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MAXIMUM LIKELIHOOD ESTIMATION OF TRANSLATION PARAMETER OF TRUNCATED DISTRIBUTION II[1]

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Let f be a density which vanishes for negative values of its argument and varies regularly with exponent $\alpha - 1$ at zero, where $1 < \alpha < 2$. Further, let f_θ denote f translated by θ. We find and study the asymptotic distribution of the MLE $\hat{\theta}_n$ based on a sample size n as $n \to \infty$.

1. Introduction. Let $X_1, \cdots, X_n$ be independent random variables with common density f_θ, where θ is unknown and

$$f_\theta(x) = f(x - \theta), \qquad -\infty < x, \theta < \infty .$$

We shall consider here the case that f is a known, uniformly continuous density which vanishes on the interval $(-\infty, 0)$ and is positive on the interval $(0, \infty)$, and we will be particularly interested in the case that

$$f(x) \sim \alpha x^{\alpha-1} L(x) \qquad \text{as } x \to 0, \tag{1.1}$$

where $1 < \alpha < 2$ and $L(x)$ varies slowly as $x \to 0$. In particular, this includes the case that

$$f(x) \sim c\alpha x^{\alpha-1} \qquad \text{as } x \to 0 \tag{1.1$'$}$$

with $c > 0$ and $1 < \alpha < 2$.

Let $\hat{\theta}_n$ denote the MLE (maximum likelihood estimate) of θ and let γ_n be a sequence of positive numbers for which

$$n\gamma_n{}^\alpha L(\gamma_n) \to 1, \tag{1.2}$$

where L is as in (1.1). (We may take $\gamma_n{}^{-\alpha} = nc$ in the special case that $L(x) \to c > 0$ as $x \to 0$.) In this paper we shall show that

$$(\hat{\theta}_n - \theta)/\gamma_n$$

has a limiting distribution H, under some regularity conditions which imply (1.1). We shall also study this limiting distribution function.

It is interesting to remark that the minimum

$$M_n = \min(X_1, \cdots, X_n)$$

also converges to θ at the rate γ_n if relation (1.1) is satisfied. In fact, it is easily deduced from Lemma 4.1 of this paper and Example (b) of [5], page 270, that

$$\lim \Pr[\gamma_n{}^{-1}(M_n - \theta) > t] = e^{-t^\alpha} \tag{1.3}$$

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for every $t > 0$ if (1.1) is satisfied. The case that relation (1.1) is satisfied, therefore, stands in contrast to the case that either $\beta = \lim f'(x)$ exists as $x \to 0$ and $0 < \beta < \infty$ or the Fisher Information is finite, for in these cases the MLE converges to θ strictly faster than does M_n ([8] and [9]).

2. Conditions and theorems. We shall impose the following regularity conditions on f.

C_1: f is a uniformly continuous density which vanishes on $(-\infty, 0)$ and is positive on $(0, \infty)$.

C_2: f is continuously differentiable on $(0, \infty)$ with derivative f'; and f' is absolutely continuous on every compact subinterval of $(0, \infty)$ with derivative f''.

C_3: C_2 is satisfied, and $f''(x) = -\alpha(\alpha - 1)(2 - \alpha)x^{\alpha-3}L(x)$ where $L(x)$ varies slowly as $x \to 0$.

If C_1 is satisfied, then $g(x) = \log f(x)$ is well defined for $x > 0$. Moreover, if both C_1 and C_2 are satisfied, then g will be continuously differentiable on $(0, \infty)$ with derivative $g' = f'/f$; and g'' will be absolutely continuous on every compact subinterval of $(0, \infty)$ with derivative $g'' = (ff'' - f'^2)/f^2$. We also require

C_4: $$\int_0^\infty -g(x)f(x)\,dx < \infty\,.$$

C_5: For every $\delta > 0$, there is an $\varepsilon > 0$, for which

$$\int_\delta^\infty \sup_{|s| \leq \varepsilon} (g'(x - s)^2 + |g''(x - s)|)f(x)\,dx < \infty\,.$$

Conditions C_1 and C_4 insure the existence and consistency of the MLE (see below), and condition C_3 is simply (1.1) differentiated twice (see Lemma 4.1). Conditions C_2 and C_5 are similar to the classical conditions of Cramér ([4], page 500).

If g'' is continuous, then we may replace "for every $\delta > 0$" in C_5 by "for some $\delta > 0$."

EXAMPLE 1. If f is a Gamma density, say

$$f(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1}e^{-x}\,, \qquad x > 0\,,$$

where $1 < \alpha < 2$, then conditions $C_1, \cdots, C_5$ are all satisfied.

EXAMPLE 2. They are also satisfied if f is a Pareto density, say

$$f(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{x^{\alpha-1}}{(1 + x)^{\alpha+\beta}}\,, \qquad x > 0\,,$$

where $1 < \alpha < 2$ and $\beta > 0$.

EXAMPLE 3. If $f(x) \sim a/x \log^2 x$ as $x \to \infty$, then condition C_4 is violated.

If condition C_1 is satisfied, then the likelihood function

$$L_n(t) = \prod_{i=1}^n f_t(X_i)$$

will attain its maximum at a point $\hat{\theta}_n$ in the interval $(-\infty, M_n)$, and it is easily seen that $\hat{\theta}_n$ may be selected to depend on $X_1, \cdots, X_n$ in a measurable manner. Moreover, if C_4 is also satisfied, then $\hat{\theta}_n$ will be a consistent sequence of estimates of θ ([7]).

Let $G_n(t) = -\log L_n(t)$ for $t < M_n$. If conditions C_1, C_2 and C_4 are all satisfied, then $\hat{\theta}_n$, $n \geqq 1$, will form a consistent sequence of roots of the likelihood equation

$$\hat{\theta}_n < M_n \quad \text{and} \quad G_n'(\hat{\theta}_n) = 0 . \tag{2.1}$$

In Section 4 we shall prove the following lemma.

