The Annals of Statistics
1994, Vol. 22, No. 2, 968–979

ON THE BERRY–ESSEEN BOUND FOR L-STATISTICS IN THE NON-I.D. CASE WITH APPLICATIONS TO THE ESTIMATION OF LOCATION PARAMETERS[1]

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In this paper, two versions of the Berry–Esseen theorems are established for L-statistics in the non-identically distributed case. One theorem, which requires $E|X_i|^3 < \infty$, is an extension of the classical Berry–Esseen theorem. Another, proved under the condition $E|X_i|^\alpha < \infty$ for some $\alpha \in (0, 1]$, seems to be of more interest for statistical inference. Some applications are also discussed.

1. Introduction and main results. Linear combinations of order statistics, exhibiting desirable robustness, play an important role in the theory of estimation. After asymptotic normality has been established for such a statistic, one often needs more precise information and may try to find suitable bounds for the error in the normal approximation for finite sample sizes. Let $X_1, X_2, \dots, X_n$ be n independent (but not necessarily identically distributed) random variables with distribution functions $G_1, G_2, \dots, G_n$, where $G_k(x) = P(X_k \le x)$. Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ denote the order statistics of the sample $X_1, X_2, \dots, X_n$. Define $\overline{F}_n(x) = (1/n)\sum_{k=1}^n G_k(x)$, $F_n(x) = (1/n)\sum_{k=1}^n I_{(X_k \le x)}$ and $\mu(J, \overline{F}_n) = \int_{-\infty}^{\infty} x J(\overline{F}_n(x))\, d\overline{F}_n(x)$. Consider linear functions of order statistics of the form

$$S_n = \frac{1}{n}\sum_{k=1}^n J\left(\frac{k}{n}\right)X_{(k)} = \int_{-\infty}^{\infty} x J\big(F_n(x)\big)\, dF_n(x), \tag{1.1}$$

where $J(x)$ is a specified weight function defined on $[0, 1]$. Several authors [e.g., Shorack (1973), Stigler (1974) and Xiang (1991a)] have established the asymptotic normality of L-statistics in the non-i.d. case under different sets of conditions on J and X_i. The aim of the present paper is to obtain a Berry–Esseen bound for these statistics. Two examples illustrative of the present results are given in Section 4.

Received October 1991; revised May 1992.
[1]This manuscript was prepared using computer facilities supported in part by NSF Grants DMS-89-05292, DMS-87-03942, DMS-86-01732 and DMS-84-04941 awarded to the Department of Statistics at the University of Chicago, and by the University of Chicago Block Fund.
AMS 1991 *subject classifications*. Primary 60F05, 62F12, 62G30.
Key words and phrases. L-statistics, Berry–Esseen bound, estimation of location parameter from non-i.d. sample.

Let

$$T_i = -\frac{1}{n}\int_{-\infty}^{\infty} J(\overline{F}_n(x))\big(I_{(X_i\le x)} - G_i(x)\big)\,dx$$
$$-\frac{1}{2n^2}\int_{-\infty}^{\infty} J'(\overline{F}_n(x))\big(I_{(X_i\le x)} - G_i(x)\big)^2\,dx$$
$$+\frac{1}{2n^2}\int_{-\infty}^{\infty} J'(\overline{F}_n(x))G_i(x)\big(1 - G_i(x)\big)\,dx,$$
$$h(X_i,X_j) = -\frac{1}{n^2}\int_{-\infty}^{\infty} J'(\overline{F}_n(x))\big(I_{(X_j\le x)} - G_j(x)\big)\big(I_{(X_i\le x)} - G_i(x)\big)\,dx, \qquad i\ne j,$$
$$\widehat{S}_n = \sum_{i=1}^{n} T_i, \qquad \Delta_n = \sum_{i=2}^{n}\sum_{j=1}^{i-1} h(X_i, X_j).$$

From the properties $ET_i = 0,\ i = 1,2,3,\dots,n,\ Eh(X_i, X_j)h(X_j, X_k) = 0,\ 1 \le i < j < k \le n,\ Ef(T_i)h(X_j, X_k) = 0,\ 1\le i\le n,\ 1 \le j < k \le n$, for each measurable function $f(x)$ (real or complex), we have $\sigma^2(\widehat{S}_n + \Delta_n) = \sigma^2(\widehat{S}_n) + \sigma^2(\Delta_n)$. Write $\sigma_n^2 = \sigma^2(\widehat{S}_n)$. Let $H_{kn} = (1/n)\Sigma_{i=1}^n E|X_i|^k,\ k = 1,2,3,\ L_n = \max(H_{1n}, H_{2n}, H_{3n})$ and $M_n = \max(1, \|J'\|L_n)$ with $\|J'\| = \max_{0\le x\le 1}|J'(x)|$. Let C denote a positive universal constant throughout this paper. If M_n is finite, we wish to show

$$\sup_x\Big|P\Big(\sigma_n^{-1}(S_n - \mu(J,\overline{F}_n)) \le x\Big) - \Phi(x)\Big| \le C\frac{\Gamma_n}{\sigma_n^3}M_n,$$

where $\Gamma_n = \Sigma_{i=1}^n E|T_i|^3$.

