	1	3	1	1	1	3	1	1	1	3	1	1
diag $D_{12}^{\frac{1}{2}}$:	$\sqrt{12}$	$\sqrt{60}$	$\sqrt{12}$	$\sqrt{60}$	$\sqrt{8}$	$\sqrt{40}$	$\sqrt{8}$	$\sqrt{40}$	$\sqrt{24}$	$\sqrt{120}$	$\sqrt{24}$	$\sqrt{120}$

Also, from (1.8)

$$K = \operatorname{diag}(12^{\frac{1}{2}}, 4^{\frac{1}{2}}, 4^{\frac{1}{2}}, 3^{\frac{1}{2}}, 3^{\frac{1}{2}}, 3^{\frac{1}{2}}, 1, 1, 1, 1, 1, 1)\,. \tag{1.16}$$

Next, let

$$\mathbf{b} = (b_0, \mathbf{b}_1', \mathbf{b}_2', \mathbf{b}_{12}')'\,,$$

where

$$\begin{aligned} b_0 &= \mu + 4^{-\frac{1}{2}}a_1(0) + 3^{-\frac{1}{2}}a_2(0) + a_{12}(0, 0) \\ \mathbf{b}_1 &= \mathbf{a}_1 + 4^{-\frac{1}{2}}\mathbf{a}_{12}(i_1, 0) \\ \mathbf{b}_2 &= \mathbf{a}_2 + 3^{-\frac{1}{2}}\mathbf{a}_{12}(0, i_2) \\ \mathbf{b}_{12} &= \mathbf{a}_{12}\,. \end{aligned} \tag{1.17}$$

In this example,

$$\begin{aligned} \hat{\eta} &= S^{-1}X'\mathbf{y} = \text{vector of class means} \\ &= (2.6667, 5.0000, 3.6667, 4.5000, 3.5000, 5.0000, 6.0000, \\ &\qquad 6.3333, 5.0000, 7.3333, 5.5000, 8.0000)'\,. \end{aligned} \tag{1.18}$$

So, from (1.14), we obtain:

$$\hat{\mathbf{b}} = (5.2083, 1.7678, 0.0000, 1.5528, -0.4167, 1.0559, 0.4743, -0.6999, 0.5893, 0.2041, 0.4216, 1.3389)' . \tag{1.19}$$

Confusion sometimes arises in regard to the method of "fitting constants" analysis and the "weighted squares of means" analysis which were presented by Yates [13]. The former terminology is often used in the context of no interactions between two or more factors while the latter is used when interaction is present and one or more observations are available on each treatment of the factorial. Both procedures "fit constants" but for different models. This estimation of parameters aspect has caused no confusion but the computation of sums of squares has. For a two-way classification and disproportionate subclass numbers, the sum of squares for one main effect free of the mean, the other main effect, and the interaction is different from the sum of squares for the main effect computed under the assumption of no interaction. Both sums of squares are useful when used in connection with the correct parametric model. These comments apply equally well to the n-way classification. Also, it should be noted from the above that $\hat{\mathbf{b}}$ from either equation (1.13) or (1.14) appears to be the same; these two methods, however, will not always give the same result as illustrated below. Suppose that some of the parameters $\mathbf{b}_s, \mathbf{b}_{rs}, \cdots, \mathbf{b}_{12\cdots n}$ are zero in an n-factor factorial experiment; then, omitting the zero parameters from the parameter vector $\mathbf{b}$, deleting the corresponding columns from the matrix A, and denoting the new parameter vector and matrix by $\mathbf{b}^*$ and A^*, we obtain the following solutions:

Method I.

$$\hat{\mathbf{b}}^* = (A^{*\prime}SA^*)^{-1}A^{*\prime}X'\mathbf{y} . \tag{1.20}$$

Method II.

$$\begin{aligned} \hat{\mathbf{b}}^* &= (A^{*\prime}A^*)^{-1}A^{*\prime}\hat{\boldsymbol{\eta}} \\ &= (A^{*\prime}A^*)^{-1}A^{*\prime}S^{-1}X'\mathbf{y} . \end{aligned} \tag{1.21}$$

(See Bush and Anderson [2] for a comparison of these two methods.)

2. Analysis of variance. In the present paper, the sums of squares used in the analysis of variance are similar to those given by the "weighted squares of means" procedure (see Yates [13] and Zelen and Federer [3, 15]).

In the $m_1 \times m_2$ factorial experiment, let

$$\begin{aligned} W_1 &= [W(1,0), W(2,0), \cdots, W(m_1-1, 0)] , \\ W_2 &= [W(0,1), W(0,2), \cdots, W(0, m_2-1)] , \qquad \text{and} \\ W_{12} &= [W(1,1), W(1,2), \cdots, W(m_1-1, m_2-1)] , \end{aligned} \tag{2.1}$$

then the matrix W^* can be expressed as $[W(0,0), W_1, W_2, W_{12}]$.

The resulting quadratic forms as sums of squares for main effects, two-factor interactions are:

Sums of squares	Degrees of freedom
$SS(\mathbf{b}_s) = \hat{\eta}' W_s[W_s' S^{-1} W_s]^{-1} W_s' \hat{\eta}$	$m_s - 1$
$SS(\mathbf{b}_{12}) = \hat{\eta}' W_{12}[W_{12}' S^{-1} W_{12}]^{-1} W_{12}' \hat{\eta}$	$(m_1 - 1)(m_2 - 1)$

The sums of squares for single degree of freedom contrasts are:

$$SS[b_1(i_1)] = \hat{\eta}' W(i_1, 0)[W(i_1, 0)' S^{-1} W(i_1, 0)]^{-1} W(i_1, 0)' \hat{\eta}$$
$$SS[b_2(i_2)] = \hat{\eta}' W(0, i_2)[W(0, i_2)' S^{-1} W(0, i_2)]^{-1} W(0, i_2)' \hat{\eta}$$
$$SS[b_{12}(i_1, i_2)] = \hat{\eta}' W(i_1, i_2)[W(i_1, i_2)' S^{-1} W(i_1, i_2)]^{-1} W(i_1, i_2)' \hat{\eta} ,$$

where $i_1 = 1, 2, \cdots, m_1 - 1$; $i_2 = 1, 2, \cdots, m_2 - 1$.

The within subclass sum of squares is given by

$$SS_e = \mathbf{y}'(I - XS^{-1}X')\mathbf{y} ,$$

and expected value of SS_e, which has $(N - t)$ degrees of freedom, is $(N - t)\sigma^2$. However, if some of the elements of the parameter vector are zero as in (1.21), then

$$SS_e = \mathbf{y}'[I - XS^{-1}A^*(A^{*\prime}S^{-1}A^*)^{-1}A^{*\prime}S^{-1}X']\mathbf{y} ,$$

and the expected value of SS_e, which has (N — number of columns in A^*) degrees of freedom, is (N — number of columns in A^*)σ^2.

In the numerical example 1.1, from (1.18) and (2.1),

$$W_1'\hat{\eta} = 4^{\frac{1}{2}}(1.7678, 0.0000)' , \qquad W_2'\hat{\eta} = 3^{\frac{1}{2}}(1.5528, -0.4167, 1.0559)' ,$$
$$W_{12}'\hat{\eta} = (0.4743, -0.6999, 0.5893, 0.2041, 0.4216, 1.3389)' .$$

After calculating the inverse matrices of $W_1'S^{-1}W_1$, $W_2'S^{-1}W_2$, and $W_{12}'S^{-1}W_{12}$, we obtain the following sums of squares:

$$SS(\mathbf{b}_1) = SS(A_1) = 30.0000 \qquad \text{d.f.} = 2$$
$$SS(\mathbf{b}_2) = SS(A_2) = 26.2185 \qquad \text{d.f.} = 3$$
$$SS(\mathbf{b}_{12}) = SS(A_1 \times A_2) = 5.5557 \qquad \text{d.f.} = 6$$

and since $[W(0, 1)'S^{-1}W(0, 1)]^{-1} = 2.3529$, $[W(0, 2)'S^{-1}W(0, 2)]^{-1} = 2.1818$, and $[W(0, 3)'S^{-1}W(0, 3)]^{-1} = 2.0339$,

$$SS[b_2(1)] = (2.3529)(3)(1.5528)^2 = 17.0207: \quad SS \text{ of linear effect of } A_2 ,$$
$$SS[b_2(2)] = (2.1818)(3)(-0.4167)^2 = 1.1364: \quad SS \text{ of quadratic effect of } A_2 ,$$
$$SS[b_2(3)] = (2.0339)(3)(1.0599)^2 = 6.8032: \quad SS \text{ of cubic effect of } A_2 .$$

In the next section, we obtain the expectations for the above sums of squares.

3. Expectation of sum of squares.

3.1. *Fixed effects case.* If all effects in (1.3) are fixed, we may make the assumption that effects sum to zero over the levels of any factor without loss of generality. Then, from (1.5), we obtain $a_s(0) = 0$ for $s = 1, 2$; $\mathbf{a}_{12}(i_1, 0) = \mathbf{a}_{12}(0, i_2) = 0$ and from (1.10)

$$b_0 = \mu\,, \qquad \mathbf{b}_s = \mathbf{a}_s\,, \qquad \mathbf{b}_{12} = \mathbf{a}_{12}\,.$$

Hence, from (1.14)

$$(\hat{\mu}, \hat{\mathbf{a}}_1', \hat{\mathbf{a}}_2', \hat{\mathbf{a}}_{12}')' = K^{-1}W^{*\prime}\hat{\eta}$$

and we obtain, from (1.5)

$$\hat{\boldsymbol{\alpha}}_s = P_s D_s^{-\frac{1}{2}} \begin{bmatrix} 0 \\ \hat{\mathbf{a}}_s \end{bmatrix}, \qquad \hat{\boldsymbol{\alpha}}_{12} = P_{12} D_{12}^{-\frac{1}{2}} \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{a}}_{12} \end{bmatrix}.$$

Also, in the fixed effects model,

$$\begin{aligned} E[\hat{\eta}\hat{\eta}'] &= E[(S^{-1}X'\mathbf{y})(S^{-1}X'\mathbf{y})'] \\ &= A(\mathbf{bb}')A' + S^{-1}\sigma^2\,, \qquad \text{since} \quad \mathbf{y} = XA\mathbf{b} + \boldsymbol{\varepsilon}\,. \end{aligned}$$

We now present the expectations of the sums of squares for the model of this subsection:

$$\begin{aligned} E[SS(\mathbf{a}_s)] &= \text{main effect of} \quad A_s \\ &= E[\hat{\eta}' W_s(W_s'S^{-1}W_s)^{-1}W_s'\hat{\eta}] \\ &= \operatorname{tr}[W_s(W_s'S^{-1}W_s)^{-1}W_s'E(\hat{\eta}\hat{\eta}')] \\ &= \operatorname{tr}[(W_s'S^{-1}W_s)^{-1}W_s'A(\mathbf{bb}')A'W_s] + \operatorname{tr}[(W_s'S^{-1}W_s)^{-1}W_s'S^{-1}W_s]\sigma^2 \\ &= m_u \operatorname{tr}[(W_s'S^{-1}W_s)^{-1}\mathbf{a}_s\mathbf{a}_s'] + (m_s - 1)\sigma^2\,, \end{aligned}$$

where $u = 2$ if $s = 1$ and $u = 1$ if $s = 2$.