LEMMA 2.1. *Let conditions* C_1, C_2, C_3, *and* C_5 *be satisfied. Then, for sufficiently small* $\varepsilon > 0$, *there are events* A_n, $n = 1, 2, \cdots$ *for which* $\lim P(A_n) = 1$ *as* $n \to \infty$ *and* A_n *implies for* $n = 1, 2, \cdots$

$$G_n''(t) > 0 , \qquad \theta - \varepsilon \leqq t < M_n .$$

Now suppose that $C_1, \cdots, C_5$ are all satisfied. Then with probability approaching one, G_n' will be an increasing function on the interval $[\theta - \varepsilon, M_n)$ for sufficiently small $\varepsilon > 0$, and $\hat{\theta}_n$, $n \geqq 1$ will be a consistent sequence of roots of the likelihood equation (2.1). It follows easily that as $n \to \infty$

$$\Pr(\hat{\theta}_n \leqq t) = \Pr(G_n'(t) \geqq 0) + o(1) , \tag{2.2a}$$

where $o(1)$ is uniform in t for $\theta - \varepsilon \leqq t \leqq \theta$ and

$$\Pr(\theta_n > t) = \Pr(G_n'(t) < 0, M_n > t) + o(1) , \tag{2.2b}$$

where $o(1)$ is uniform in t for $t > \theta$. Let γ_n be chosen as in (1.2). Then, relation (2.2) may also be written as

$$\Pr[\gamma_n^{-1}(\hat{\theta}_n - \theta) \leqq -t] = \Pr[Z_{nt} \geqq 0] + o(1) , \tag{2.3a}$$

$$\Pr[\gamma_n^{-1}(\hat{\theta}_n - \theta) > t] = \Pr[Z_{nt}^* < 0 \mid M_n^* > t] \Pr[M_n^* > t] + o(1) , \tag{2.3b}$$

as $n \to \infty$ for $t \geqq 0$ and $t > 0$, respectively, where

$$\begin{aligned} Z_{nt} &= t\gamma_n G_n'(\theta - t\gamma_n) , & t > 0 , \\ Z_{n0} &= \gamma_n G_n'(\theta) , & \\ Z_{nt}^* &= t\gamma_n G_n'(\theta + t\gamma_n) , & 0 < t < M_n^* , \\ M_n^* &= (M_n - \theta)/\gamma_n . & \end{aligned}$$

The Z_{nt}^* are only defined on the event that $M_n^* > t$.

The limiting distribution of $(\hat{\theta}_n - \theta)/\gamma_n$ may now be deduced from the following theorems.

THEOREM 2.1. *Let conditions* C_1, C_2, C_3, *and* C_5 *be satisfied. Then* Z_{n0} *converges in distribution to a random variable* Z_0 *which has a stable distribution. The characteristic function of* Z_0 *is given by*

$$E(e^{i\lambda Z_0}) = \exp\left\{-d|\lambda|^\alpha\left(1 + i\,\mathrm{sign}(\lambda)\tan\left(\frac{\pi\alpha}{2}\right)\right)\right\} , \tag{2.4}$$

where $d > 0$.

THEOREM 2.2. *Let conditions* C_1, C_2, C_3, *and* C_5 *be satisfied. Then, for* $t > 0$, Z_{nt} *converges in distribution to a random variable* Z_t *with characteristic function*

$$E(e^{i\lambda Z_t}) = e^{-t^\alpha \Psi(\lambda)}, \qquad \lambda \in R, \tag{2.5}$$

where

$$\Psi(\lambda) = i\lambda m_\alpha + \int_0^{\alpha-1} [e^{i\lambda x} - 1 - i\lambda x]\, dF_\alpha(x)$$

with

$$m_\alpha = \alpha\Gamma(\alpha)\Gamma(2-\alpha)$$

and

$$F_\alpha(x) = \left[\frac{\alpha-1}{x} - 1\right]^\alpha$$

for $0 < x < \alpha - 1$.

THEOREM 2.3. *Let conditions* C_1, C_2, C_3, *and* C_5 *be satisfied. Then, for* $t > 0$, *the conditional distribution function of* Z_{nt}^*, *given* $M_n^* > t$, *converges completely to the distribution function of a random variable* Z_t^* *with characteristic function*

$$E(e^{i\lambda Z_t^*}) = e^{t^\alpha \Psi^*(\lambda)} \tag{2.6}$$

where

$$\Psi^*(\lambda) = i\lambda m_\alpha^* - \int_0^\infty [e^{i\lambda x} - 1 - i\lambda \sin(x)]\, dF_\alpha^*(x)$$

with

$$m_\alpha^* = \alpha \int_1^\infty \left[\sin\left(\frac{\alpha-1}{x-1}\right) - \frac{\alpha-1}{x}\right] x^{\alpha-1}\, dx - \alpha$$

and

$$F_\alpha^*(x) = \left[\frac{\alpha-1}{x} + 1\right]^\alpha, \qquad x > 0.$$

Lemma 2.1 and Theorems 2.1, 2.2, and 2.3 will be proved in Section 4. As a corollary to them, we shall now prove

THEOREM 2.4. *Let conditions* $C_1, \cdots, C_5$ *be satisfied. Then*

$$(\hat{\theta}_n - \theta)/\gamma_n$$

has a limiting distribution function H *as* $n \to \infty$, *where*

$$H(-t) = \Pr(Z_t \geqq 0), \qquad t \geqq 0,$$
$$1 - H(t) = \Pr(Z_t^* < 0)e^{-t^\alpha}, \qquad t > 0,$$

with Z_t *and* Z_t^* *as in the statements of Theorems* 2.1, 2.2, *and* 2.3.

PROOF. In view of equation (2.3), it will suffice to show that the distribution functions of Z_t and Z_t^* are continuous at zero. This result is well known for Z_0. For Z_t, $t > 0$, it may be established as follows.

$$\mathscr{R}\Psi(\lambda) = \alpha(\alpha-1)\int_0^{\alpha-1}[1-\cos(\lambda x)]\left[\frac{\alpha-1}{x} - 1\right]^{\alpha-1}\frac{1}{x^2}\,dx$$
$$= \alpha(\alpha-1)|\lambda|^\alpha \int_0^{(\alpha-1)|\lambda|}(1-\cos(x))\left[\frac{\alpha-1}{x} - \frac{1}{|\lambda|}\right]^{\alpha-1}\frac{1}{x^2}\,dx$$

for $\lambda \neq 0$. Therefore, as $|\lambda| \to \infty$,

$$|\lambda|^{-\alpha} \cdot \mathscr{R}\Psi(\lambda) \to \alpha(\alpha - 1)^{\alpha} \int_0^{\infty} (1 - \cos(x)) \frac{1}{x^{\alpha+1}} dx ,$$

which is positive. It follows that $\exp(-t^{\alpha}\Psi)$ is integrable, so that the distribution of Z_t is, in fact, absolutely continuous.

A similar argument will establish the absolute continuity of the distribution of Z_t^* to complete the proof of the theorem.

3. The limiting distribution: efficiency. In this section we shall study the limiting distribution function H of Theorem 2.4 and the efficiency of $\hat{\theta}_n$.