Let $\Gamma_{ij}(n) = \Sigma_{k=1,\dots,n,\ k\ne i,j} E|T_k|^3,\ \sigma_{ij}^2(n) = \Sigma_{k=1,\dots,n,\ k\ne i,j}\sigma^2(T_k)$. We abbreviate these to Γ_{ij} and σ_{ij}^2. The first main result of this paper is the following.

THEOREM 1. *Suppose that* $E|X_i|^3 < \infty,\ i = 1,2,\dots,n$, *and there exist positive numbers* α_1, α_2, *and* α_3 *which are independent of* n, *such that the following hold*: (i) $n\sigma_n^2 \ge \alpha_1$; (ii) $\Gamma_n \le \alpha_2\min_{i,j}\Gamma_{ij}$ *and* $\sigma_n^2 \le \alpha_3\min_{i,j}\sigma_{ij}^2$. *Then* $J'(x)\in \mathrm{Lip}(\delta),\ \delta > \frac{1}{3}$, *implies*

$$\sup_x\Big|P\Big(\sigma_n^{-1}(S_n - \mu(J,\overline{F}_n)) \le x\Big) - \Phi(x)\Big| \le C\frac{\Gamma_n}{\sigma_n^3}M_n. \tag{1.2}$$

REMARK 1. If $J(x)\equiv 1$, then $T_i = (1/n)(X_i - EX_i)$, $\|J'\| = 0$ and the above result becomes the classical Berry–Esseen theorem. It follows, in particular, that the moment conditions $E|X_i|^3 < \infty,\ i = 1,2,\dots,n$, are necessary. A previous extension to the non-i.d. case was given by Friedrich (1989), who required that $E|X_i|^4,\ i = 1,2,\dots,n$, be finite.

REMARK 2. If $\{L_n\}$ is bounded, by Lemma 1 in Section 2, $n^2\Gamma_n$ is bounded. This and condition (i) of Theorem 1 imply $(\Gamma_n/\sigma_n^3)M_n = O(1/\sqrt{n})$. Friedrich

(1989) obtained this rate by imposing two additional conditions [his (C2) and (C3)]. Here those two conditions, which impose stringent uniformity upon the X_i, are relaxed.

REMARK 3. In the i.i.d. case, Berry–Esseen bounds for L-statistics with a bounded smooth weight function have been studied by many authors. Helmers (1977) first established a Berry–Esseen bound for L-statistics. The best result to date has been obtained by Helmers (1981, 1982), also by van Zwet (1984) as an application of his Theorem 1.1. Serfling (1980) established Berry–Esseen bounds for L-statistics by treating L-statistics as a functional of the empirical distribution function. Our approach in this paper is essentially based on Serfling's method, which is convenient for the present non-id case. If the weight function is allowed to be unbounded, a Berry–Esseen bound has been obtained by Helmers and Hušková (1984).

Another goal of this paper is to obtain a Berry–Esseen bound when the moment conditions $E|X_n|^3 < \infty$, $n = 1, 2, \dots$, are replaced by

$$I_n = \int_{-\infty}^{\infty} \left[\overline{F}_n(x)(1 - \overline{F}_n(x))\right]^{\alpha} dx < \infty, \qquad n = 1, 2, \dots \tag{1.3}$$

for some $\alpha \geq 1$. We will show that $E|X_i|^{1/\alpha} < \infty$ for $i = 1, 2, \dots, n$ implies (1.3) (see Lemma 3). In this case, we choose the weight function $J(x)$ to be zero near 0 and 1, and the L-statistics is like a smoothly trimmed mean [see Stigler (1973)]. Our result is the following.

THEOREM 2. *Assume the following*: (i) *For some* $\alpha \geq 1$, (1.3) *holds.* (ii) *Each* $G_k(x)$ *is continuous.* (iii) $J'(x) \in \mathrm{Lip}\left(\frac{1}{3}\right)$, $J(x) = 0$ *for* $x \in [0, \varepsilon] \cup [1 - \varepsilon, 1]$, $\varepsilon \in (0, \frac{1}{2})$. *Then, if conditions* (i) *and* (ii) *of Theorem* 1 *hold,*

$$\sup_x \left| P\left(\sigma_n^{-1}(S_n - \mu(J, \overline{F}_n)) \leq x\right) - \Phi(x)\right| \leq C \frac{\Gamma_n}{\sigma_n^3} \overline{M}_n, \tag{1.4}$$

where $\overline{M}_n = \max(1, I_n^3)$.

REMARK 4. If $\{I_n\}$ is bounded, we obtain the convergence rate $O(1/\sqrt{n})$. If $X_1, \dots, X_n$ are i.i.d. random variables, Egorov and Nevzorov (1974) obtained the normal approximation for classical trimmed mean with rate $O(\sqrt{(\log n)/n})$ and Bjerve (1977) obtained the rate $O(1/\sqrt{n})$.