Similarly,

$$\begin{aligned} E[SS(a_{12})] &= \operatorname{tr}[(W_{12}'S^{-1}W_{12})^{-1}\mathbf{a}_{12}\mathbf{a}_{12}'] + (m_1 - 1)(m_2 - 1)\sigma^2 \\ E[SS(a_1(i_1))] &= m_2[W(i_1, 0)'S^{-1}W(i_1, 0)]^{-1}[a_1(i_1)]^2 + \sigma^2 \\ E[SS(a_2(i_2))] &= m_1[W(0, i_2)'S^{-1}W(0, i_2)]^{-1}[a_2(i_2)]^2 + \sigma^2 \\ E[SS(a_{12}(i_1, i_2))] &= [W(i_1, i_2)'S^{-1}W(i_1, i_2)]^{-1}[a_{12}(i_1, i_2)]^2 + \sigma^2\,. \end{aligned}$$

3.2. *Random effects case.* If all effects (except mean effect μ) in (1.3) are random, we do not assume that $\sum_{i_s} \alpha_s(i_s) = 0$, $\sum_{i_1} \alpha_{12}(i_1, i_2) = \sum_{i_2} \alpha_{12}(i_1, i_2) = 0$. In this case, however, we may assume that all expectations of parameters are zero and that

$$\begin{aligned} E[\boldsymbol{\alpha}_s\boldsymbol{\alpha}_s'] &= I_{m_s}\sigma_s^2 \qquad \text{for} \quad s = 1, 2\,, \\ E[\boldsymbol{\alpha}_{12}\boldsymbol{\alpha}_{12}'] &= I_{m_1 m_2}\sigma_{12}^2\,. \end{aligned}$$

We shall use the notation $\sigma_{b_0}^2, \sigma_{b_s}^2, \sigma_{b_{12}}^2$ for the variances of b_0, $b_s(i_s)$, and $b_{12}(i_1, i_2)$, respectively; they may be calculated as follows:

$$\begin{aligned} \sigma_{b_0}^2 &= E[b_0^2] - \mu^2 = \textstyle\sum_s m_s^{-1}\sigma_s^2 + (m_1 m_2)^{-1}\sigma_{12}^2\,, \\ \sigma_{b_s}^2 &= E[(b_s(i_s))^2] = \sigma_s^2 + m_u^{-1}\sigma_{12}^2\,, \\ &\qquad\qquad \text{where} \quad u = 2 \ \text{if} \ s = 1 \ \text{and} \ u = 1 \ \text{if} \ s = 2\,, \\ \sigma_{b_{12}}^2 &= E[(b_{12}(i_1, i_2))^2] = \sigma_{12}^2\,. \end{aligned}$$

Under this random model, $E[\mathbf{y}] = \mu\mathbf{1}_N = \mu X\mathbf{1}_t$ and variance of $\mathbf{y}$ is

$$V = E[(\mathbf{y} - E\mathbf{y})(\mathbf{y} - E\mathbf{y})']$$
$$= XAE[\mathbf{bb}']A'X'\mu^2 X\mathbf{1}_t\mathbf{1}_t'X' + I\sigma^2 ,$$

where $\mathbf{1}_t$ is a $t \times 1$ column vector with elements unity. Hence,

$$E[\hat{\boldsymbol{\eta}}\hat{\boldsymbol{\eta}}'] = AE(\mathbf{bb}')A' + S^{-1}\sigma^2 .$$

Note that

$$E[\mathbf{bb}'] = \operatorname{diag}(\mu^2 + \sigma_{b_0}^2, I_{(m_1-1)}\sigma_{b_1}^2, I_{(m_2-1)}\sigma_{b_0}^2, I_{(m_1-1)(m_2-1)}\sigma_{12}^2) .$$

Then, the expectations of the sums of squares are:

$$E[SS(\mathbf{b}_s)] = E[\hat{\boldsymbol{\eta}}'W_s(W_s'S^{-1}W_s)^{-1}W_s'\hat{\boldsymbol{\eta}}]$$
$$= \operatorname{tr}[W_s(W_s'S^{-1}W_s)^{-1}W_s'E(\hat{\boldsymbol{\eta}}\hat{\boldsymbol{\eta}}')]$$
$$= \operatorname{tr}[(W_s'S^{-1}W_s)^{-1}W_s'AE(\mathbf{bb}')A'W_s] + \operatorname{tr}[(W_s'S^{-1}W_s)^{-1}W_s'S^{-1}W_s]\sigma^2$$
$$= m_u \operatorname{tr}[W_s'S^{-1}W_s]^{-1}\sigma_{b_s}^2 + (m_s - 1)\sigma^2 ,$$

where $u = 2$ if $s = 1$ and $u = 1$ if $s = 2$.

Similarly,

$$E[SS(\mathbf{b}_{12})] = \operatorname{tr}[W_{12}'S^{-1}W_{12}]^{-1}\sigma_{b_{12}}^2 + (m_1 - 1)(m_2 - 1)\sigma^2 .$$

3.3. *Mixed effects case*. Suppose that $\boldsymbol{\alpha}_1$ is a random effect and that $\boldsymbol{\alpha}_2$ is a fixed effect in equation (1.3). For this case, we do not assume that $\sum_{i_1} \alpha_1(i_1) = 0$, but we will assume that $\sum_{i_2} \alpha_2(i_2) = 0$, and $\sum_{i_2} \alpha_{12}(i_1, i_2) = 0$. Then, $a_2(0) = 0$, and

$$a_{12}(i_1, 0) = [d_1(i_1)d_2(0)]^{-1}[p_1(0, i_1), p_1(1, i_1), \cdots, p_1(m_1 - 1, i_1)]$$
$$\otimes [p_2(1, 0), p_2(2, 0), \cdots, p_2(m_2 - 1, 0)]\boldsymbol{\alpha}_{12}$$
$$= [d_1(i_1)d_2(0)]^{-1}[p_1(0, i_1), p_1(1, i_1), \cdots, p_1(m_1 - 1, i_1)]$$
$$\otimes [1, 1, \cdots, 1]\boldsymbol{\alpha}_{12}$$
$$= [d_1(i_1)d_2(0)]^{-1} \textstyle\sum_{k=0}^{m_1-1} p_1(k, i_1) \sum_{i_2=0}^{m_2-1} \alpha_{12}(k, i_2) = 0 ,$$
$$\text{since } \textstyle\sum_{i_2} \alpha_{12}(k, i_2) = 0 ,$$

where $d_1(i_1)$ is the $i_1 + 1$th diagonal element of D_1, $d_2(0)$ is the first diagonal element of D_2. However, in this model, $a_1(0) \neq 0$, $a_{12}(0, i_2) \neq 0$. So, using (1.10)

$$b_0 = \mu + m_1^{-\frac{1}{2}}a_1(0) , \qquad \mathbf{b}_1 = \mathbf{a}_1 , \tag{3.1}$$
$$\mathbf{b}_2 = \mathbf{a}_2 + m_1^{-\frac{1}{2}}\mathbf{a}_{12}(0, i_2) , \qquad \mathbf{b}_{12} = \mathbf{a}_{12} .$$

Also, in this mixed model, we may assume that:

$$E(\boldsymbol{\alpha}_1) = 0 \quad \text{and} \quad E(\boldsymbol{\alpha}_{12}) = 0 , \qquad \text{so} \tag{3.2}$$
$$E(\mathbf{a}_1) = 0 \quad \text{and} \quad E(\mathbf{a}_{12}) = 0 .$$

Further, we assume that $E[\boldsymbol{\alpha}_1\boldsymbol{\alpha}_1'] = I_{m_1}\sigma_1^2$; that is, $E[\mathbf{a}_1\mathbf{a}_1'] = I_{(m_1-1)}\sigma_1^2$. Define $\sigma_{12}^2 = (m_2 - 1)^{-1} \sum_{i_2=0}^{m_2-1} \operatorname{Var}(a_{12}(i_1, i_2))$ and assume that $\operatorname{Var}(a_{12}(i_1, i_2))$ are identical for $i_2 = 0, 1, \cdots, m_2 - 1$, then

$$\begin{aligned} E[\alpha_{12}(i_1, i_2)\alpha'_{12}(i_1', i_2')] &= \frac{m_2 - 1}{m_2}\sigma_{12}^2 \quad \text{if} \quad i_1 = i_1' \quad \text{and} \quad i_2 = i_2' , \\ &= -\frac{1}{m_2}\sigma_{12}^2 \quad \text{if} \quad i_1 = i_1' \quad \text{and} \quad i_2 \neq i_2' . \\ &= 0 \quad \text{otherwise,} \end{aligned} \tag{3.3}$$

and then

$$E[\boldsymbol{\alpha}_{12}\boldsymbol{\alpha}'_{12}] = [m_2^{-1}I_{m_1m_2} - m_2^{-1}\operatorname{diag}(J_{m_2}, J_{m_2}, \cdots, J_{m_2})]\sigma_{12}^2 \tag{3.4}$$

where J_{m_2} is an $m_2 \times m_2$ matrix with all elements unity. So, we obtain

$$\begin{aligned} E[\mathbf{a}_{12}(0, i_2)\mathbf{a}_{12}(0, i_2)'] &= I_{(m_2-1)}\sigma_{12}^2 , \\ E[\mathbf{a}_{12}\mathbf{a}'_{12}] &= I_{(m_1-1)(m_2-1)}\sigma_{12}^2 , \end{aligned} \tag{3.5}$$

and using (3.1) and (3.2)

$$\begin{aligned} &\sigma_{b_1}^2 = \sigma_1{}^2 , \qquad \sigma_{b_2}^2 = m_1^{-1}\sigma_{12}^2 , \qquad \text{i.e.,} \\ &E[\mathbf{b}_2\mathbf{b}_2'] = \mathbf{a}_2\mathbf{a}_2' + m_1^{-1}I_{(m_2-1)}\sigma_{12}^2 , \qquad \text{and} \\ &\sigma_{b_{12}}^2 = \sigma_{12}^2 . \end{aligned} \tag{3.6}$$