The probability that a stable random variable exceeds 0 was computed by Zolotorov [10] for configurations of the parameters that include (2.4). Applying his result to Z_0 yields

$$H(0) = \Pr(Z_0 \geqq 0) = \alpha^{-1} .$$

Thus, $H(0)$ always exceeds one half and $H(0) \to 1$ as $\alpha \to 1$.

We shall now study the left tail of H. The function Ψ of Theorem 2.2 admits an analytic extension to the entire complex plane. Therefore, for $t > 0$, Z_t has a moment generating function given by

$$E(e^{\lambda Z_t}) = e^{-t^{\alpha}\phi(\lambda)} , \qquad \lambda \in R ,$$

where

$$\phi(\lambda) = m_{\alpha} \lambda + \int_0^{\alpha-1} (e^{\lambda x} - 1 - \lambda x) \, dF_{\alpha}(x)$$

with m_{α} and F_{α} as in the statement of Theorem 2.2. In particular, the mean and variance of Z_t are

$$E(Z_t) = -t^{\alpha} m_{\alpha} \qquad \text{and} \qquad D(Z_t) = t^{\alpha}\sigma_{\alpha}^2$$

where

$$\sigma_{\alpha}^2 = \int_0^{\alpha-1} -x^2 \, dF_{\alpha}(x) = \alpha(\alpha - 1)^2\Gamma(\alpha)\Gamma(2 - \alpha) . \tag{3.1}$$

(To evaluate the integral, make the change of variable $x = (\alpha - 1)/(1 + y)$.) Let

$$\rho = \max_{\lambda} \phi(\lambda) .$$

Then

$$\begin{aligned} \rho &\geqq \max_{\lambda} [m_{\alpha} \lambda - \tfrac{1}{2}\lambda^2\sigma_{\alpha}^2 e^{\lambda(\alpha-1)}] \\ &\geqq \tfrac{1}{2} m_{\alpha}/\alpha(\alpha - 1) \end{aligned} \tag{3.2}$$

by an obvious Taylor Series expansion, and

$$H(-t) \leqq e^{-t^{\alpha}\rho} , \qquad t > 0 ,$$

by Bernstein's Inequality. In fact, the latter estimate is precise in the following sense.

THEOREM 3.1. *$\lim t^{-\alpha} \log H(-t) = -\rho$ as $t \to \infty$.*

PROOF. When t^{α} is an integer, Z_t has the distribution of the sum of t^{α} independent random variables with common moment generating function $\exp(-\phi)$.

It follows easily from a standard theorem on large deviations (e.g., [3], pages 1017–1018) that $\lim t^{-\alpha} \log H(-t) = -\rho$ as $t \to \infty$ through values of t for which t^α is an integer. For general $t > 0$, choose t_1 and t_2 for which $t_1 \leqq t < t_2$, t_1^α and t_2^α are integers, and $t_2^\alpha - t_1^\alpha = 1$. Then,

$$t^{-\alpha} \log H(-t) \leqq t_2^{-\alpha} \log H(-t_1) = \left(\frac{t_1}{t_2}\right)^\alpha t_1^{-\alpha} \log H(-t_1),$$

so that $\limsup t^{-\alpha} \log H(-t) \leqq -\rho$ as $t \to \infty$. A similar argument will show that $\liminf t^{-\alpha} \log H(-t) \geqq -\rho$ to complete the proof.

Analysis of the right tail is similar. The function Ψ^* of Theorem 2.3 admits an analytic extension to the half plane $\mathscr{I}(\lambda) > 0$. Therefore, for $t > 0$, Z_t^* has a Laplace transform given by

$$E(e^{-\lambda Z_t^*}) = e^{t^\alpha \phi^*(\lambda)}$$

where

$$\phi^*(\lambda) = -m_\alpha^* \lambda - \int_0^\infty [e^{-\lambda x} - 1 + \lambda \sin(x)]\, dF_\alpha^*(x)$$

with m_α^* and F_α^* as in the statement of Theorem 2.3.

We shall need to know the behavior of $\phi^*(\lambda)$ as $\lambda \to 0$.

LEMMA 3.1. *As* $\lambda \to 0$, $\phi^*(\lambda) \sim -\alpha(\alpha - 1)\lambda \log \lambda^{-1}$.

PROOF. For $\lambda > 0$, we may write

$$\begin{aligned}\phi^*(\lambda) &= -m_\alpha^*\lambda + \alpha(\alpha-1)\int_0^\infty [e^{-\lambda x} - 1 + \lambda\sin(x)]\left[\frac{\alpha-1}{x} + 1\right]^{\alpha-1} \frac{1}{x^2}\, dx \\ &= -m_\alpha^*\lambda + \alpha(\alpha-1)\lambda \int_0^\infty (e^{-x} - 1 + \sin(x))\left[\frac{(\alpha-1)\lambda}{x} + 1\right]^{\alpha-1}\frac{1}{x^2}\, dx \\ &\quad + \alpha(\alpha-1)\lambda\int_0^\infty [\lambda \sin(x\lambda^{-1}) - \sin(x)]\left[\frac{(\alpha-1)\lambda}{x} + 1\right]^{\alpha-1}\frac{1}{x^2}\, dx \\ &= -m_\alpha^*\lambda + \alpha(\alpha-1)\lambda[I_1 + I_2], \quad \text{say.}\end{aligned}$$

Let $b_\lambda(x)$ denote the integrand in I_2. Then, simple applications of the dominated convergence theorem yield

$$\lim I_1 = \int_0^\infty (e^{-x} - 1 + \sin(x)) \frac{1}{x^2}\, dx \tag{3.3}$$

and

$$\lim \int_\delta^\infty b_\lambda(x)\, dx = \int_\delta^\infty -\sin(x)\frac{1}{x^2}\, dx \tag{3.4}$$

as $\lambda \to 0$ for any $\delta > 0$. In particular, the left sides of (3.3) and (3.4) remain bounded as $\lambda \to 0$ for any $\delta > 0$. Similarly, since $|\sin(x) - x| \leqq x^3$ for $x > 0$,

$$\left|\int_0^{a\lambda} b_\lambda(x)\, dx\right| \leqq \int_0^{a\lambda} (1 + \lambda^{-2})\left[\frac{(\alpha-1)\lambda}{x} + 1\right]^{\alpha-1} x\, dx, \tag{3.5}$$

which remains bounded as $\lambda \to 0$ for any $a > 0$.