The following section contains several lemmas which are essential to this paper. The proofs of the theorems are given in Section 3. In Section 4 two examples are discussed.

2. Lemmas.

LEMMA 1. *Let the random variable X have distribution F and satisfy $E|X|^k < \infty$, where k is a positive integer. Then, for $m = 1, 2, 3, \dots,$*

$$E\left(\int_{-\infty}^{\infty} |I_{(X \le y)} - F(y)|^m \, dy\right)^k \le 2^k E|X|^k.$$

PROOF. As $|I_{(X \le y)} - F(y)| \le 1$, for any $m \ge 1$,

$$E\left(\int_{-\infty}^{\infty} |I_{(X \le y)} - F(y)|^m \, dy\right)^k \le E\left(\int_{-\infty}^{\infty} |I_{(X \le y)} - F(y)| \, dy\right)^k \le 2^k E|X|^k. \quad \square$$

LEMMA 2. *Let $X_1, X_2, \dots, X_n$ be n independent random variables. If, for a positive integer $k, E|X_i|^k < \infty$, $i = 1, 2, \dots, n$, then*

$$E\left(\int_{-\infty}^{\infty} (F_n(x) - \overline{F}_n(x))^2 \, dx\right)^k \le Cn^{-k}\left(\frac{1}{n}\sum_{i=1}^{n} E|X_i|^k\right).$$

PROOF. From the decomposition,

$$\begin{aligned}
&\int_{-\infty}^{\infty} (F_n(x) - \overline{F}_n(x))^2 \, dx \\
&\qquad = n^{-2}\sum_{i=1}^{n}\int_{-\infty}^{\infty} (I_{(X_i \le x)} - G_i(x))^2 \, dx \\
&\qquad\quad + 2n^{-2}\sum_{i=2}^{n}\sum_{j=1}^{i-1}\int_{-\infty}^{\infty} (I_{(X_i \le x)} - G_i(x))(I_{(X_j \le x)} - G_j(x)) \, dx \\
&\qquad = \Delta_{1n} + 2\Delta_{2n},
\end{aligned}$$

$$E\left(\int_{-\infty}^{\infty} (F_n(x) - \overline{F}_n(x))^2 \, dx\right)^k \le 2^k \sum_{i=0}^{k}\binom{k}{i}(E\Delta_{1n}^k)^{i/k}(E\Delta_{2n}^{2k})^{(k-i)/2k}. \tag{2.1}$$

Using the inequality $|\sum_{i=1}^{n} a_i|^l \le n^{l-1}\sum_{i=1}^{n}|a_i|^l$, $a_i \in R^1$, $l \ge 1$, and Lemma 1,

$$E\Delta_{1n}^k \le Cn^{-k}\left(\frac{1}{n}\sum_{i=1}^{n} E|X_i|^k\right). \tag{2.2}$$

Let

$$g(X_i, X_j) = \frac{1}{n^2}\int_{-\infty}^{\infty} (I_{(X_i \le x)} - G_i(x))(I_{(X_j \le x)} - G_j(x)) \, dx, \qquad 1 \le j < i,$$

and define $\psi_1 = 0$, $\psi_i = \sum_{j=1}^{i-1} g(X_i, X_j)$, $i = 2, \dots, n$. Then $\{\psi_i,\ i = 1, 2, \dots, n\}$ is a martingale difference sequence and so is $\{g(X_i, X_j),\ 1 \le j < i\}$ for each fixed i. Hence, applying a theorem of Dharmadhikari, Fabian and Jogdeo (1968) twice, we obtain

$$E\Delta_{2n}^{2k} = E\left|\sum_{i=2}^{n} \psi_i\right|^{2k} \le Cn^{k-1}\sum_{i=2}^{n} E|\psi_i|^{2k} \le Cn^{2k-2}\sum_{i=2}^{n}\sum_{j=1}^{i-1} E|g(X_i, X_j)|^{2k}.$$

By the Schwarz inequality and Lemma 1, $E|g(X_i,X_j)|^{2k} \le Cn^{-4k}E|X_i|^k E|X_j|^k$. Hence

$$(2.3)\qquad E\Delta_{2n}^{2k} \le Cn^{-2k-2}\sum_{i=1}^{n}\sum_{j=1}^{n} E|X_i|^k E|X_j|^k \le Cn^{-2k}\left(\frac{1}{n}\sum_{i=1}^{n} E|X_i|^k\right)^2.$$

Substituting (2.2) and (2.3) in (2.1), we obtain the desired statement. □

LEMMA 3. *Suppose H is a distribution function, and, for $0 < \gamma \le 1$, $a_\gamma^\gamma = \int |x|^\gamma \, dH(x) < \infty$. Then*

$$\int_{-\infty}^{\infty} [H(x)1 - H(x)]^{1/\gamma} \, dx \le \frac{2}{\gamma} a_\gamma.$$