We now present the expectation of the sums of squares for the mixed model:

$$\begin{aligned} E[SS(\mathbf{b}_1)] &= E[\hat{\boldsymbol{\eta}}'W_1(W_1'S^{-1}W_1)^{-1}W_1'\hat{\boldsymbol{\eta}}] \\ &= \operatorname{tr}[W_1(W_1'S^{-1}W_1)^{-1}W_1'E(\hat{\boldsymbol{\eta}}\hat{\boldsymbol{\eta}}')] \end{aligned}$$

where

$$E[\hat{\boldsymbol{\eta}}\hat{\boldsymbol{\eta}}'] = AE(\mathbf{bb}')A' + S^{-1}\sigma^2 ;$$

then

$$\begin{aligned} E[SS(\mathbf{b}_1)] &= \operatorname{tr}[(W_1'S^{-1}W_1)^{-1}W_1'AE(\mathbf{bb}')A'W_1] + (m_1 - 1)\sigma^2 \\ &= m_2 \operatorname{tr}[W_1'S^{-1}W_1]^{-1}\sigma_{b_1}^2 + (m_1 - 1)\sigma^2 \\ &= m_2 \operatorname{tr}[W_1'S^{-1}W_1]^{-1}\sigma_1{}^2 + (m_1 - 1)\sigma^2 . \end{aligned} \tag{3.7}$$

Similarly,

$$\begin{aligned} E[SS(\mathbf{b}_2)] &= m_1 \operatorname{tr}[(W_2'S^{-1}W_2)^{-1}E(\mathbf{b}_2\mathbf{b}_2')] + (m_2 - 1)\sigma^2 \\ &= m_1 \operatorname{tr}[(W_2'S^{-1}W_2)^{-1}\mathbf{a}_2\mathbf{a}_2'] + \operatorname{tr}[W_2'S^{-1}W_2]^{-1}\sigma_{12}^2 + (m_2 - 1)\sigma^2 , \\ E[SS(b_2(i_2))] &= m_1[W(0, i_2)'S^{-1}W(0, i_2)]^{-1}(a_2(i_2))^2 \\ &\quad + [W(0, i_2)S^{-1}W(0, i_2)]^{-1}\sigma_{12}^2 + \sigma^2 , \\ E[SS(\mathbf{b}_{12})] &= \operatorname{tr}[W'_{12}S^{-1}W_{12}]^{-1}\sigma_{12}^2 + (m_1 - 1)(m_2 - 1)\sigma^2 . \end{aligned}$$

In numerical example 1.1, we may assume that α_1 is a random effect associated with ovens, and α_2 is a fixed effect associated with temperature. We summarize the analysis of variance from the results in Section 2 and expectations of mean squares calculated by the above formulas in the following table:

TABLE 1.3
Analysis of variance

Source of variation	d.f.	SS	EMS
Total	29	922.0000	
CFM	1	796.6897	
Error	17	50.1667	σ^2
A_1	2	30.0000	$8.80\sigma_1^2 + \sigma^2$
A_2	3	26.2185	$^*f(\mathbf{a}_2) + 2.24\sigma_{12}^2 + \sigma^2$
$A_2(L)$	1	17.0207	$^{**}7.06(a_2(1))^2 + 2.35\sigma_{12}^2 + \sigma^2$
$A_2(Q)$	1	1.1364	$6.54(a_2(2))^2 + 2.18\sigma_{12}^2 + \sigma^2$
$A_2(C)$	1	6.8032	$6.10(a_2(3))^2 + 2.03\sigma_{12}^2 + \sigma^2$
$A_1 \times A_2$	6	5.5557	$2.30\sigma_{12}^2 + \sigma^2$

* $f(\mathbf{a}_2) = \text{tr}\,[(W_2'S^{-1}W_2)^{-1}\mathbf{a}_2\mathbf{a}_2']$.
** $m_1\,\text{tr}\,[W_2(1)'S^{-1}W_2(1)]^{-1} = 3(2.3529) = 7.0587$.

REMARK. Gaylor, Lucas and Anderson [4] present a computational procedure to obtain the "fitting constants" analysis of variance and expectations of mean squares for fixed, mixed, and random models, using the same convention as ours for nonrandom effects. Using their method, we obtain $SS(A_1, \text{adj}. A_2) = 32.2002$, $SS(A_2, \text{adj}. A_1) = 27.5331$, $SS(A_2(L)) = 17.5485$, $SS(A_2(Q)) = 0.8375$, $SS(A_2(C)) = 9.1471$, $SS(A_1 \times A_2) = 5.5557$, where the linear effect is not adjusted, quadratic is adjusted for linear, and cubic is adjusted for linear and quadratic. The expectation of the A_2(adj.) mean square was determined for this method to be $f_1(\mathbf{a}_2) + 2.46\sigma_{12}^2 + \sigma^2$ as compared with $f(\mathbf{a}_2) + 2.24\sigma_{12}^2 + \sigma^2$ for our method. Note the large difference in the sum of squares for the cubic by the two methods.

4. Some of the effect parameters equal to zero. In Section 1, we described the following procedure if some of the parameters $\mathbf{b}_s, \mathbf{b}_{rs}, \cdots, \mathbf{b}_{12\cdots n}$ are zero in an n-factor factorial experiment: omit the zero parameters from the parameter vector $\mathbf{b}$, delete the corresponding columns from the matrix A and obtain the solution. Now suppose, in the mixed model, that there are no interactions between a specified random effect (possibly blocking effect) and other effects which have all main effects and interactions among them. For example, suppose y_{vjg} is the jth observation made on the treatment combination (i_1, i_2) in the gth block in an $m_1 \times m_2$ factorial experiment, where $v = 0, 1, \cdots, t-1$; $j = 1, 2, \cdots, r_{vg}$ ($r_{vg} \neq 0$ for some g); $g = 1, 2, \cdots, h$ and v is defined in (1.1). In this case, we may apply a procedure combining Methods I and II as described in Section 1. Using the notation in (1.2) and (1.3), let

$$y_{vjg} = \beta_g + \eta_v + \varepsilon_{vjg}\,, \tag{4.1}$$

where β_g is a random effect. $E[\beta_g] = 0$, $\text{Var}\,(\beta_g) = \sigma_\beta^2$ for $g = 1, 2, \cdots, h$, all effects in η_v are fixed, and N is the total number of observations.

Using matrix notation,

$$\mathbf{y} = X[\boldsymbol{\beta}', \boldsymbol{\eta}']' + \boldsymbol{\varepsilon}\,; \qquad E[\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'] = I_N\sigma^2\,,$$

then

$$E[\mathbf{y}] = X[\mathbf{0}', \boldsymbol{\eta}']', \qquad V(\mathbf{y}) = X_1 X_1' \sigma_\beta^2 + I_N \sigma^2, \tag{4.2}$$

where $X = [X_1, X_2]$, X_1 is an $N \times h$ matrix and X_2 is an $N \times t$ matrix. By the least squares procedure, we obtain

$$\hat{\boldsymbol{\eta}} = GX_2[I - X_1(X_1'X_1)^{-1}X_1']\mathbf{y} + [H - I]\mathbf{z}, \tag{4.3}$$

where G is a g-inverse of $X_2'[I - X_1(X_1'X_1)^{-1}X_1']X_2$, $H = GX_2'[I - X_1(X_1'X_1)^{-1}X_1']X_2$, and $\mathbf{z}$ is an arbitrary $t \times 1$ vector of components $z_0, z_1, \cdots, z_{t-1}$.

Now, we may apply Method II as described in Section 1 to obtain $\hat{\mathbf{b}}$ and analysis of variance using the procedures in Section 2. A theorem related to this case will be proved in Section 6.3.

5. Some of the subclasses empty. In the event that some of the subclass numbers are zero, a fractional replicate results with respect to all of the combinations. In model (1.2), suppose that $\alpha = t - f$, $f < t$, subclasses have no observations, i.e., some of r_v's are zero; then using the notation defined in Section 1:

$$\mathbf{y} = X\boldsymbol{\eta}_f + \varepsilon, \qquad E[\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'] = I_N \sigma^2, \tag{5.1}$$

where $\mathbf{y}$ and $\boldsymbol{\varepsilon}$ are $N \times 1$ column vectors, $\boldsymbol{\eta}_f$ is an $f \times 1$ parameter vector, and X is an $N \times f$ design matrix.

Let A_f be an $f \times t$ submatrix of A, as defined in (1.7), and let A_f be constructed by deleting some of the d rows from the matrix A such that

$$\boldsymbol{\eta}_f = A_f \mathbf{b} \tag{5.2}$$

is satisfied; let $A_f = W_f^* K_f$, where W_f^* and K_f are $f \times t$ and $f \times f$ submatrices of W^* and K, respectively.

Suppose that after rearranging the column order in W_f^*, the partitioned matrix $[W_{11,f}, W_{12,f}]$ results and is obtained such that $W_{11,f}$ is an $f \times f$ nonsingular matrix and $W_{12,f}$ is an $f \times d$ matrix. For convenience we shall assume that the matrix $W_{11,f}$ is constructed from the first f columns in W_f^* and $d < (m_1 - 1)(m_2 - 1)$. Then, the matrix A_f may be partitioned as:

$$A_f = [A_f^*, W_{12,f}].$$

Now, rewrite (5.2) as:

$$\boldsymbol{\eta}_f = [A_f^*, W_{12,f}'][\mathbf{b}_f', \mathbf{b}_d']', \tag{5.3}$$

and note that the matrix A_f^* and vector $\mathbf{b}_d$ will be dependent upon the choice of the matrix $W_{11,f}$.