Let us now consider $\int_{a\lambda}^{\delta} b_\lambda \, dx$. Clearly,

$$\int_{a\lambda}^{\delta} |\lambda \sin (x\lambda^{-1})| \left[\frac{(\alpha - 1)\lambda}{x} + 1\right]^{\alpha-1} \frac{1}{x^2} \, dx \tag{3.6}$$

$$\leqq \left(\frac{\alpha - 1}{a} + 1\right)^{\alpha-1} \lambda \int_{a\lambda}^{\delta} x^{-2} \, dx \leqq \frac{1}{a} \left(\frac{\alpha - 1}{a} + 1\right)^{\alpha-1}$$

for any $a > 0$ and $\delta > 0$. Let $1 - \varepsilon$ be the minimum of $\sin (x)/x$ for $0 < x \leqq \delta$. Then, we have the inequalities

$$\int_{a\lambda}^{\delta} \sin (x) \left[\frac{(\alpha - 1)\lambda}{x} + 1\right]^{\alpha-1} \frac{1}{x^2} \, dx \tag{3.7a}$$

$$\geqq (1 - \varepsilon) \int_{a\lambda}^{\delta} x^{-1} \, dx = (1 - \varepsilon)[\log \delta - \log a\lambda]$$

and

$$\int_{a\lambda}^{\delta} \sin (x) \left[\frac{(\alpha - 1)\lambda}{x} + 1\right]^{\alpha-1} \frac{1}{x^2} \, dx \tag{3.7b}$$

$$\leqq \left(\frac{\alpha - 1}{a} + 1\right)^{\alpha-1} [\log \delta - \log a\lambda]$$

for $\delta > 0$ and $a > 0$. Since a and δ are arbitrary, it now follows easily from (3.4)—(3.7) that $I_2 \sim -\log \lambda^{-1}$ as $\lambda \to 0$; and since I_1 remains bounded as $\lambda \to 0$ by (3.3), the lemma follows.

As a consequence of Lemma 3.1 we shall now prove

THEOREM 3.2. *For every* $k > 0$, $\Pr (Z_t^* \leqq 0) = O(t^{-k})$ *as* $t \to \infty$.

PROOF. For every $\lambda \geqq 1$, we have

$$\Pr (Z_t^* \leqq 0) \leqq e^{t^\alpha \phi^*(\lambda t^{-\alpha})}$$

for all $t > 0$ by Bernstein's Inequality. Moreover, by Lemma 3.1

$$t^\alpha \phi^*(\lambda t^{-\alpha}) \leqq -\tfrac{1}{2}\alpha^2(\alpha - 1)\lambda \log t$$

for sufficiently large $t > 0$. The theorem follows easily.

It is always hard to compare distributions of different shapes, and it is especially hard when one of the distributions is as complicated as is H. Nevertheless, we shall attempt a comparison of H with the limiting distribution (1.3). The following lemma is relevant.

LEMMA 3.2. *Let* ρ *be as in Theorem* 3.1. *Then,* $\rho > 1$ *for all* α, $1 < \alpha < 2$, *and* $\rho \to \infty$ *as* $\alpha \to 1$ *or* $\alpha \to 2$.

PROOF. By equations (3.1) and (3.2), we have

$$\rho \geqq \frac{\Gamma(\alpha)\Gamma(2 - \alpha)}{2(\alpha - 1)}$$

which diverges to ∞ as $\alpha \to 1$ or $\alpha \to 2$. Moreover, since $(\alpha - 1)(2 - \alpha) \leqq \frac{1}{4}$ for $1 < \alpha < 2$, we have $\rho \geqq 2\Gamma(\alpha)\Gamma(3 - \alpha)$. Finally, since $\Gamma(x) \geqq .88$ for all $x > 0$ ([1], page 259), the lemma follows.

To compare $\hat{\theta}_n$ with M_n, we shall use the following definition of asymptotic relative efficiency, which is adapted from [2]. For each $n \geqq 1$, let $T_n = T_n(X_1, \cdots, X_n)$ be a translation invariant estimate of θ and suppose that T_n, $n \geqq 1$, is a consistent sequence of estimates. For $\varepsilon > 0$ and $0 < \delta < 1$, let $\alpha_n(\varepsilon) = \Pr(|T_n - \theta| \geqq \varepsilon)$ and

$$N(\varepsilon, \delta) = \text{least} \quad n \geqq 1 \quad \text{for which} \quad \alpha_n(\varepsilon) \leqq \delta .$$

Thus, $N(\varepsilon, \delta)$ is the sample size required to attain a fixed precision with confidence at least $1 - \delta$. If T_n', $n \geqq 1$, is another consistent sequence of translation invariant estimates, then we define the asymptotic relative δ-efficiency of T_n with respect to T_n', $n \geqq 1$, to be

$$\text{eff}(\delta) = \lim_{\varepsilon \to 0} N'(\varepsilon, \delta)/N(\varepsilon, \delta)$$

provided, of course that the limit exists. The following lemma will be useful in computations.

LEMMA 3.3. *Let $T_n = T_n(X_1, \cdots, X_n)$ be a consistent sequence of translation invariant estimates of θ. Let $a_n \to \infty$ with a_n/a_{n+1} as $n \to \infty$ and suppose that $a_n(T_n - \theta)$ has a limiting distribution function K. Let $K_0(x) = K(-x) + (1 - K(x))$ for $x \geqq 0$ and suppose that K_0 is continuous and strictly increasing on $[0, \infty)$. Then*

$$\lim \varepsilon a_{N(\varepsilon, \delta)} = K_0^{-1}(\delta)$$

as $\varepsilon \to 0$ for every δ, $0 < \delta < 1$.

The proof of the lemma is quite pedestrian and will be omitted. It follows from the lemma that if $n^{\frac{1}{2}}(T_n - \theta)$ is asymptotically normal with mean 0 and variance v, and if $n^{\frac{1}{2}}(T' - \theta)$ is asymptotically normal with mean 0 and variance v', then $\text{eff}(\delta) = v'/v$ for $0 < \delta < 1$.

We shall now compare the MLE $\hat{\theta}_n$ with the minimum M_n. More generally, we shall compare $T_n = \hat{\theta}_n$ with systematic statistics of the form

$$T_n' = \textstyle\sum_{i=1}^{k} c_{ni} X_{ni} - d_n , \tag{3.8}$$

where $X_{n1}, \cdots, X_{nn}$ are the order statistics of $X_1, \cdots, X_n$, $c_{n1}, \cdots, c_{nk}$ are non-negative constants for which $c_{n1} + \cdots + c_{nk} = 1$, and d_n are constants. Estimates of the form (3.8) were considered in [6] under regularity conditions which are compatible with ours, and the following result may be deduced from [6], pages 46–56. If conditions C_1, C_2, and C_3 are satisfied, if $d_n/\gamma_n \to d$, and if $(c_{n1}, \cdots, c_{nk}) \to (c_1, \cdots, c_k)$ as $n \to \infty$, then $(T_n' - \theta)/\gamma_n$ converges in distribution to

$$Y = \textstyle\sum_{j=1}^{k} c_j S_j^{1/\alpha} - d ,$$

where $S_j = E_1 + \cdots + E_j$ and $E_1, \cdots, E_k$ are independent standard exponential random variables.