PROOF. We have

$$\begin{aligned}\int_0^\infty (1 - H(x))^{1/\gamma} \, dx &\le \int_0^\infty \left(\int_x^\infty y^\gamma \, dH(y)\right)^{1/\gamma - 1} x^{\gamma-1}(1 - H(x)) \, dx \\ &\le a_\gamma^{1-\gamma} \int_0^\infty x^{\gamma-1}(1 - H(x)) \, dx \\ &\le \frac{1}{\gamma} a_\gamma^{1-\gamma} \int_0^\infty |x|^\gamma \, dH(x) = \frac{1}{\gamma} a_\gamma.\end{aligned}$$

Similarly for $\int_{-\infty}^0 H(x)^{1/\gamma} \, dx$. □

REMARK. If $H(x) = \overline{F}_n(x)$, $a_\gamma \le (1/n)\sum_{i=1}^n (E|X_i|^\gamma)^{1/\gamma}$, as $0 < \gamma \le 1$.

3. Proofs of theorems. In the following proofs, we always assume that $\|J'\| > 0$.

PROOF OF THEOREM 1. From Remark 5 of Xiang (1991b), Theorem 1 holds if we can show (1.2) with S_n replaced by $\int_0^1 F_n^{-1}(x) J(x) \, dx$, say T_n. With the same argument as that in Serfling [(1980), page 289], we may write

$$(3.1)\qquad \begin{aligned} S_n - \mu(J, \overline{F}_n) &= -\int_{-\infty}^{\infty} J(\overline{F}_n(x))(F_n(x) - \overline{F}_n(x)) \, dx \\ &\quad - \tfrac{1}{2}\int_{-\infty}^{\infty} J'(\overline{F}_n(x))(F_n(x) - \overline{F}_n(x))^2 \, dx + R_n \\ &= \widehat{S}_n + \Delta_n + \overline{R}_n + R_n, \end{aligned}$$

where

$$\overline{R}_n = -\frac{1}{2n^2}\sum_{i=1}^{n}\int_{-\infty}^{\infty} J(\overline{F}_n(x))G_i(x)(1-G_i(x))\,dx, \tag{3.2}$$

and it holds that

$$|R_n| \le C\int_{-\infty}^{\infty}\left|F_n(x)-\overline{F}_n(x)\right|^{2+\delta}dx \le C\|F_n-\overline{F}_n\|_\infty^\delta\|F_n-\overline{F}_n\|_{L_2}^2, \tag{3.3}$$

where

$$\|F_n-\overline{F}_n\|_\infty = \sup_x\left|F_n(x)-\overline{F}_n(x)\right| \quad \text{and}$$

$$\|F_n-\overline{F}_n\|_{L_2} = \left(\int_{-\infty}^{\infty}\left|F_n(x)-\overline{F}_n(x)\right|^2dx\right)^{1/2}.$$

Let $\{c_n\}$ be a sequence $c_n \le n$ and $c_n \to \infty$ as $n \to \infty$. Write $\Delta_n = \overline{\Delta}_n + \widetilde{\Delta}_n$ with $\overline{\Delta} = \sum_{1\le j<i\le c_n} h(X_i, X_j)$ and $\widetilde{\Delta}_n = \sum_{i=c_n+1}^{n}\sum_{j=1}^{i-1} h(X_i, X_j)$. The precise form of c_n will be given later on. This decomposition was first used by Chan and Wierman (1977) and then by Callaert and Janssen (1978). If we can show

$$\sup_x\left|P(\sigma_n^{-1}(\widehat{S}_n+\overline{\Delta}_n)\le x)-\Phi(x)\right| \le O\left(M_n\frac{\Gamma_n}{\sigma_n^3}\right), \tag{3.4}$$

$$P\left(\sigma_n^{-1}|\widetilde{\Delta}_n| \ge M_n\frac{\Gamma_n}{\sigma_n^3}\right) = O\left(M_n\frac{\Gamma_n}{\sigma_n^3}\right), \tag{3.5}$$

$$P\left(\sigma_n^{-1}|R_n| \ge M_n\frac{\Gamma_n}{\sigma_n^3}\right) = O\left(M_n\frac{\Gamma_n}{\sigma_n^3}\right), \tag{3.6}$$

$$\left|\sigma_n^{-1}\overline{R}_n\right| = O\left(M_n\frac{\Gamma_n}{\sigma_n^3}\right), \tag{3.7}$$

then the theorem follows from a well-known lemma [cf. Serfling (1980), page 228]. First, (3.7) follows easily from $|\sigma_n^{-1}\overline{R}_n| \le (C/n\sigma_n)H_{1n}$, assumption (i) and the inequality

$$\frac{\Gamma_n}{\sigma_n^3} \ge \frac{\sum_{i=1}^n E|T_i|^3}{\sqrt{n}\sum_{i=1}^n\left(E|T_i|^2\right)^{3/2}} \ge \frac{\sum_{i=1}^n E|T_i|^3}{\sqrt{n}\sum_{i=1}^n E|T_i|^3} \ge \frac{1}{\sqrt{n}}. \tag{3.8}$$