Let

$$Z' = W_{11,f}^{-1}. \tag{5.4}$$

For the column vector of $Z = (W_{11,f}^{-1})'$, we shall use the following notation:

$$Z = [Z_0, Z_1(1), \cdots, Z_1(m_1 - 1), Z_2(1), \cdots, Z_2(m_2 - 1),$$
$$Z_{12}(1), \cdots, Z_{12}((m_1 - 1)(m_2 - 1) - d)],$$

and let $Z_1 = [Z_1(1), \cdots, Z_1(m_1-1)]$, $Z_2 = [Z_2(1), \cdots, Z_2(m_2-1)]$, $Z_w = [Z_{12}(1), Z_{12}(2), \cdots, Z_{12}((m_1-1)(m_2-1)-d)]$. Then,

$$Z'A_f^* = \text{diag}\,[(m_1m_2)^{\frac{1}{2}}, m_2^{\frac{1}{2}}I_{(m_1-1)}, m_1^{\frac{1}{2}}I_{(m_2-1)}, I_{(m_1-1)(m_2-1)-d}]\,. \tag{5.5}$$

Next, let

$$\mathbf{b}_f^* = \mathbf{b}_f + (Z'A_f^*)^{-1}Z'W_{12,f}\mathbf{b}_d = (b_0^f, \mathbf{b}_1^{f\prime}, \mathbf{b}_2^{f\prime}, \mathbf{b}_w^{f\prime})'\,, \tag{5.6}$$

then, from (5.3)

$$\mathbf{b}_f^* = (Z'A^*)^{-1}Z'\boldsymbol{\eta}_f\,.$$

Since the least squares estimate of $\boldsymbol{\eta}$ is $\hat{\boldsymbol{\eta}} = S^{-1}X'\mathbf{y}$,

$$\hat{\mathbf{b}}_f^* = (Z'A^*)^{-1}Z'\hat{\boldsymbol{\eta}}_f\,. \tag{5.7}$$

We shall use the following quadratic forms as sums of squares for main effects, two-factor interactions, and so on.

Sums of squares	*Degrees of freedom*
$SS(\mathbf{b}_s^f) = \hat{\boldsymbol{\eta}}_f'Z_s[Z_s'S^{-1}Z_s]^{-1}Z_s'\hat{\boldsymbol{\eta}}_f$	$m_s - 1$
$SS(\mathbf{b}_w^f) = \hat{\boldsymbol{\eta}}_f'Z_w[Z_w'S^{-1}Z_w]^{-1}Z_w'\hat{\boldsymbol{\eta}}_f$	$(m_1-1)(m_2-1)$

The sums of squares for single degree of freedom contrasts are:

$$SS(b_s^f(i_s)) = \hat{\boldsymbol{\eta}}_f'Z_s(i_s)[Z_s(i_s)'S^{-1}Z_s(i_s)\,]^{-1}Z_s(i_s)'\hat{\boldsymbol{\eta}}_f\,,$$
$$SS(b_{12}^f(i_1, i_2)) = \hat{\boldsymbol{\eta}}_f'Z_{12}(i_1, i_2)[Z_{12}(i_1, i_2)'S^{-1}Z_{12}(i_1, i_2)]^{-1}Z_{12}(i_1, i_2)'\hat{\boldsymbol{\eta}}_f\,.$$

In this case, the expected values for the sums of squares in the above analysis of variance for fixed, mixed, and random models may be obtained in the same manner as described in Section 3. This will become evident from the numerical example given below.

In example 1.1, suppose that there were no observations for the combination (2, 2), i.e., $y_{10,1} = 5$, $y_{10,2} = 6$ in Table 1.1; then $S = X'X = \text{diag}\,(3, 2, 3, 2, 2, 3, 1, 3, 2, 3, 3)$, $\hat{\boldsymbol{\eta}}_f = (2.6667, 5.0000, 3.6667, 4.5000, 3.5000, 5.0000, 6.0000, 6.3333, 5.0000, 7.3333, 8.0000)'$. Consider the following orthogonal matrix which is constructed by rearranging the rows and columns of W^*,

$$\begin{bmatrix} W_{11,f} & W_{12,f} \\ W_{21,f} & W_{22,f} \end{bmatrix}.$$

In this example,

$$W'_{12,f} = (-1, 3, -3, 1, 2, -6, 6, -2, -1, 3, 1)/2(30)^{\frac{1}{2}},$$
$$W_{21,f} = (1/2(3)^{\frac{1}{2}}, 1/2(15)^{\frac{1}{2}}, -1/2(3)^{\frac{1}{2}}, -3/2(15)^{\frac{1}{2}}, 1/2(2)^{\frac{1}{2}}, 1/2(10)^{\frac{1}{2}}, -1/2(2)^{\frac{1}{2}}, -3/2(10)^{\frac{1}{2}}\ 1/2(6)^{\frac{1}{2}}, 1/2(30)^{\frac{1}{2}}, -1/2(5)^{\frac{1}{2}})'\,,$$

and

$$W_{22,f} = -3/2(30)^{\frac{1}{2}},$$

then

$$Z' = W_{11,f}^{-1}$$

may be obtained by the following formula (see Banerjee and Federer [1]):

$$Z = W_{11,f} - W_{12,f}W_{22,f}^{-1}W_{21,f}$$

$$= \begin{bmatrix} 2 & -10 & 4 & 0 & -2 & 4 & -1 & 1 & 1 & -5 & 2 \\ 6 & 0 & -6 & 0 & 0 & 3 & 0 & -3 & 3 & 0 & -3 \\ 0 & 0 & 0 & 0 & -3 & -3 & 3 & 3 & 0 & 0 & 0 \\ 4 & 10 & 2 & 0 & -1 & -4 & -2 & -1 & 2 & 5 & 1 \\ 5 & -7 & 1 & -3 & 1 & 1 & -1 & -1 & -2 & 10 & -4 \\ -3 & -9 & 3 & 9 & -3 & -3 & 3 & 3 & -6 & 0 & 6 \\ 9 & 9 & -9 & -9 & 3 & 3 & -3 & -3 & 0 & 0 & 0 \\ 1 & 7 & 5 & 3 & -1 & -1 & 1 & 1 & -4 & -10 & -2 \\ 2 & -10 & 4 & 0 & 1 & -5 & 2 & 0 & 1 & -5 & 2 \\ 6 & 0 & -6 & 0 & 3 & 0 & -3 & 0 & 3 & 0 & -3 \\ 4 & 10 & 2 & 0 & 2 & 5 & 1 & 0 & 2 & 5 & 1 \end{bmatrix} D^{-1},$$

$$= [Z(0), Z_2(1), Z_2(2), Z_2(3), Z_1(1), Z_1(2), Z_{12}(1, 1), Z_{12}(1, 2), Z_{12}(1, 3), Z_{12}(2, 1), Z_{12}(2, 2)],$$

where $D = \operatorname{diag}(6(3)^{\frac{1}{2}}, 6(15)^{\frac{1}{2}}, 6(3)^{\frac{1}{2}}, 2(15)^{\frac{1}{2}}, 3(2)^{\frac{1}{2}}, 3(10)^{\frac{1}{2}}, 3(2)^{\frac{1}{2}}, 10^{\frac{1}{2}}, 3(6)^{\frac{1}{2}}, 3(30)^{\frac{1}{2}}, 3(6)^{\frac{1}{2}})$. Therefore, for example.

$$Z_2(1)'\hat{\eta}_f = (-26.667 + 45 - 24.5 - 45 + 54 + 44.3331 - 50 + 80)/6(15)^{\frac{1}{2}}$$

and

$$[Z_2(1)'S^{-1}Z_2(1)]^{-1} = 1.7116 ;$$

so we obtain

(i) $SS(b_2{}^f(1)) = \hat{\eta}_f{}'Z_2(1)[Z_2(1)'S^{-1}Z_2(1)]^{-1}Z_2(1)'\hat{\eta}_f$
$= [(77.1661)/6(15)^{\frac{1}{2}}]^2(1.7116) = 18.8738$.
$E[SS(b_2{}^f(1))] = m_1[Z_2(1)'S^{-1}Z_2(1)]^{-1}(a_2(1))^2 + ([Z_2(1)'S^{-1}Z_2(1)]^{-1}$
$+ W'_{12,f}Z_2(1)[Z_2(1)'S^{-1}Z_2(1)]^{-1}Z_2(1)'W_{12,f})\sigma_{12}^2 + \sigma^2$.

Similarly,

(ii) $SS(b_2{}^f(2)) = 3.5233$, and
$E[SS(b_2{}^f(2))] = 3(0.7742)(a_2(2))^2 + (0.7742 + 0.9602)\sigma_{12}^2 + \sigma^2$.

(iii) $SS(b_2{}^f(3)) = 0.0022$;
$E[SS(b_2{}^f(3))] = 3(0.5195)(a_2(3))^2 + (0.5195 + 1.0390)\sigma_{12}^2 + \sigma^2$.

(iv) $SS(\mathbf{b}_2{}^f) = 27.2308$;
$E[SS(\mathbf{b}_2{}^f)] = m_1 \operatorname{tr}[(Z_2'S^{-1}Z_2)^{-1}\mathbf{a}_2\mathbf{a}_2'] + \operatorname{tr}[(Z_2'S^{-1}Z_2)^{-1}$
$+ W'_{12,f}Z_2(Z_2'S^{-1}Z_2)^{-1}Z_2'W_{12,f}]\sigma_{12}^2 + (m_2 - 1)\sigma^2$
$= 3 \operatorname{tr}[(Z_2'S^{-1}Z_2)^{-1}\mathbf{a}_2\mathbf{a}_2'] + (5.2569 + 1.1174)\sigma_{12}^2 + \sigma^2$,

where $Z_2 = [Z_2(1), Z_2(2), Z_2(3)]$.