THEOREM 3.3. *Let conditions $C_1, \cdots, C_5$ be satisfied. Suppose also that $(c_{n1}, \cdots, c_{nk}) \to (c_1, \cdots, c_k)$ and $d_n/\gamma_n \to d$ as $n \to \infty$. Then the asymptotic relative*

δ-efficiency of $T_n = \hat{\theta}_n$ with respect to T_n' is

$$\text{eff}(\delta) = \left(\frac{J_0^{-1}(\delta)}{H_0^{-1}(\delta)}\right)^\alpha, \tag{3.9}$$

for $0 < \delta < 1$, where J denotes the distribution function of Y. Moreover, $\liminf \text{eff}(\delta) \geqq 1$ *as* $\delta \to 0$.

PROOF. That $\text{eff}(\delta)$ exists and is given by (3.9) follows immediately from Lemma 3.3 and the fact that $n\gamma_n^\alpha$ varies slowly as $n \to \infty$.

To see that $\liminf \text{eff}(\delta) \geqq 1$ as $\delta \to 0$, observe first that, by Theorems 3.1 and 3.2 and Lemma 3.2, we have

$$H_0(t) = H(-t) + (1 - H(t)) = o(e^{-t^\alpha})$$

as $t \to \infty$. Thus, $\limsup H_0^{-1}(\delta)^\alpha/(-\log \delta) \leqq 1$ as $\delta \to 0$. Also,

$$Y > \sum_{j=1}^k c_j E_1^{1/\alpha} - d,$$

so that $J_0(t) \geqq \Pr(Y > t) \geqq \exp\{-(t + d)^\alpha\}$, and consequently,

$$\liminf J_0^{-1}(\delta)^\alpha/(-\log \delta) \geqq 1$$

as $\delta \to 0$. The theorem follows.

Theorem 3.3, of course, is a very weak result. In particular, there is no guarantee that $\text{eff}(\delta) \geqq 1$ for any positive δ.

4. Proofs. In this section we shall prove Lemma 2.1 and Theorems 2.1, 2.2, and 2.3. Since θ is a translation parameter, it will suffice to prove them in the special case that $\theta = 0$. We shall, therefore, assume that $\theta = 0$ throughout this section. We shall also assume conditions C_1, C_2, C_3, and C_5 throughout this section.

Let f be as in the statement of $C_1, \cdots, C_5$ and let F denote the distribution function of f. Further, let L be as in the statement of condition C_3. We shall have several occasions to use the following lemma.

LEMMA 4.1. *For $x > 0$ we may write* i) $f'(x) = \alpha(\alpha - 1)x^{\alpha-2}L_2(x)$; $f(x) = \alpha x^{\alpha-1}L_1(x)$; *and* $F(x) = x^\alpha L_0(x)$, *where* $L_0(x) \sim L_1(x) \sim L_2(x) \sim L(x)$ *as* $x \to 0$. *In particular,* L_0, L_1, L_2 *all vary slowly as* $x \to 0$.

Moreover, the relations

$$g'(x) \sim \frac{\alpha - 1}{x} \quad \text{and} \quad g''(x) \sim -\frac{(\alpha - 1)}{x^2}$$

hold as $x \to 0$.

PROOF. The first assertion of the lemma follows easily from Theorem 1 of [5], page 273. Thereafter, the second follows directly from the relations $g' = f'/f$ and $g'' = (ff'' - f'^2)/f^2$.

LEMMA 4.2. *Let h_γ, $\gamma > 0$, be measurable functions on $(0, \infty)$. Also, let $0 < d \leqq \infty$ and let K be a measurable function on $(0, d)$ which is bounded on $[\varepsilon, d)$ for* *every* $\varepsilon > 0$. *Suppose that* $K(x)$ *varies slowly as* $x \to 0$ *and that* $h_\gamma \to h_0$ a.e. *on* $(0, \infty)$ *as* $\gamma \to 0$. *Suppose also that* h_γ, $\gamma > 0$, *are dominated by a measurable function* h *for which*

$$\int_0^\infty (x^\beta + x^{-\beta}) h(x)\, dx < \infty$$

for some $\beta > 0$. *Then as* $\gamma \to 0$

$$\lim \int_0^{d\gamma^{-1}} h_\gamma(x) \frac{K(\gamma x)}{K(\gamma)}\, dx = \int_0^\infty h_0(x)\, dx\ .$$

PROOF. We may write

$$K(x) = a(x) \exp\left\{\int_x^1 \frac{\varepsilon(y)}{y}\, dy\right\}$$

where $a(x) \to a > 0$ and $\varepsilon(x) \to 0$ as $x \to 0$. Therefore,

$$\frac{K(\gamma x)}{K(\gamma)} = \frac{a(\gamma x)}{a(\gamma)} \exp\left\{\int_x^1 \frac{\varepsilon(\gamma y)}{y}\, dy\right\}\ .$$

Now, since $\varepsilon(x) \to 0$ as $x \to 0$, there is a $\delta > 0$ for which $|\varepsilon(x)| \leqq \beta$ for $0 < x \leqq \delta$, and it follows that for $x \leqq \delta\gamma^{-1}$

$$\frac{K(\gamma x)}{K(\gamma)} \leqq \frac{a(\gamma x)}{a(\gamma)} (x^\beta + x^{-\beta})\ .$$

Therefore,

$$\lim \int_0^{\delta\gamma^{-1}} h_\gamma(x) \frac{K(\gamma x)}{K(\gamma)}\, dx = \int_0^\infty h_0(x)\, dx$$

as $\gamma \to 0$ by the dominated convergence theorem. Moreover, letting b be an upper bound for $|K|$ on (δ, d), we have

$$\left|\int_{\delta\gamma^{-1}}^{d\gamma^{-1}} h_\gamma(x) \frac{K(\gamma x)}{K(\gamma)}\, dx\right| \leqq \frac{b}{|K(\gamma)|} \left(\frac{\gamma}{\delta}\right)^\beta \int_{\delta\gamma^{-1}}^\infty x^\beta h(x)\, dx\ ,$$

which tends to zero as $\gamma \to 0$.