To prove (3.6), again by (i) and (3.8) we have from (3.3), for some universal constant $a > 0$,

$$P\left(\sigma_n^{-1}|R_n| \ge M_n\frac{\Gamma_n}{\sigma_n^3}\right) \le P\left(n^{1/6}\|F_n-\overline{F}_n\|_\infty^\delta \ge \frac{a}{C}\right)$$
$$+P\left(n^{5/6}\|F_n-\overline{F}_n\|_{L_2}^2 \ge 1\right) = P_{1n}+P_{2n}.$$

Using $\delta > \frac{1}{3}$ and an inequality of Bretagnolle [cf. Shorack and Wellner (1986), page 797], $P_{1n} = O(M_n\Gamma_n/\sigma_n^3)$. By Lemma 2, $P_{2n} \le n^{5/2}E\|F_n - \overline{F}_n\|_{L_2}^6 \le CM_n\Gamma_n/\sigma_n^3$. Hence (3.6) holds.

Write $\gamma_n = \sigma_n^3/\Gamma_n$ and set $c_n = [n - 3n(\log \gamma_n)/\gamma_n]$ the integer part of $n - 3n(\log \gamma_n)/\gamma_n$. For convenience, we assume that $\gamma_n \geq 3 \log 3$, and Theorem 1 also holds for all γ_n by taking $C \geq 3 \log 3$ in (1.2). Then (3.5) follows from

$$\begin{aligned} P\left(\sigma_n^{-1}|\widetilde{\Delta}_n| \geq M_n \frac{\Gamma_n}{\sigma_n^3}\right) &\leq P\big(n|\widetilde{\Delta}_n| \geq CM_n\big) \\ &\leq CM_n^{-1} n^6 (n - c_n)^2 \sum_{i=c_n+1}^{n} E\left|\sum_{j=1}^{i-1} h(X_i, X_j)\right|^6 \\ &\leq CM_n^{-1} n^{10} \gamma_n^{-2} (\log \gamma_n)^2 \sum_{i,j=1,\dots,n,\ i\neq j} E|h(X_i, X_j)|^6 \\ &\leq C\gamma_n^{-2}(\log \gamma_n)^2 M_n \leq O\left(M_n \frac{\Gamma_n}{\sigma_n^3}\right). \end{aligned}$$

We now prove (3.4). For $t \in (-\infty, \infty)$, denote by $\phi_n(t) = E\exp\{it\sigma_n^{-1}(\widehat{S}_n + \overline{\Delta}_n)\}$ the characteristic function of $\sigma_n^{-1}(\widehat{S}_n + \overline{\Delta}_n)$. Also, let $\widetilde{\phi}_n(t) = E\exp\{it\sigma_n^{-1}(\widehat{S}_n)\}$, $\phi(t) = \exp(-t^2/2)$. According to Esseen's smoothing lemma [cf. Feller (1971), page 538],

$$\sup_x \left|P\big(\sigma_n^{-1}(\widehat{S}_n + \overline{\Delta}_n) \leq x\big) - \Phi(x)\right| \leq \int_{-H}^{H} \left|\frac{\phi_n(t) - \phi(t)}{t}\right| dt + 4H^{-1}, \tag{3.9}$$

where we choose $H = \frac{8}{9}\gamma_n$. From the proof of the classical Berry–Esseen theorem we conclude that

$$\int_{-H}^{H} \left|\frac{\widetilde{\phi}_n(t) - \phi(t)}{t}\right| dt \leq C\gamma_n^{-1}, \tag{3.10}$$

$$\sup_x \left|P\big(\sigma_n^{-1}(\widehat{S}_n + \overline{\Delta}_n) \leq x\big) - \Phi(x)\right| \leq C\gamma_n^{-1} + \int_{-H}^{H} \left|\frac{\phi_n(t) - \widetilde{\phi}_n(t)}{t}\right| dt. \tag{3.11}$$

Hence it suffices to show that

$$\int_{\gamma_n^{1/2}}^{H} \left|\frac{\phi_n(t) - \widetilde{\phi}_n(t)}{t}\right| dt \leq CM_n\gamma_n^{-1}, \tag{3.12}$$

$$\int_{|t| \leq \gamma_n^{1/2}} \left|\frac{\phi_n(t) - \widetilde{\phi}_n(t)}{t}\right| dt \leq CM_n\gamma_n^{-1}. \tag{3.13}$$

From the inequality [cf. Billingsley (1986), page 353]

$$\left|\exp(ix) - \sum_{k=0}^{m} \frac{(ix)^k}{k!}\right| \leq \frac{1}{(m+1)!}|x|^{m+1},$$

we have

$$\phi_n(t) = \widetilde{\phi}_n(t) + it\sigma_n^{-1} E\exp\{it\sigma_n^{-1}\widehat{S}_n\}\overline{\Delta}_n + O\big(t^2\sigma_n^{-2}E\overline{\Delta}_n^2\big). \tag{3.14}$$