(v) $SS(\mathbf{b}_1{}^f) = 22.3308$;

$$E[SS(\mathbf{b}_1{}^f)] = m_2 \operatorname{tr}(Z_1'S^{-1}Z_1)^{-1}\sigma_1{}^2 + \operatorname{tr}[W'_{12,f}Z_1(Z_1'S^{-1}Z_1)^{-1}Z_1'W_{12,f}]\sigma_{12}^2 + (m_1 - 1)\sigma^2$$
$$= 4(2.9197)\sigma_1{}^2 + 1.7044\sigma_{12}^2 + 2\sigma^2,$$

where $Z_1 = [Z_1(1), Z_2(2)]$.

(vi) $SS(\mathbf{b}_{12}^f) = 3.6832$;

$$E[SS(\mathbf{b}_{12}^f)] = \operatorname{tr}[(Z_w'S^{-1}Z_w)^{-1} + W'_{12,f}Z_w(Z_w'S^{-1}Z_w)^{-1}Z_w'W_{12,f}]\sigma_{12}^2 + ((m_1 - 1)(m_2 - 1) - d)\sigma^2$$
$$= (11.5510)\sigma_{12}^2 + 5\sigma^2,$$

where $Z_w = [Z_{12}(1, 1), Z_{12}(1, 2), Z_{12}(1, 3), Z_{12}(2, 1), Z_{12}(2, 2)]$.

We now obtain the following analysis of variance table.

TABLE 1.4
Analysis of variance

Source of variation	d.f.	SS	EMS
Total	27	861.00	
CFM	1	736.3333	
Error	16	49.6687	σ^2
A_1	2	22.3308	$5.84\sigma_1{}^2 + 0.85\sigma_{12}^2 + \sigma^2$
A_2	3	27.2308	$^*f(\mathbf{a}_2{}^f) + 2.12\sigma_{12}^2 + \sigma^2$
$A_2(L)$	1	18.8738	$5.13(a_2(1))^2 + 2.09\sigma_{12}^2 + \sigma^2$
$A_2(Q)$	1	3.5233	$2.33(a_2(2))^2 + 1.63\sigma_{12}^2 + \sigma^2$
$A_2(C)$	1	0.0022	$1.56(a_2(3))^2 + 1.56\sigma_{12}^2 + \sigma^2$
$A_1 \times A_2$	5	3.6832	$2.31\sigma_{12}^2 + \sigma^2$

* $f(\mathbf{a}_2) = \operatorname{tr}[(Z_2'S^{-1}Z_2)^{-1}\mathbf{a}_2\mathbf{a}_2']$.

6. Generalization to an *n*-way classification. Consider an asymmetric factorial experiment with n factors $\{A_s: s = 1, 2, \cdots, n\}$ such that the sth factor A_s has m_s levels. The space of treatments, Z, is represented by the set $Z = \{(i_1, i_2, \cdots, i_n): i_s = 0, 1, \cdots, m_s - 1 \text{ for all } s = 1, 2, \cdots, n\}$ which contains $t = \prod_{s=1}^n m_s$ points. The order of the points in Z is given by the relationship between the coordinate of a point $z_v = (i_1, i_2, \cdots, i_n)$, $v = 0, 1, \cdots, t - 1$, and the order subscript

$$v = [\textstyle\sum_{s=1}^{n-1} (\prod_{k=s+1}^{n} m_k)i_s] + i_n . \tag{6.1}$$

Let y_{vj} be the jth observation made on the vth treatment combination, $(i_1, i_2, \cdots, i_n)$, where $j = 1, 2, \cdots, r_v$, $(r_v \geqq 1)$, and let N be the total number of observations; then y_{vj} is expressed as:

$$y_{vj} = \eta_v + \varepsilon_{vj}, \tag{6.2}$$

where

$$\eta_v = \mu + \sum_{s=1}^{n} \alpha_s(i_s) + \sum_{1 \leq r < s \leq n}^{r} \sum^{s} \alpha_{rs}(i_r, i_s) + \cdots + \alpha_{12\cdots n}(i_1, i_2, \cdots, i_n) . \tag{6.3}$$

Using matrix notation,

$$\mathbf{y} = X\boldsymbol{\eta} + \boldsymbol{\varepsilon} , \tag{6.4}$$

where X is an $N \times t$ design matrix.

Using the matrix defined in Section 1, let $P_{12\cdots n} = P_1 \otimes P_2 \otimes \cdots \otimes P_n$, $W = P_{12\cdots n}(P'_{12\cdots n} P_{12\cdots n})^{-\frac{1}{2}}$, and we describe the column vectors $W(\alpha_1, \alpha_2, \cdots, \alpha_n)$ of the $t \times t$ orthogonal matrix W by considering the space of the t points where $\{(\alpha_1, \alpha_2, \cdots, \alpha_n) : \alpha_s = 0, 1, \cdots, m_s - 1 \text{ for all } s = 1, 2, \cdots, n\}$. Let W^* be the column order rearranged matrix from the matrix W in the following way, i.e., $W(0, 0, \cdots, 0)$, $W(1, 0, \cdots, 0), \cdots, W(m_1 - 1, 0, \cdots, 0)$, $W(0, 1, 0, \cdots, 0), \cdots, W(m_1 - 1, m_2 - 1, \cdots, m_n - 1)$. For simplicity, we may use the following notation: $W_0 = W(0, 0, \cdots, 0)$, $W_1(i_1) = W(i_1, 0, \cdots, 0)$, $W_2(i_2) = W(0, i_2, 0, \cdots, 0)$, $W_{12}(i_1, i_2, 0, \cdots, 0)$, etc., and $W_1 = [W_1(1), W_1(2), \cdots, W_1(m_1 - 1)]$, $W_2 = [W_2(1), W_2(2), \cdots, W_2(m_2 - 1)], \cdots, W_{12\cdots n} = [W(1, 1, \cdots, 1), W(1, 1, \cdots, 2), \cdots, W(m_1 - 1, m_2 - 1, \cdots, m_n - 1)]$.

Let

$$A = W^*K , \tag{6.5}$$

where

$$K = \operatorname{diag}\left(\prod_{s=1}^{n} m_s^{\frac{1}{2}}, \prod_{s=2}^{n} m_s^{\frac{1}{2}} I_{(m_1-1)}, \cdots, m_1^{\frac{1}{2}} I_{\prod_{s=2}^{n}(m_s-1)}, I_{\prod_{s=1}^{n}(m_s-1)}\right) . \tag{6.6}$$

Consider the orthogonal transformed parameter vector $(\mathbf{a}^{*\prime}, \mathbf{a}')'$ from the parameters $\boldsymbol{\alpha}$'s, as in Section 1, and let

$$\mathbf{b} = (b_0, \mathbf{b}_1', \mathbf{b}_2', \cdots, \mathbf{b}_n', \mathbf{b}_{12}', \cdots, \mathbf{b}_{23\cdots n}', \mathbf{b}_{12\cdots n}')' , \tag{6.7}$$

where

$$\begin{aligned}
b_0 &= \mu + \sum_{s=1}^{n} m_s^{-\frac{1}{2}} a_s(0) + \sum_{1 \leq r < s \leq n}^{r} \sum^{s} (m_r m_s)^{-\frac{1}{2}} a_{rs}(0, 0) + \cdots \\
&\quad + \prod_{s=1}^{n} m_s^{-\frac{1}{2}} a_{12\cdots n}(0, 0, \cdots, 0) , \\
\mathbf{b}_s &= \mathbf{a}_s + \sum_{u=1}^{s-1} m_u^{-\frac{1}{2}} \mathbf{a}_{us}(0, i_s) + \sum_{u=s+1}^{n} m_u^{-\frac{1}{2}} \mathbf{a}_{su}(i_s, 0) , \\
\mathbf{b}_{rs} &= \mathbf{a}_{rs} + \sum_{u=1}^{r-1} m_u^{-\frac{1}{2}} \mathbf{a}_{usr}(0, i_r, i_s) + \sum_{u=r+1}^{s-1} m_u^{-1} \mathbf{a}_{rus}(i_r, 0, i_s) \\
&\quad + \sum_{u=s+1}^{n} m_u^{-\frac{1}{2}} \mathbf{a}_{rsu}(i_r, i_s, 0) + \cdots \\
&\quad + \prod_{u=1}^{r-1} m_u^{-\frac{1}{2}} \prod_{u=r+1}^{s-1} m_u^{-\frac{1}{2}} \prod_{u=s+1}^{n} m_u^{-\frac{1}{2}} \\
&\quad \times \mathbf{a}_{12\cdots n}(0, \cdots, i_r, 0, \cdots, i_s, 0, \cdots, 0) \\
&\quad \cdot \quad \cdot \quad \cdot \\
\mathbf{b}_{23\cdots n} &= \mathbf{a}_{23\cdots n} + m_1^{-\frac{1}{2}} \mathbf{a}_{12\cdots n}(0, i_2, i_3, \cdots, i_n) , \\
\mathbf{b}_{12\cdots n} &= \mathbf{a}_{12\cdots n} .
\end{aligned} \tag{6.8}$$

Then, the vector $\boldsymbol{\eta}$ in (6.4) can be written as:

$$\boldsymbol{\eta} = A\mathbf{b} ,$$

and from (1.14)

$$\hat{\mathbf{b}} = A^{-1}\hat{\boldsymbol{\eta}} = K^{-1}W^{*\prime}S^{-1}X'\mathbf{y}\,. \tag{6.9}$$

6.1. *Fixed effects case.* If all effects in (6.3) are fixed $b = \mu$, $\mathbf{b}_s = \mathbf{a}_s$, $\mathbf{b}_{rs} = \mathbf{a}_{rs}, \cdots, \mathbf{b}_{12\cdots n} = \mathbf{a}_{12\cdots n}$ from (6.8); then, from (6.9), we obtain $\hat{\boldsymbol{\alpha}}_s = Q_s\hat{\mathbf{a}}_s$, $\hat{\boldsymbol{\alpha}}_{rs} = Q_r \otimes Q_s\hat{\mathbf{a}}_{rs}, \cdots, \hat{\boldsymbol{\alpha}}_{12\cdots n} = (\prod_{s=1}^n \otimes Q_s)\hat{\mathbf{a}}_{12\cdots n}$, where Q_s is obtained by partitioning the matrix P_s such that $P_s = [\mathbf{1}, P_s^*]$ and letting $Q_s = [P_s^{*\prime}P_s^*]^{-\frac{1}{2}}P_s^*$.