We shall now prove Lemma 2.1. Recall that $G_n(t) = -\log L_n(t)$, $t < M_n$, so that

$$G_n(t) = \sum_{i=1}^n -g(X_i - t)$$

for $t < M_n$. Of course, the sum may be differentiated termwise twice. Let $\delta > 0$ be so small that $-g''(x) \geqq (\alpha - 1)/2x^2$ for $0 < x \leqq 2\delta$. Then, for $\varepsilon < \delta$ we have

$$\text{(4.1)}\qquad \min_{-\varepsilon < t < M_n} \left(\frac{1}{n}\right) G_n''(t) \geqq \left(\frac{\alpha_i - 1}{2n}\right) \sum_0^\delta (X_i + \varepsilon)^{-2} - \left(\frac{1}{n}\right) \sum_\delta^\infty \sup_{|t| \leqq \varepsilon} |g''(X_i - t)|\ ,$$

where $\sum_a^b$ denotes summation over $i \leqq n$ for which $a \leqq X_i < b$. As $n \to \infty$, the right side of (4.1) converges in probability to

$$\left(\frac{\alpha - 1}{2}\right) \int_0^\delta (x + \varepsilon)^{-2} f(x)\, dx - \int_\delta^\infty \sup_{|t| \leqq \varepsilon} |g''(x - t)| f(x)\, dx\ ,$$

which, in turn, diverges to ∞ as $\varepsilon \to 0$. Lemma 2.1 follows easily.

We shall now prove Theorem 2.1. Recall that the sequence $\gamma_1, \gamma_2, \cdots$ is so chosen that $n\gamma_n{}^\alpha L(\gamma_n) \to 1$, and let $Y_i = g'(X_i)$, $i = 1, 2, \cdots$. Then $Y_1, Y_2, \cdots$ are i.i.d. with common expectation

$$E(Y_i) = \textstyle\int_0^\infty g'(x)f(x)\,dx = 0\,,$$

and Theorem 2.1 asserts that $Z_{n0} = \gamma_n(Y_1 + \cdots + Y_n)$ has an asymptotic stable distribution with characteristic function (2.4) as $n \to \infty$. Therefore, by Theorem 2 of [5], page 546, it will suffice to show that

$$\Pr[Y_1 < -y] = o[y^{-\alpha}L(y^{-1})] \tag{4.2a}$$

$$\Pr[Y_1 > y] \sim (\alpha - 1)^\alpha y^{-\alpha}L(y^{-1}) \tag{4.2b}$$

as $y \to \infty$.

We may establish (4.2) as follows. For $0 < \varepsilon < 1$, there is, by Lemma 4.1, a $\delta = \delta(\varepsilon)$ for which

$$\left|g'(x) - \frac{\alpha - 1}{x}\right| \leqq \frac{(\alpha - 1)\varepsilon}{x} \tag{4.3}$$

for $0 < x \leqq 2\delta$. In particular, taking $\varepsilon = \frac{1}{2}$, we see that $Y_1 < 0$ implies $X_1 \geqq \delta_0 = \delta(\frac{1}{2})$. It follows that for $y > 0$, $\Pr(Y_1 < -y) = \Pr(Y_1 < -y, X_1 \geqq \delta_0)$ which does not exceed

$$y^{-2}\textstyle\int_{\delta_0}^\infty g'(x)^2 f(x)\,dx = o[y^{-\alpha}L(y^{-1})]$$

by Markov's Inequality and C_5. This establishes (4.2a). For (4.2b) let $\varepsilon > 0$ be given and choose $\delta = \delta(\varepsilon)$ as in (4.3). Then for $y > 0$, $Y_1 > y$ and $X_1 \leqq \delta$ imply $X_1 \leqq (\alpha - 1)(1 + \varepsilon)/y$. Therefore,

$$\begin{aligned}\Pr(Y_1 > y) &= \Pr(Y_1 > y, X_1 \leqq \delta) + \Pr(Y_1 > y, X_1 > \delta)\\ &\leqq F[(\alpha - 1)(1 + \varepsilon)y^{-1}] + o[y^{-\alpha}L(y^{-1})]\,,\end{aligned}$$

where F denotes the distribution of X_1 and we have again used Markov's Inequality and C_5. Moreover, by Lemma 4.1

$$F[(\alpha - 1)(1 + \varepsilon)y^{-1}] \sim (\alpha - 1)^\alpha(1 + \varepsilon)^\alpha y^{-\alpha}L(y^{-1})$$

as $y \to \infty$. Since $\varepsilon > 0$ was arbitrary, it now follows easily that

$$\limsup y^\alpha \Pr(Y_1 > y)/L(y^{-1}) \leqq (\alpha - 1)^\alpha$$

as $y \to \infty$; and a similar argument shows that $\liminf y^\alpha \Pr(Y_1 > y)/L(y^{-1}) \geqq (\alpha - 1)^\alpha$ as $y \to \infty$ to complete the proof of Theorem 2.1.

We shall now prove Theorem 2.2. Let $t > 0$ and for $i = 1, \cdots, n$ let

$$Y_{ni} = t\gamma_n g'(X_i + t\gamma_n)\,,$$

where γ_n are chosen to satisfy (1.2), as above. Then Theorem 2.2 asserts that $Z_{nt} = Y_{n1} + \cdots + Y_{nn}$ converges in distribution to Z_t as $n \to \infty$, where Z_t has characteristic function given by (2.5). Therefore, since $Y_{n1}, \cdots, Y_{nn}$ are i.i.d.

for each $n = 1, 2, \cdots$, it will suffice to show that as $n \to \infty$

$$\lim nE(Y_{n1}) = -m_\alpha t^\alpha \tag{4.4}$$

$$\lim n \Pr(Y_{n1} < -y) = 0\,, \qquad y > 0 \tag{4.5a}$$

$$\lim n \Pr(Y_{n1} > y) = t^\alpha F_\alpha(y)\,, \qquad 0 < y < (\alpha - 1) \tag{4.5b}$$

with F_α and m_α as in the statement of Theorem 2.2 and that

$$\lim_{\tau\to 0} \limsup_{n\to\infty} \int_{|Y_{n1}|\leqq\tau} n Y_n{}^2\, dP = 0\,. \tag{4.6}$$

The sufficiency of (4.4), (4.5) and (4.6) may be deduced from Section 17.1 of [5].