Let $\phi_{ij}(t) = E\exp\{it\sigma_n^{-1}\Sigma_{k=1,\ldots,n,\ k\neq i,j}T_k\}$. Then

$$E\exp\{it\sigma_n^{-1}\widehat{S}_n\}\overline{\Delta}_n = \sum_{1\le j<i\le c_n}\phi_{ij}(t)E\exp\{it\sigma_n^{-1}(T_i+T_j)\}h(X_i,X_j). \tag{3.15}$$

Let $\|T_1\|_2 = (E|T_1|^2)^{1/2}$. Then, from

$$E\exp\{it\sigma_n^{-1}(T_i+T_j)\}h(X_i,X_j) = O\big(t^2\sigma_n^{-2}\|T_i\|_2\|T_j\|_2\|h(X_i,X_j)\|_2\big),$$

we have

$$\left|E\exp\{it\sigma_n^{-1}\widehat{S}_n\}\overline{\Delta}_n\right| = O\left(t^2\sigma_n^{-2}\sum_{1\le j<i\le c_n}|\phi_{ij}(t)|\|T_i\|_2\|T_j\|_2\|h(X_i,X_j)\|_2\right).$$

Thus, by (3.14),

$$\left|\frac{\phi_n(t)-\widetilde{\phi}_n(t)}{t}\right| \le O\left(t^2\sigma_n^{-3}\sum_{1\le j<i\le c_n}|\phi_{ij}(t)|\|T_i\|_2\|T_j\|_2\|h(X_i,X_j)\|_2\right) + O\big(|t|\sigma_n^{-2}E\overline{\Delta}_n^2\big). \tag{3.16}$$

Let $\gamma_{ij} = \sigma_{ij}^3/\Gamma_{ij}$. From condition (ii) of the theorem, for some $\alpha_4 > 0$, $\gamma_{ij} \ge \alpha_4\gamma_n$, $i,j = 1,2,\ldots,n$. Suppose that the constant C in (1.2) is larger than $16\alpha_4^{-2}$. Then, for $\gamma_n^{-1} \ge \alpha_4^2/16$, the conclusion of the theorem always holds. Hence we may assume that $\gamma_n \ge 16\alpha_4^{-2}$. Using a lemma [cf. Chung (1974), page 229],

$$|\phi_{ij}(t)| \le \exp\left\{-\frac{(t\sigma_n^{-1}\sigma_{ij})^2}{3}\right\}, \qquad |t| \le \frac{1}{4}\gamma_{ij}\frac{\sigma_n}{\sigma_{ij}}. \tag{3.17}$$

This and $\frac{1}{4}\gamma_{ij}\sigma_n/\sigma_{ij} \ge \frac{1}{4}\alpha_4\gamma_n \ge \gamma_n^{1/2}$ imply

$$\int_0^{\gamma_n^{1/2}} t^2|\phi_{ij}(t)|\,dt \le \int_0^{\infty} t^2\exp\left\{-\frac{t^2}{3\alpha_3}\right\}dt < \infty.$$

Hence, by $\|T_i\|_2\|T_j\|_2\|h(X_i,X_j)\|_2 \le n^{-4}\|X_i\|_3^{3/2}\|X_j\|_3^{3/2}$,

$$\int_0^{\gamma_n^{1/2}}\left(t^2\sigma_n^{-3}\sum_{1\le j<i\le c_n}|\phi_{ij}|\|T_i\|_2\|T_j\|_2\|h(X_i,X_j)\|_2\right)dt \le Cn^{-1/2}H_{3n} \le CM_n\gamma_n^{-1}. \tag{3.18}$$

On the other hand, we have

$$\int_0^{\gamma_n^{1/2}} t\sigma_n^{-2}E\overline{\Delta}_n^2\,dt \le C\sigma_n^{-2}\gamma_n n^{-2}H_{2n} \le CM_n\gamma_n^{-1}. \tag{3.19}$$

Combining (3.16), (3.18) and (3.19), relation (3.13) is proved.

Finally to prove (3.12) we note that

$$\left|\phi_n(t) - \widetilde{\phi}_n(t)\right| \le |t|\sigma_n^{-1}E|\overline{\Delta}_n| \prod_{k=c_n+1}^{n} \left|E\exp\left\{it\sigma_n^{-1}T_k\right\}\right|.$$

Let $\delta_j = \frac{1}{2}\sigma_n^{-2}\sigma^2(T_j) - \frac{3}{8}\sigma_n^{-3}E|T_j|^3H$, $j = 1, 2, \dots, n$. Then, following Feller [(1971), page 544], $|E\exp\{it\sigma_n^{-1}T_j\}| \le \exp\{-\delta_j t^2\}$, $|t| \le H$, $j = 1, 2, \dots, n$, and $\prod_{k=c_n+1}^{n} |E\exp\{it\sigma_n^{-1}T_k\}| \le \exp\{-t^2\sum_{k=c_n+1}^{n}\delta_k\}$. Without loss of generality, we assume $\delta_n \ge \delta_{n-1} \ge \cdots \ge \delta_1$. Then $\sum_{j=1}^{n}\delta_j = \frac{1}{6}$ implies

$$\exp\left\{-t^2\sum_{k=c_n+1}^{n}\delta_k\right\} \le \exp\left\{-\frac{n-c_n}{6n}t^2\right\} \le \exp\left\{-\frac{\log\gamma_n}{2\gamma_n}t^2\right\}. \tag{3.20}$$