We assume that $\mathbf{y}$ is a vector of random variables following a multivariate normal distribution with mean $\boldsymbol{\eta}$ and covariance $I\sigma^2$. With the above notation, we now state and prove a theorem on the distribution of quadratic forms.

THEOREM 6.1. *In the case of fixed effects, the quadratic forms* $\hat{\boldsymbol{\eta}}'W_s(W_s'S^{-1} \times W_s)^{-1}W_s'\sigma^{-2}\hat{\boldsymbol{\eta}}$, $\hat{\boldsymbol{\eta}}'W_{rs}(W_{rs}'S^{-1}W_{rs})^{-1}W_{rs}'\sigma^{-2}\hat{\boldsymbol{\eta}}, \cdots,$ *and* $\hat{\boldsymbol{\eta}}'W_{12\cdots n}(W_{12\cdots n}'S^{-1}W_{12\cdots n})^{-1} \times W_{12\cdots n}'\sigma^{-2}\hat{\boldsymbol{\eta}}$ *subject to restrictions* $\alpha_s(0) = \alpha_s(1) = \cdots = \alpha_s(m_s - 1)$, $\alpha_{rs}(0, 0) = \alpha_{rs}(0, 1) = \cdots = \alpha_{rs}(m_r - 1, m_s - 1), \cdots,$ *and* $\alpha_{12\cdots n}(0, 0, \cdots, 0) = \alpha_{12\cdots n}(0, 0, \cdots, 1) = \cdots = \alpha_{12\cdots n}(m_1 - 1, m_2 - 1, \cdots, m_n - 1)$ *are distributed as* χ^2 *with* $m_s - 1$, $(m_r - 1)(m_s - 1), \cdots,$ *and* $\prod_{s=1}^n (m_s - 1)$ *degrees of freedom, respectively, and the quadratic forms* $\hat{\boldsymbol{\eta}}'W_s(i_s)(W_s(i_s)'S^{-1}W_s(i_s))^{-1}W_s(i_s)'\sigma^{-2}\hat{\boldsymbol{\eta}}$, $\hat{\boldsymbol{\eta}}'W_{rs}(i_r, i_s)(W_{rs}(i_r, i_s)' \times S^{-1}W_{rs}(i_r, i_s))^{-1}W_{rs}(i_r, i_s)'\sigma^{-2}\hat{\boldsymbol{\eta}}, \cdots,$ *and* $\hat{\boldsymbol{\eta}}'W(i_1, i_2, \cdots, i_n)(W(i_1, i_2, \cdots, i_n)'S^{-1} \times W(i_1, i_2, \cdots, i_n))^{-1}W(i_1, i_2, \cdots, i_n)'\sigma^{-2}\hat{\boldsymbol{\eta}}$ *subject to* $a_s(i_s) = 0$, $a_{rs}(i_r, i_s) = 0, \cdots,$ *and* $a_{12\cdots n}(i_1, i_2, \cdots, i_n) = 0$ *are distributed as* χ^2 *with one degree of freedom, respectively.*

PROOF. We shall prove one case, e.g., $\hat{\boldsymbol{\eta}}'W_s(W_s'S^{-1}W_s)^{-1}W_s'\sigma^{-2}\hat{\boldsymbol{\eta}}$: The other cases may be proved similarly.

Since $\hat{\boldsymbol{\eta}} = S^{-1}X'\mathbf{y}$, Cov $(\hat{\boldsymbol{\eta}}) = S^{-1}\sigma^2$, then $V(W_s'\hat{\boldsymbol{\eta}}) = W_s'S^{-1}W_s\sigma^2$, and $\alpha_s(0) = \alpha_s(1) = \cdots = \alpha_s(m_s - 1)$ implies $\mathbf{a}_s = 0$, i.e., $W_s'\boldsymbol{\eta} = 0$, so $E[W_s'\hat{\boldsymbol{\eta}}] = W_s'\boldsymbol{\eta} = 0$. Furthermore, the rank of the matrix $W_s'S^{-1}W_s$ is clearly $m_s - 1$. Hence, the quadratic form $\hat{\boldsymbol{\eta}}'W_s(W_s'S^{-1}W_s)^{-1}W_s'\sigma^{-2}\hat{\boldsymbol{\eta}}$ subject to $\alpha_s(0) = \alpha_s(1) = \cdots = \alpha_s(m_s - 1)$ is distributed as χ^2 with $m_s - 1$ degrees of freedom.

6.2. *Random effects case.* If all effects (except mean effect μ) in (6.3) are random, we may assume that all expectations of parameters are zero and that

$$\begin{aligned} E[\boldsymbol{\alpha}_s\boldsymbol{\alpha}_s'] &= I_{m_s}\sigma_s^2 \qquad \text{for} \quad s = 1, 2, \cdots, n\,, \\ E[\boldsymbol{\alpha}_{rs}\boldsymbol{\alpha}_{rs}'] &= I_{m_r m_s}\sigma_{rs}^2 \qquad \text{for} \quad r = 1, 2, \cdots, n-1\,; \\ & \qquad\qquad\qquad s = 2, 3, \cdots, n\,; \quad r < s\,, \\ & \qquad \cdots \\ E[\boldsymbol{\alpha}_{12\cdots n}\boldsymbol{\alpha}_{12\cdots n}'] &= I_{\prod_{s=1}^n m_s}\sigma_{12\cdots n}^2\,. \end{aligned}$$

The variances $\sigma_{b_0}^2, \sigma_{b_s}^2, \cdots, \sigma_{b_{12\cdots n}}^2$ for the $b_0, b_s(i_s), \cdots, b_{12\cdots n}(i_1, i_2, \cdots, i_n)$, respectively, may be calculated as follows:

$$\begin{aligned} \sigma_{b_0}^2 = E[b_0^2] - \mu^2 = \textstyle\sum_{s=1}^n m_s^{-1}\sigma_s^2 + \displaystyle\sum_{\substack{r \quad s \\ 1 \leq r < s \leq n}}\sum (m_r m_s)^{-1}\sigma_{rs}^2 + \cdots \\ + (\textstyle\prod_{i=1}^n m_s)^{-1}\sigma_{12\cdots n}^2\,, \end{aligned}$$

$$\sigma_{b_1}^2 = E[(b_s(i_s))^2] = \sigma_s{}^2 + \textstyle\sum_{u=1}^{s-1} m_u{}^{-1}\sigma_{us}^2 + \sum_{u=s+1}^{n} m_u{}^{-1}\sigma_{su}^2 + \cdots$$
$$+ \textstyle\prod_{i=1}^{s-1} m_i{}^{-1} \prod_{j=s+1}^{n} m_j{}^{-1}\sigma_{12\cdots n}^2 ,$$
$$\sigma_{b_{rs}}^2 = E[(b_{rs}(i_r, i_s))^2] = \sigma_{rs}^2 + \textstyle\sum_{u=1}^{r-1} m_u{}^{-1}\sigma_{urs}^2 + \sum_{u=r+1}^{s-1} m_u{}^{-1}\sigma_{rus}^2$$
$$+ \textstyle\sum_{u=s+1}^{n} m_u{}^{-1}\sigma_{rsu}^2 + \cdots + \prod_{i=1}^{r-1} m_i{}^{-1} \prod_{j=r+1}^{s-1} m_j{}^{-1} \prod_{k=1}^{n} m_k{}^{-1}\sigma_{12\cdots n}^2 ,$$
$$\cdots$$
$$\sigma_{b_{23\cdots n}}^2 = E[(b_{23\cdots n}(i_2, i_3, \cdots, i_n))^2] = \sigma_{23\cdots n}^2 + m_1{}^{-1}\sigma_{12\cdots n}^2 ,$$
$$\sigma_{b_{12\cdots n}}^2 = \sigma_{12\cdots n}^2 .$$

Under this random model, $E[\mathbf{y}] = \mu\mathbf{1}_N = \mu X\mathbf{1}_t$ and variance $V = XAE[\mathbf{bb}']A'X' - \mu^2 X\mathbf{1}_t\mathbf{1}_t{}'X' + I\sigma^2$. Hence,

$$V(\hat{\eta}) = S^{-1}X'VXS^{-1}$$
$$= AE[\mathbf{bb}']A' + \mathbf{11}'\mu^2 + S^{-1}\sigma^2 .$$

Note that

$$E[\mathbf{bb}'] = \operatorname{diag}(\mu^2 + \sigma_{b_0}^2, I_{(m_1-1)}\sigma_{b_1}^2, \cdots, I_{\prod_{s=2}^{n}(m_s-1)}\sigma_{b_{23\cdots n}}^2, I_{\prod_{s=1}^{n}(m_s-1)}\sigma_{12\cdots n}^2) .$$

Hence we have

$$V(W_s{}'\hat{\eta}) = (\textstyle\prod_{i=1}^{s-1} m_i \prod_{j=s+1}^{n} m_j) I_{(m_s-1)}\sigma_{b_s}^2 + W_s{}'S^{-1}W_s\sigma^2 ,$$

since $W^{*\prime}A = K$. Similarly,

$$V(W_{rs}'\hat{\eta}) = (\textstyle\prod_{i=1}^{r-1} m_i \prod_{j=r+1}^{s-1} m_j \prod_{k=s+1}^{n} m_k) I_{(m_r-1)(m_s-1)}\sigma_{b_{rs}}^2 + W_{rs}'S^{-1}W_{rs}\sigma^2 ,$$
$$\cdots$$
$$V(W_{23\cdots n}'\hat{\eta}) = m_1 I_{\prod_{s=2}^{n}(m_s-1)}\sigma_{23\cdots n}^2 + W_{23\cdots n}'S^{-1}W_{23\cdots n}\sigma^2 ,$$
$$V(W_{12\cdots n}'\hat{\eta}) = I_{\prod_{s=1}^{n}(m_s-1)}\sigma_{12\cdots n}^2 + W_{12\cdots n}'S^{-1}W_{12\cdots n}\sigma^2 .$$