The proof of (4.5) is similar to that of (4.2). Let $\alpha < \beta < 2$ and choose $\delta = \delta_0$ as in (4.3) with $\varepsilon = \frac{1}{2}$. Further, let n_0 be so large that $2t\gamma_n \leqq \delta_0$ for $n \geqq n_0$. Then, for $n \geqq n_0$ and $y > 0$ we have

$$n \Pr(Y_{n1} < -y) = n \Pr(Y_{n1} < -y, X_1 \geqq \delta) \leqq bn\gamma_n{}^\beta y^{-\beta} \tag{4.7}$$

with

$$b = \sup_{n\geqq n_0} t^\alpha \int_{\delta_0}^\infty |g'(x + t\gamma_n)|^\beta f(x)\, dx\,; \tag{4.8}$$

and since $n\gamma_n{}^\beta \to 0$ as $n \to \infty$ for $\beta > \alpha$, (4.5a) follows. To establish (4.5b), let $\varepsilon > 0$ be given and choose $\delta = \delta(\varepsilon)$ as in (4.3). Further, let n_1 be so large that $2t\gamma_n \leqq \min(\delta, \delta_0)$ for $n \geqq n_1$. Then for $n \geqq n_1$ and $0 < y < \alpha - 1$, $Y_{n1} > y$ and $X_1 \leqq \delta$ imply $X_1 \leqq t\gamma_n z$, where

$$z = \left[\frac{(\alpha - 1)(1 + \varepsilon)}{y} - 1\right].$$

Therefore, for $n \geqq n_1$ and $0 < y < \alpha - 1$, we have

$$\begin{aligned} n \Pr(Y_{n1} > y) &\leqq n \Pr(Y_{n1} > y, X_1 \leqq \delta) + n \Pr(Y_{n1} > y, X_1 \geqq \delta) \\ &\leqq nF(t\gamma_n z) + bn\gamma_n{}^\beta y^{-\beta} \\ &= d_n t^\alpha z^\alpha \frac{L_0(t\gamma_n z)}{L_0(\gamma_n)} + bn\gamma_n{}^\beta y^{-\beta} \end{aligned} \tag{4.9}$$

with b as in (4.8) and $d_n = n\gamma^\alpha L_0(\gamma_n)$. Now as $n \to \infty$, $d_n \to 1$ and $L_0(\gamma_n z)/L(\gamma_n) \to 1$ by Lemma 4.1, and $n\gamma_n{}^\beta \to 0$, as above. Therefore, since $\varepsilon > 0$ was arbitrary, we have $\limsup n \Pr(Y_{n1} > y) \leqq t^\alpha F_\alpha(y)$ as $n \to \infty$; and a similar argument will show that $\liminf n \Pr(Y_{n1} > y) \geqq t^\alpha F_\alpha(y)$ as $n \to \infty$. This establishes (4.5).

Relation (4.6) may now be deduced from the inequalities (4.7) and (4.9) with $\varepsilon = \frac{1}{2}$. In fact, we have

$$\int_{|y_{n1}|\leqq\tau} nY_{n1}^2\, dP \leqq \int_0^\tau 2yn \Pr(|Y_{n1}| > y)\, dy\,,$$

which, for $n \geqq n_1$, does not exceed

$$d_n t^\alpha \int_0^\tau 2yz^\alpha L_0(t\gamma_n z)L_0(\gamma_n)^{-1}\, dy + 4bn\gamma_n{}^{-\beta} \int_0^\tau y^{1-\beta}\, dy = I_n + I_n'\,, \quad \text{say,}$$

with b as in (4.8), d_n as in (4.9), and $z = [(3(\alpha - 1)/2y) - 1]$. Since $\beta < 2$ by selection, $I_n' \to 0$ as $n \to \infty$ uniformly in $\tau \leqq 1$. Moreover, letting

$z^* = [(3(\alpha - 1)/2\tau) - 1]$, we find that

$$I_n = d_n t^\alpha \int_{z^*}^\infty \frac{5(\alpha - 1)^2}{(1 + z)^3} z^\alpha \frac{L_0(t\gamma_n z)}{L_0(\gamma_n)} dz .$$

Finally, since $L_0(x) \leqq x^{-\alpha}$, $x > 0$, it follows from Lemma 4.2 that I_n converges as $n \to \infty$ to

$$5(\alpha - 1)^2 t^\alpha \int_{z^*}^\infty \frac{z^\alpha}{(1 + z)^3} dz ,$$

which in turn, tends to zero as $\tau \to 0$. Relation (4.6) follows.

Finally, we must establish (4.4). We have

$$nE(Y_{n1}) = tn\gamma_n \int_0^\infty [g'(x + t\gamma_n) - g'(x)]f(x)\, dx .$$

Moreover, for any $\delta > 0$, we have

$$\begin{aligned} tn\gamma_n \int_\delta^\infty |g'(x + t\gamma_n) - g'(x)|f(x)\, dx & \\ \leqq t^2 n\gamma_n^2 \cdot \int_\delta^\infty \sup_{0 \leqq s \leqq t} |g''(x + s\gamma_n)|f(x)\, dx , \end{aligned}$$

which tends to zero as $n \to \infty$ by C_5 and choice of γ_n. Let $\varepsilon > 0$ be given and let $\delta > 0$ be so small that

$$\left| g''(x) + \frac{\alpha - 1}{x^2} \right| \leqq \frac{(\alpha - 1)\varepsilon}{x^2}$$

for $0 < x \leqq 2\delta$. Then, for n so large that $t\gamma_n \leqq \delta$, we have

$$\begin{aligned} nt\gamma_n \int_0^\delta & [g'(x + t\gamma_n) - g'(x)]f(x)\, dx \\ & \geqq -(\alpha - 1)(1 + \varepsilon)nt^2\gamma_n^2 \int_0^\delta x^{-1}(x + \gamma_n)^{-1}f(x)\, dx \\ & = -(\alpha - 1)(1 + \varepsilon)t^\alpha \alpha n\gamma_n^\alpha L_1(\gamma_n) \int_0^{\delta/t\gamma_n} \left(\frac{x^{\alpha-2}}{1 + x}\right) \frac{L_1(t\gamma_n x)}{L_1(\gamma_n)} dx . \end{aligned}$$

Moreover, by Lemma 4.2, the latter integral converges to

$$-\alpha(\alpha - 1)t^\alpha \int_0^\infty \left(\frac{x^{\alpha-2}}{1 + x}\right) dx = -t^\alpha m_\alpha$$

as $n \to \infty$ and $\varepsilon \to 0$ in that order. It follows that $\liminf nE(Y_{n1}) \geqq -t^\alpha m_\alpha$ as $n \to \infty$; and a similar argument will show that $\limsup nE(Y_{n1}) \leqq -t^\alpha m_\alpha$ as $n \to \infty$ to complete the proof of Theorem 2.2.