Therefore, (3.12) follows from

$$\int_{\gamma_n^{1/2}}^{H} \exp\left\{-\frac{\log\gamma_n}{2\gamma_n}t^2\right\}dt \le \frac{C}{\log\gamma_n} \quad \text{and} \quad \sigma_n^{-1}E|\overline{\Delta}_n| \le CM_n\gamma_n^{-1}. \qquad \square$$

PROOF OF THEOREM 2. We note first that the representation (3.1) holds. Let $\overline{F}_n^{-1}(\beta)$ be the β-th quantile of $\overline{F}_n(x)$ for $\beta \in (0,1)$. Then assumptions (i) and (ii) of the theorem yield

$$\left|\overline{F}_n^{-1}(1-\varepsilon) - \overline{F}_n^{-1}(\varepsilon)\right| \le \left(\frac{2}{\varepsilon}\right)^{\alpha} \int_{-\infty}^{\infty} \left[\overline{F}_n(x)(1-\overline{F}_n(x)\right]^{\alpha} dx. \tag{3.21}$$

With arguments similar to the proof of Theorem 1, it is easy to show $|\sigma_n^{-1}\overline{R}_n| \le (C/\sqrt{n})\overline{M}_n$ and

$$E|h(X_i, X_j)|^6 \le \frac{C}{n^{12}}\overline{M}_n^2, \qquad P\left(\sigma_n^{-1}|\widetilde{\Delta}_n| \ge \frac{\Gamma_n}{\sigma_n^3}\overline{M}_n\right) \le O\left(\frac{\Gamma_n}{\sigma_n^3}\overline{M}_n\right).$$

Let $\Delta(J, \overline{F}_n, u, x) = J(u) - J(\overline{F}_n(x)) - J'(\overline{F}_n(x))(u - \overline{F}_n(x))$. For the rate of R_n, we have

$$\begin{aligned}
|R_n| &\le \int_{-\infty}^{\infty} I_{(|F_n(x) - \overline{F}_n(x)| \le \varepsilon/2)} \left| \int_{\overline{F}_n(x)}^{F_n(x)} \left|\Delta(J, \overline{F}_n, u, x)\right| du \right| dx \\
&\quad + \int_{-\infty}^{\infty} I_{(|F_n(x) - \overline{F}_n(x)| > \varepsilon/2)} \left| \int_{\overline{F}_n(x)}^{F_n(x)} \left|\Delta(J, \overline{F}_n, u, x)\right| du \right| dx \\
&= I_{1n} + I_{2n}.
\end{aligned}$$

Since $J(x) = 0$ for $x \in [0, \varepsilon] \cup [1-\varepsilon, 1]$,

$$I_{1n} \le C\int_{\overline{F}_n^{-1}(\varepsilon/2)}^{\overline{F}_n^{-1}(1-\varepsilon/2)} \left|F_n(x) - \overline{F}_n(x)\right|^{2+1/3} dx.$$

Hence by the Hölder inequality, $P(I_{1n} \geq n^{-1}) \leq (C/\sqrt{n})\overline{M}_n$. For I_{2n}, we have

$$\begin{aligned} P(I_{2n} \geq n^{-1}) &\leq Cn \int_{\overline{F}_n^{-1}(1-\varepsilon/3)}^{\infty} P\Big(\big|F_n(x) - \overline{F}_n(x)\big| > \frac{\varepsilon}{2}\Big)\,dx \\ &\quad + Cn \int_{\overline{F}_n^{-1}(\varepsilon/3)}^{\overline{F}_n^{-1}(1-\varepsilon/3)} P\Big(\big|F_n(x) - \overline{F}_n(x)\big| > \frac{\varepsilon}{2}\Big)\,dx \\ &\quad + Cn \int_{-\infty}^{\overline{F}_n^{-1}(\varepsilon/2)} P\Big(\big|F_n(x) - \overline{F}_n(x)\big| > \frac{\varepsilon}{2}\Big)\,dx \\ &= I_{2n}^{(1)} + I_{2n}^{(2)} + I_{2n}^{(3)}, \end{aligned}$$

and it is easy to see that $I_{2n}^{(2)} \leq (C/\sqrt{n})\overline{M}_n$. To estimate $I_{2n}^{(1)}$, let $k = [\alpha] + 1$ and assume that

$$n - \left[n\left(1 - \frac{\varepsilon}{2}\right)\right] \geq 2k + 1, \qquad n\varepsilon \geq 12. \tag{3.22}$$