Therefore, for the random effects case, *if we assume that* $\mathbf{y}$ *has a multivariate normal distribution with mean* $\mu\mathbf{1}_N$ *and variance* V, *then Theorem* 6.1 *may be restated with the restrictions that* $\sigma_{b_s}^2 = 0, \sigma_{b_{rs}}^2 = 0, \cdots, \sigma_{b_{12\cdots n}}^2 = 0$ *in the place of those used in the theorem.*

The expectations of sums of squares are:

$$E[SS(\mathbf{b}_s)] = \textstyle\prod_{i=1}^{s-1} m_i \prod_{j=s+1}^{n} m_j \operatorname{tr}[W_s{}'S^{-1}W_s]^{-1}\sigma_{b_s}^2 + (m_s - 1)\sigma^2 ,$$
$$E[SS(\mathbf{b}_{rs})] = \textstyle\prod_{i=1}^{r-1} m_i \prod_{j=r+1}^{s-1} m_j \prod_{k=s+1}^{n} m_k \operatorname{tr}[W_{rs}'S^{-1}W_{rs}]^{-1}\sigma_{b_{rs}}^2$$
$$+ (m_r - 1)(m_s - 1)\sigma^2 ,$$
$$\cdots$$
$$E[SS(b_{23\cdots n})] = m_1 \operatorname{tr}[W_{23\cdots n}'S^{-1}W_{23\cdots n}']^{-1}\sigma_{b_{23\cdots n}}^2 + \textstyle\prod_{s=2}^{n}(m_s - 1)\sigma^2 ,$$
$$E[SS(b_{12\cdots n})] = \operatorname{tr}[W_{12\cdots n}'S^{-1}W_{12\cdots n}]^{-1}\sigma_{12\cdots n}^2 + \textstyle\prod_{s=1}^{n}(m_s - 1)\sigma^2 .$$

6.3. *Mixed effects case.* Suppose that $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$, and $\boldsymbol{\alpha}_{12}$ are random effects and that other effects are fixed in (6.3). For this case, we do not assume that $\sum_{i_1}\alpha_1(i_1) = \sum_{i_2}\alpha_2(i_2) = 0$, $\sum_{i_1}\alpha_{12}(i_1, i_2) = \sum_{i_2}\alpha_{12}(i_1, i_2) = 0$, but we will assume that $\sum_{i_s=0}^{m_s-1}\alpha_s(i_s) = 0$ for $s \geqq 3$, $\sum_{i_s=0}^{m_s-1}\alpha_{ks}(i_k, i_s) = 0$ for $k = 1, 2$, $s \geqq 3$; $\sum_{i_s=0}^{m_s-1}\alpha_{12s}(i_1, i_2, i_s) = 0$ for $s \geqq 3$; $\cdots$; $\sum_{i_3=0}^{m_3-1}\alpha_{12\cdots n}(i_1, i_2, \cdots, i_n) = \sum_{i_n=0}^{m_n-1}\alpha_{12\cdots n}(i_1, i_2, \cdots, i_n) = 0$.

Then, $a_{1s}(i_1, 0) = 0$ for $s \geqq 3$, $a_{2s}(i_2, 0) = 0$ for $s \geqq 3$, $a_{1rs}(i_1, 0, 0) = 0$ for

$r \geqq 3$, $s \geqq 4$, $a_{1rs}(i_1, i_r, 0) = 0$ for $r \geqq 3$, $s \geqq 4, \cdots$, $a_{134\cdots n}(i_1, 0, 0, \cdots, 0) = 0, \cdots$, $a_{134\cdots n}(i_1, i_3, i_4, \cdots, i_{n-1}, 0) = 0$; $a_{2rs}(i_2, 0, 0) = 0$ for $r \geqq 3$, $s \geqq 4, \cdots$, $a_{234\cdots n}(i_2, i_3, i_3, \cdots, i_{n-1}, 0) = 0$; $a_{12s}(i_1, i_2, 0) = 0$ for $s \geqq 3, \cdots$, $a_{123\cdots n}(i_1, i_2, 0, \cdots, 0) = 0$, $a_{123\cdots n}(i_1, i_2, i_3, \cdots, i_{n-1}, 0) = 0$, but $a_1(0) \neq 0$, $a_2(0) \neq 0$, $a_{12}(0, 0) \neq 0$; $a_{12}(i_1, 0) \neq 0$; $a_{1s}(0, i_s) \neq 0$ for $s \geqq 3$ and $i_s \neq 0$, $a_{134\cdots n}(0, i_3, i_4, \cdots, i_n) \neq 0$, $a_{234\cdots n}(0, i_3, i_4, \cdots, i_n) \neq 0$ for $i_3 = \cdots = i_n \neq 0$; $a_{12s}(0, 0, i_s) \neq 0$ for $s \geqq 3$, $i_s \neq 0$ and $a_{123\cdots n}(0, 0, i_3, \cdots, i_n) \neq 0$ for $i_3 = \cdots = i_n \neq 0$. Also, since the other effects are fixed, we make the assumption that the effects sum to zero over the levels of any factor of the fixed effects.

In the above mixed model case, then, we have

$$\begin{aligned}
b_0 &= \mu + m_1^{-\frac{1}{2}} a_1(0) + m_2^{-\frac{1}{2}} a_2(0) + (m_1 m_2)^{-\frac{1}{2}} a_{12}(0, 0)\,, \\
\mathbf{b}_1 &= \mathbf{a}_1 + m_2^{-\frac{1}{2}} \mathbf{a}_{12}(i_1, 0)\,, \\
\mathbf{b}_2 &= \mathbf{a}_2 + m_1^{-\frac{1}{2}} a_{12}(0, i_2)\,, \\
\mathbf{b}_{12} &= \mathbf{a}_{12}\,, \\
\mathbf{b}_s &= \mathbf{a}_s + \textstyle\sum_{k=1}^{2} m_k^{-\frac{1}{2}} \mathbf{a}_{ks}(0, i_s) + \sum_{k=1}^{2} \sum_{u=s+1}^{n} (m_k m_u)^{-\frac{1}{2}} \mathbf{a}_{ksu}(0, i_s, 0) \\
&\quad + (m_1 m_2)^{-\frac{1}{2}} \mathbf{a}_{12s}(0, 0, i_s) + \cdots + (m_1 m_2 \textstyle\prod_{i=3}^{s-1} m_i \prod_{j=s+1}^{n} m_j)^{-\frac{1}{2}} \\
&\quad \times \mathbf{a}_{12\cdots n}(0, \cdots, i_s, 0, \cdots, 0) \qquad \text{for} \quad s \geqq 3\,, \\
\mathbf{b}_{ks} &= \mathbf{a}_{ks} \qquad \text{for} \quad k = 1, 2, \quad \text{and} \quad s \geqq 3\,, \\
&\cdots \\
\mathbf{b}_{k34\cdots n} &= \mathbf{a}_{k34\cdots n} \qquad \text{for} \quad k = 1, 2, \\
\mathbf{b}_{12s} &= \mathbf{a}_{12s} \qquad \text{for} \quad s \geqq 3\,, \\
&\cdots \\
\mathbf{b}_{123\cdots n} &= \mathbf{a}_{123\cdots n}\,, \\
\mathbf{b}_{su} &= \mathbf{a}_{su} + \textstyle\sum_{k=1}^{2} m_k^{-\frac{1}{2}} \mathbf{a}_{ksu}(0, i_s, i_u) + (m_1 m_2)^{-\frac{1}{2}} \mathbf{a}_{12su}(0, 0, i_s, i_u) + \cdots \\
&\quad + (m_1 m_2 \textstyle\prod_{i=3}^{s-1} m_i \prod_{j=s+1}^{u-1} m_j \prod_{k=u+1}^{n} m_k)^{-\frac{1}{2}} \\
&\quad \times \mathbf{a}_{12\cdots n}(0, 0, \cdots, i_s, \cdots, i_u, 0, \cdots, 0) \qquad \text{for} \quad s \geqq 3\,, \quad u > s\,, \\
&\cdots \\
\mathbf{b}_{34\cdots n} &= \mathbf{a}_{34\cdots n} + \textstyle\sum_{k=1}^{2} m_k^{-\frac{1}{2}} \mathbf{a}_{k34\cdots n}(0, i_3, i_4, \cdots, i_n) \\
&\quad + (m_1 m_2)^{-\frac{1}{2}} \mathbf{a}_{12\cdots n}(0, 0, i_3, i_4, \cdots, i_n)\,.
\end{aligned}$$

Also, in the above mixed model case, we may assume that $E[\mathbf{a}_1] = 0$, $E[\mathbf{a}_2] = 0$, $E[\mathbf{a}_{1s}] = 0$, $E[\mathbf{a}_{2s}] = 0$, $E[\mathbf{a}_{12s}] = 0, \cdots$, $E[\mathbf{a}_{12\cdots n}] = 0$. Further, we assume the effects $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$ and $\boldsymbol{\alpha}_{12}$ are mutually independent random vectors such that $E(\boldsymbol{\alpha}_k \boldsymbol{\alpha}_k') = I_{m_k} \sigma_k^2$ for $k = 1, 2$, and $E[\boldsymbol{\alpha}_{12} \boldsymbol{\alpha}_{12}'] = I_{m_1 m_2} \sigma_{12}^2$, then $E[\mathbf{a}_k \mathbf{a}_k'] = I_{(m_k - 1)} \sigma_k^2$ for $k = 1, 2$, and $E[\mathbf{a}_{12} \mathbf{a}_{12}'] = I_{(m_1-1)(m_2-1)} \sigma_{12}^2$.

Define $\sigma_{13}^2 = (m_3 - 1)^{-1} \sum_{i_3=0}^{m_3-1} \operatorname{Var}(a_{13}(i_1, i_3))$ and assume that $\operatorname{Var}(a_{13}(i_1, i_3))$ are identical for $i_3 = 0, 1, \cdots, m_3 - 1$, then we have $E[\mathbf{a}_{13} \mathbf{a}_{13}'] = I_{(m_1-1)(m_3-1)} \sigma_{13}^2$. By similarly defining σ_{23}^2, $\sigma_{123}^2, \cdots$, and $\sigma_{123\cdots n}^2$, we obtain $E[\mathbf{a}_{23} \mathbf{a}_{23}'] = I_{(m_2-1)(m_3-1)} \sigma_{23}^2$, $E[\mathbf{a}_{123} \mathbf{a}_{123}'] = I_{(m_1-1)(m_2-1)(m_3-1)} \sigma_{123}^2, \cdots$, $E[\mathbf{a}_{12\cdots n} \mathbf{a}_{12\cdots n}'] = I_{\prod_{s=1}^{n}(m_s-1)} \sigma_{12\cdots n}^2$. Furthermore, we assume that $\boldsymbol{\alpha}_{13}$, $\boldsymbol{\alpha}_{23}$, $\boldsymbol{\alpha}_{123}, \cdots$, and $\boldsymbol{\alpha}_{12\cdots n}$ are independent of each other.