Finally, we must prove Theorem 2.3. As in the proof of Theorem 2.3, we may write

$$Z_{nt}^* = \sum_{i=1}^n Y_{ni}^*$$

for $t > 0$, where

$$Y_{ni}^* = t\gamma_n g'(X_i - t\gamma_n)$$

if $M_n^* = \gamma_n^{-1} M_n > t$ and Y_{ni}^* is undefined otherwise. Now, the conditional distribution of $X_1, \cdots, X_n$, given $M_n^* > t$, is that of independent random variables with common density

$$\begin{aligned} f^*(x) &= c_n^{-1} f(x) : \quad x \geqq t\gamma_n \\ &= 0 : \qquad \text{otherwise} \end{aligned}$$

where c_n is a normalizing constant and $c_n \to 1$ as $n \to \infty$. Therefore, the conditional distribution of Z^*_{nt}, given $M_n^* > t$, is that of the sum of n independent, identically distributed random variables. Therefore, to prove Theorem 2.3, it will suffice to show that as $n \to \infty$

$$(4.10) \qquad nE^*(\sin (Y^*_{n1})) \to t^\alpha m_\alpha^* ,$$

$$(4.11\text{a}) \qquad n \Pr{}^* (Y^*_{n1} < -y) \to 0 , \qquad y > 0 ,$$

$$(4.11\text{b}) \qquad n \Pr{}^* (Y_{n1} > y) \to t^\alpha F_\alpha^*(y) , \qquad y > 0 ,$$

with m_α^* and F_α^* as in the statement of Theorem 2.3, and that

$$(4.12) \qquad \lim_{\tau \to 0} \limsup_{n \to \infty} \int_{|Y^*_{n1}| \leq \tau} nY^{*2}_{n1}\, dP^* = 0 .$$

Here P^* and E^* denote conditioned probability and expectation given $M_n^* > t$. Again, the sufficiency of (4.10), (4.11), and (4.12) may be deduced from Section 17.1 of [5].

The proofs of (4.11) and (4.12) are too similar to those of (4.5) and (4.6) to warrant repetition. To establish (4.10), let $\gamma = t\gamma_n$ and write

$$\begin{aligned} nE(\sin (Y^*_{n1})) &= nc_n^{-1} \int_\gamma^\infty [\sin (\gamma g'(x - \gamma)) - \sin (\gamma g'(x))] f(x)\, dx \\ &\quad + nc_n^{-1} \int_\gamma^\infty [\sin (\gamma g'(x)) - \gamma g'(x)] f(x)\, dx + \gamma n c_n^{-1} \int_\gamma^\infty g'(x) f(x)\, dx \\ &= I_1 + I_2 + I_3 , \quad \text{say.} \end{aligned}$$

Since

$$|\sin (\gamma g'(x - \gamma)) - \sin (\gamma g'(x))| \leq \gamma |g'(x - \gamma) - g'(x)| \leq \gamma^2 \sup_{|s| \leq \gamma} |g''(x - s))| ,$$

for any $x > \gamma$, we have

$$n \int_\delta^\infty |\sin (\gamma g'(x - \gamma)) - \sin (\gamma g'(x))| f(x)\, dx \leq n\gamma^2 \int_\delta^\infty \sup_{|s| \leq \gamma} |g''(x - s)| f(x)\, dx ,$$

which tends to zero as $n \to \infty$ for any $\delta > 0$ by C_5. Consider

$$\begin{aligned} &n \int_\gamma^\delta [\sin (\gamma g'(x - \gamma)) - \sin (\gamma g'(x))] f(x)\, dx \\ &\quad = \int_1^{\delta\gamma^{-1}} d_n'[\sin (\gamma g'(\gamma(x - 1))) - \sin (\gamma g'(\gamma x))] x^{\alpha - 1} \frac{\alpha L_1(\gamma x)}{L_1(\gamma)}\, dx \end{aligned}$$

with $d_n' = n\gamma^\alpha L_1(\gamma)$. If $\delta > 0$ is so small that $|g''(x)| \leq 2(\alpha - 1)x^{-2}$ for $0 \leq x \leq \delta$, then we must have

$$(4.13) \qquad |\sin (\gamma g'[\gamma(x - 1)]) - \sin (\gamma g'(\gamma x))| \leq \gamma \int_{\gamma(x-1)}^{\gamma x} \frac{2(\alpha - 1)}{x^2}\, dx = \frac{2(\alpha - 1)}{x(x - 1)}$$

for $1 < x < \delta\gamma^{-1}$. Since the left side of (4.13) is also bounded by 2, it follows easily from Lemma 4.2 that

$$\lim I_1 = \alpha t^\alpha \int_1^\infty \left[\sin \left(\frac{\alpha - 1}{x - 1} \right) - \sin \left(\frac{\alpha - 1}{x} \right) \right] x^{\alpha - 1}\, dx$$

as $n \to \infty$. A similar argument will show that

$$\lim I_2 = \alpha t^\alpha \int_1^\infty \left[\sin \left(\frac{\alpha - 1}{x} \right) - \frac{\alpha - 1}{x} \right] x^{\alpha - 1}\, dx$$

as $n \to \infty$, and finally we have

$$\begin{aligned} I_3 &= -n\gamma c_n^{-1} \int_0^\gamma g'(x)f(x)\,dx \\ &= -c_n^{-1} \int_0^1 \gamma g'(\gamma x) n\gamma f(\gamma x)\,dx \\ &\to -\alpha t^\alpha \int_0^1 \left(\frac{\alpha - 1}{x}\right) x^{\alpha-1}\,dx = -\alpha t^\alpha . \end{aligned}$$

Thus, $nE^*(Y_{n1}) \to m_\alpha{}^* t^\alpha$, as asserted.

5. Concluding remarks. It is possible to find the asymptotic joint distribution of Z_{ns} and Z_{nt} for $s > 0$ and $t > 0$. Indeed, their asymptotic joint distribution has characteristic function $\exp(-\Psi)$, where

$$\Psi(\lambda, \mu) = i\lambda s^\alpha m_\alpha + i\mu t^\alpha m_\alpha + \int_0^{\alpha-1} \int_0^{\alpha-1} [e^{i\lambda x + i\mu y} - 1 - (i\lambda x + i\mu y)]\,dK(x, y)$$

where $K(x, y) = \min\{s^\alpha F_\alpha(x), t^\alpha F_\alpha(y)\}$ and m_α and F_α are as in Theorem 2.2. Thus, while the marginal distributions of Z_t are those of a process with stationary independent increment, their joint distributions are not.

A similar remark applies to the $Z_t{}^*$ process.

Estimation of θ by systematic statistics in a case similar to ours has been considered by Polfeldt [5].

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