For those n which do not satisfy (3.22), Theorem 2 also holds if we take

$$C \geq \max\left(\sqrt{\frac{12}{\varepsilon}}, \sqrt{\frac{4\alpha + 6}{\varepsilon}}\right),$$

where C is the constant in (1.4). Thus, for $x \geq \overline{F}_n(1 - \varepsilon/3)$, $[n((1 - \varepsilon/2)] + 1 \leq n(1 - \varepsilon/3) - 1 \leq n\overline{F}_n(x) - 1$. This and Hoeffding [(1956), Theorem 4] give $P(|F_n(x) - \overline{F}_n(x)| > \varepsilon/2) \leq n^{2k}(1 - \overline{F}_n(x))^{2k}$. Hence

$$I_{2n}^{(1)} \leq Cn^{k+1} \int_{\overline{F}_n^{-1}(1-\varepsilon/3)}^{\infty} \big(1 - \overline{F}_n(x)\big)^k \big(E|F_n - \overline{F}_n|^{2k+3}\big)^{1/2}\,dx \leq \frac{C}{\sqrt{n}}\overline{M}_n.$$

Symmetrically, it can be shown that $I_{2n}^{(3)} \leq (C/\sqrt{n})\overline{M}_n$. Combining these results, $P(\sigma_n^{-1}|R_n| \geq \overline{M}_n/\sqrt{n}) = O(\overline{M}_n/\sqrt{n})$. Since the rest of the proof is analogous to that of Theorem 1, it is omitted. □

4. Examples. In this section, we consider two statistical models. The proofs of the conclusions stated are omitted for the sake of brevity. These proofs can be found in Xiang (1991b).

EXAMPLE 1. Consider the model $Y_i = \theta + X_i$, $1 \leq i \leq n$, where X_i are independent random variables with distribution function $G_i(x) = G(x/a_i)$, $a_i \geq \delta$ for some $\delta \in (0, 1]$ and $G(x)$ is a continuous distribution function. Further assume that $G(x) = 1 - G(-x)$, $G(x)$ is strictly increasing in a neighborhood of the origin and, for some $\alpha \in (0, 1]$, $\int_{-\infty}^{\infty} |x|^\alpha\,dG(x) < \infty$. Our goal is to estimate θ from the sample $Y_1, Y_2, \ldots, Y_n$. This model arises, for instance, if the true model is assumed to be $Y_i = \theta + X_i$, where $X_1, X_2, \ldots, X_n$ are i.i.d. with common distribution function $G(x)$, but because of the influence of perturbations the observed model is $Y_i = \theta + a_i X_i$. Let $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ be the order statistics of the sample. Choose a weight function $J(x)$, $J(x) = 0$ for $x \in [0, \varepsilon] \cup [1 - \varepsilon, 1]$, $J(x) > 0$ for $x \in (\varepsilon, 1 - \varepsilon)$, $\varepsilon \in (0, \frac{1}{2})$ and $J'(x) \in \mathrm{Lip}(\frac{1}{3})$. Further, require that $J(x)$ is increasing in $[0, \frac{1}{2}]$, $\int_0^1 J(u)\,du = 1$ and $J(x)$ is symmetric about $x = \frac{1}{2}$. The L-estimate of θ is given by (1.1). Then, from Theorem 2 and the remark following Lemma 3, if $\bar{a}_n = (1/n)\sum_{i=1}^n a_i \le M < \infty$,

$$\sup_x \left| P\left(\sigma_n^{-1}(S_n - \theta) \le x\right) - \Phi(x) \right| \le \frac{C}{\sqrt{n}}. \tag{4.1}$$

EXAMPLE 2. Consider the gross error model [see Tukey (1960), Huber (1964, 1981) and Hampel, Ronchetti, Rousseeuw and Stahel (1986)] $Y_i = \theta + X_i$, $1 \le i \le n$, where the X_i are independent random variables with distribution functions $G_i(x) = (1 - \varepsilon_i)\Phi(x) + \varepsilon_i H_i(x)$, $\varepsilon_i \in [0, \delta]$ for some $\delta \in [0, 1)$, $\Phi(x)$ denotes the standard normal distribution function and the $H_i(x)$ are symmetric, belonging to some class of distribution functions. We further require that each $H_i(x)$ is continuous and for an $\alpha \in (0, 1]$, $h_i = \int_{-\infty}^{\infty} |x|^\alpha\, dH_i(x) < \infty$, $1 \le i \le n$. Our goal is to estimate θ from the sample $Y_1, Y_2, \dots, Y_n$. This model arises, for instance, if the distribution functions of the observations are assumed to be normal with variance 1, but because of the occasional presence of gross errors they differ from the normal distribution, and these differences for different observations might not be the same. Choose $J(x)$ of the same form as that in the previous example. Then if $(1/n)\sum_{i=1}^n h_i^{1/\alpha}$ is bounded, (4.1) holds.

Acknowledgments. This paper is part of my Ph.D. dissertation under the supervision of Professor Stephen M. Stigler. I am very grateful to Professor Stigler for introducing me to this subject and for his continuous encouragement and stimulating advice, and to Professor R. R. Bahadur for helpful discussions. I would also like to thank an Associate Editor and two referees for their valuable comments, which led to this improved version of the manuscript.

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