In this case, we have

$$
\begin{aligned}
\sigma_{b_0}^2 &= E[b_0{}^2] - \mu^2 = m_1{}^{-1}\sigma_1{}^2 + m_2{}^{-1}\sigma_2{}^2 + (m_1 m_2)^{-1}\sigma_{12}^2\,,\\
\sigma_{b_1}^2 &= \sigma_1{}^2 + m_2{}^{-1}\sigma_{12}^2\,,\\
\sigma_{b_2}^2 &= \sigma_2{}^2 + m_1{}^{-1}\sigma_{12}^2\,,\\
\sigma_{b_{12}}^2 &= \sigma_{12}^2\,,\\
\sigma_{b_s}^2 &= \textstyle\sum_{k=1}^2 m_k{}^{-1}\sigma_{k3}^2 + \sum_{k=1}^2 \sum_{u=s+1}^n (m_k m_u)^{-1}\sigma_{ksu}^2\\
&\quad + (m_1 m_2)^{-1}\sigma_{123}^2 + \cdots + (m_1 m_2 \prod_{i=3}^{s-1} m_i \prod_{j=s+1}^n m_j)^{-1}\\
&\quad \times \sigma_{12\cdots n}^2\,, \qquad \text{for} \quad s \geqq 3\,,\\
\sigma_{b_{1s}}^2 &= \sigma_{1s}^2 \qquad \text{for} \quad s \geqq 3\,,\\
&\cdots\\
\sigma_{b_{34\cdots n}}^2 &= \textstyle\sum_{k=1}^2 m_k{}^{-1}\sigma_{k34\cdots n}^2 + \cdots + (m_1 m_2)^{-1}\sigma_{12\cdots n}^2\,.
\end{aligned} \tag{6.10}
$$

Now, let

$$
\begin{aligned}
V^* &= E[(\mathbf{y} - E\mathbf{y})(\mathbf{y} - E\mathbf{y})']\\
&= XA[E(\mathbf{bb}') - E(\mathbf{b})E(\mathbf{b}')]A'X' + I\sigma^2\,;
\end{aligned}
$$

then

$$
\begin{aligned}
V(\hat{\eta}) &= S^{-1}X'V^*XS^{-1}\\
&= A[E(\mathbf{bb}') - E(\mathbf{b})E(\mathbf{b}')]A' + S^{-1}\sigma^2\,.
\end{aligned}
$$

Using (6.10)

$$
\begin{aligned}
V(W_s{}'\hat{\eta}) &= (\textstyle\prod_{i=1}^{s-1} m_i \prod_{j=s+1}^n m_j) I_{(m_s-1)}\sigma_{b_s}^2 + W_s{}'S^{-1}W_s\sigma^2\,,\\
V(W_{rs}'\hat{\eta}) &= (\textstyle\prod_{i=1}^{r-1} m_i \prod_{j=r+1}^{s-1} m_j \prod_{k=s+1}^n m_k) I_{(m_r-1)(m_s-1)}\sigma_{b_{rs}}^2 + W_{rs}'S^{-1}W_{rs}\sigma^2\,,\\
&\cdots\\
V(W_{23\cdots n}'\hat{\eta}) &= m_1 I_{\prod_{s=2}^n(m_s-1)}\sigma_{23\cdots n}^2 + W_{23\cdots n}'S^{-1}W_{23\cdots n}\sigma^2\,,\\
V(W_{12\cdots n}'\hat{\eta}) &= I_{\prod_{s=1}^n(m_s-1)}\sigma_{12\cdots n}^2 + W_{12\cdots n}'S^{-1}W_{12\cdots n}\sigma^2\,.
\end{aligned}
$$

Thus, for this case also Theorem 6.1 *can be restated subject to proper restrictions.*

The expectations of the various sums of squares are given below:

$$
\begin{aligned}
E[SS(\mathbf{b}_1)] &= E[\hat{\eta}'W_1(W_1{}'S^{-1}W_1)^{-1}W_1{}'\hat{\eta}]\\
&= \operatorname{tr}[W_1(W_1{}'S^{-1}W_1)^{-1}W_1{}'E(\hat{\eta}\hat{\eta}')]\,,
\end{aligned}
$$

where

$$
E[\hat{\eta}\hat{\eta}'] = AE(\mathbf{bb}')A' + S^{-1}\sigma^2\,;
$$

then

$$
\begin{aligned}
E[SS(\mathbf{b}_1)] &= \operatorname{tr}[(W_1{}'S^{-1}W_1)^{-1}(\textstyle\prod_{s=2}^n m_s)^{\frac12} W_1{}'W_1(\mathbf{b}_1\mathbf{b}_1{}')(\prod_{s=2}^n m_s)^{\frac12}W_1{}'W_1] + (m_1 - 1)\sigma^2\\
&= (\textstyle\prod_{s=2}^n m_s)\operatorname{tr}[W_1{}'S^{-1}W_1]^{-1}\sigma_{b_1}^2 + (m_1 - 1)\sigma^2\,.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
E[SS(\mathbf{b}_2)] &= (m_1 \textstyle\prod_{s=3}^n m_s)\operatorname{tr}[W_2{}'S^{-1}W_2]^{-1}\sigma_{b_2}^2 + (m_2 - 1)\sigma^2\,,\\
E[SS(\mathbf{b}_{12})] &= (\textstyle\prod_{s=3}^n m_s)\operatorname{tr}[W_{12}'S^{-1}W_{12}]^{-1}\sigma_{12}^2 + (m_1 - 1)(m_2 - 1)\sigma^2\,,\\
E[SS(\mathbf{b}_s),\ s \geqq 3] &= (m_1 m_2 \textstyle\prod_{i=3}^{s-1} m_i \prod_{j=s+1}^n m_j)\{\operatorname{tr}[(W_s{}'S^{-1}W_s)^{-1}\mathbf{a}_s\mathbf{a}_s{}']\\
&\quad + \operatorname{tr}[W_s{}'S^{-1}W_s]^{-1}\sigma_{b_s}^2\} + (m_s - 1)\sigma^2\,,
\end{aligned}
$$

$$E[SS(\mathbf{b}_{1s}), s \geqq 3] = (m_2 \textstyle\prod_{i=3}^{s-1} m_i \prod_{j=s+1}^{n} m_j) \operatorname{tr} [W_{1s}' S^{-1} W_{1s}]^{-1} \sigma_{1s}^2 + (m_1 - 1)(m_s - 1)\sigma^2 ,$$

$$\cdots$$

$$E[SS(\mathbf{b}_{34\cdots n})] = (m_1 m_2)\{\operatorname{tr} [(W_{34\cdots n}' S^{-1} W_{34\cdots n})^{-1} \mathbf{a}_{34\cdots n} \mathbf{a}_{34\cdots n}'] + \operatorname{tr} [W_{34\cdots n}' S^{-1} W_{34\cdots n}]^{-1} \sigma_{b_{34\cdots n}}^2\} + \textstyle\prod_{s=3}^{n} (m_s - 1)\sigma^2 ,$$

$$\cdots$$

$$E[SS(\mathbf{b}_{23\cdots n})] = m_1 \operatorname{tr} [W_{23\cdots n}' S^{-1} W_{23\cdots n}]^{-1} \sigma_{23\cdots n}^2 + \textstyle\prod_{s=2}^{n} (m_s - 1)\sigma^2 ,$$

$$E[SS(\mathbf{b}_{12\cdots n})] = \operatorname{tr} [W_{12\cdots n}' S^{-1} W_{12\cdots n}]^{-1} \sigma_{12\cdots n}^2 + \textstyle\prod_{s=1}^{n} (m_s - 1)\sigma^2 .$$

In the mixed model (4.1), suppose that $\boldsymbol{\eta}$ has an n-factor factorial structure and $\mathbf{y}$ is a vector of random variables following a multivariate normal distribution with mean $X[\mathbf{0}', \boldsymbol{\eta}']'$ and variance $V = X_1'X_1\sigma_\beta^2 + I_N\sigma^2$; we now state the following lemma without proof (see Graybill [5]).

LEMMA 6.1. *If* $\mathbf{y}$ *is distributed* $N(\mathbf{m}, V)$, *then* $\mathbf{y}'B\mathbf{y}$ *is distributed as* χ^2 *with* k *degrees of freedom if and only if* $m'Bm = 0$ *and* BV *is an idempotent matrix of rank* k.

THEOREM 6.2. *In the case of equation* (4.1), *Theorem* 6.1 *can be restated by replacing* S^{-1} *by* G.

PROOF. For example, in the case of $\hat{\eta}'W_s(W_s'GW_s)^{-1}W_s'\sigma^{-2}\hat{\eta}$,

$$B = [I - X_1(X_1'X_1)X_1']X_2 = GW_s(W_s'GW_s)^{-1}W_s'GX_2'[I - X_1(X_1'X_1)^{-1}X_1']\sigma^{-2}$$

(see Searle (1966)), then

$$BV = [I - X_1(X_1'X_1)X_1']X_2GW_s(W_s'GW_s)^{-1}W_sGX_2'[I - X_1(X_1'X_1)^{-1}X_1'] .$$

It may be easily verified that

$$(BV)(BV) = BV ,$$

$$(E\mathbf{y})'B(E\mathbf{y}) = 0 \qquad \text{if} \quad \mathbf{a}_1 = W_s'\eta = 0 ,$$

and the rank of BV is $m_s - 1$. Then, by Lemma 6.1 $\hat{\eta}'W_s(W_s'GW_s)^{-1}W_s\sigma^{-2}\hat{\eta}$ subject to restriction $W_s'\eta = 0$ is distributed as χ^2 with $m_s - 1$ degrees of freedom. The other cases may be proved similarly.

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